J. Math. Soc. Japan
Vol. 43, No. 4, 1991

# A geometric characterization of the groups $M_{12}$, He and Ru

By Richard WEISS*

(Received July 25, 1990)
(Revised Nov. 2, 1990)

## 1. Introduction.

As in [1], we define a geometry $\Gamma=(\mathcal{B}_1, \cdots, \mathcal{B}_r; *)$ to be an ordered sequence of $r$ pairwise disjoint non-empty sets $\mathcal{B}_i$ together with a symmetric incidence relation $*$ on their union $\mathcal{B}=\mathcal{B}_1\cup\cdots\cup\mathcal{B}_r$ such that if $F$ is any maximal set of pairwise incident elements (i. e. a maximal flag), then $|F\cap\mathcal{B}_i|=1$ for $i=1, \cdots, r$. The number $r$ is called the rank of $\Gamma$. The geometry $\Gamma$ is called connected if the $r$-partite graph $(\mathcal{B}, *)$ is connected.

We recall that a generalized $n$-gon (for $n\geqq 2$) is a geometry $\Gamma=(\mathcal{P}, \mathcal{L}; *)$ of rank 2 such that the bipartite graph $(\mathcal{P}\cup\mathcal{L}, *)$ has diameter $n$ and girth $2n$. The elements of $\mathcal{P}$ are called points and the elements of $\mathcal{L}$ lines. A generalized $n$-gon is called *thick* if every vertex of the graph $(\mathcal{P}\cup\mathcal{L}, *)$ has at least three neighbors. If $\Pi=(\mathcal{P}, \mathcal{L}; *)$ is a thick generalized $n$-gon, we define $\Pi_0$ to be the geometry $(\mathcal{F}, \mathcal{P}\cup\mathcal{L}; *)$, where $\mathcal{F}$ is the set of maximal flags of $\Pi$ and $*$ the natural incidence relation. Then $\Pi_0$ is a generalized $2n$-gon having two lines through every point (but more than two points on a line). We will call such a generalized $2n$-gon *point-thin.*

The building attached to the group $PSp_4(p^k)$ is a generalized quadrangle. For $p=2$, this geometry, which we denote by $Q(k)$, is self-dual (see [3]), and $Q(k)_0$ is a point-thin generalized octagon on which $\mathrm{aut}(PSp_4(p^k))$ acts flag-transitively. Similarly, there is a self-dual generalized hexagon associated with the group $G_2(3^k)$, which we denote by $\mathcal{H}(k)$, such that $\mathrm{aut}(G_2(3^k))$ acts flag-transitively on the generalized dodecagon $\mathcal{H}(k)_0$. The building attached to the group ${}^2F_4(2^k)$ is a generalized octagon with $1+2^k$ points on a line. We call this octagon $\mathcal{O}(k)$ and write $\mathcal{O}(k)^\circ$ to denote its dual.

Let $F$ be a non-maximal flag of a geometry $\Gamma=(\mathcal{B}_1, \cdots, \mathcal{B}_r; *)$. The set

$$J=\{i \mid \mathcal{B}_i\cap F\neq\emptyset\}$$

is called the type of $F$. For each $m\notin J$, let $\mathcal{B}_m^F=\{u\in\mathcal{B}_m \mid u*x \text{ for all } x\in F\}$.

* Research partially supported by NSF Grant DMS-8901904.

The residue $\Gamma_F$ is defined to be the rank $r-|J|$ subgeometry of $\Gamma$ on the sets $\mathcal{B}_m^F$. We will always assume that for given type $J$, the residue $\Gamma_F$ is independent, up to isomorphism, of the given flag $F$. Then (as usual) there is a diagram with $r$ nodes associated with $\Gamma$, the links of which reflect the structure of the rank 2 residues of $\Gamma$. In particular, a link consisting either of $n-2$ strokes ($n \geqq 2$) or a single stroke labelled $(n)$ indicates a generalized $n$-gon and a link labelled $\subset$ indicates the geometry of vertices and edges of a complete graph.

An extended generalized $n$-gon is a connected rank 3 geometry $\Gamma=(\mathcal{P}, \mathcal{L}, \mathcal{C}; *)$ with diagram


where the elements of $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{C}$ are called points, lines and circles, respectively. In this paper, we will examine extended generalized $n$-gons $\Gamma$ under the assumption that there is a group $G \leqq \mathrm{aut}(\Gamma)$ acting flag-transitively on $\Gamma$. The case $n=3$ leads to the three Mathieu groups $M_{22}$, $M_{23}$ and $M_{24}$. The case $n=4$ was first examined in pioneering work of Buekenhout and Hubaut [**2**] as a special case of their locally polar spaces. The classification of extended generalized quadrangles with classical point-residues was completed in [**5**], [**14**] and [**17**]. There exit well known extended generalized $n$-gons for $n=6$ associated with the sporadic groups $J_2$ and $Suz$ [**11**] and for $n=8$ with $Ru$ [**9**]. The universal covers of these geometries are infinite [**8**], so some additional condition is needed to characterize them. In [**15**], we showed that the $J_2$- and $Suz$-geometries, together with two more having automorphism group $2^6 : G_2(2)$ and $2^7 : G_2(2)$, are the only extended generalized hexagons which have as point-residues finite classical thick generalized hexagons and which satisfy the condition that

(*) there exist triples of pairwise collinear points not lying on any circle.

In [**16**], this result was extended to a classification in the case that the point-residues are finite classical point-thin generalized hexagons. In this paper, we treat the remaining cases, those in which the point-residues are finite classical point-thin or thick generalized $n$-gons with $n \geqq 8$. (We call a point-thin generalized $2n$-gon $\Pi_0$ classical if the thick generalized $n$-gon $\Pi$ is classical.)

(1.1) THEOREM. *Let $\Gamma=(\mathcal{P}, \mathcal{L}, \mathcal{C}; *)$ be an extended generalized octagon with a flag-transitive group $G \leqq \mathrm{aut}(\Gamma)$. Suppose $\Gamma$ satisfies* (*) *and that for each $P \in \mathcal{P}$, the residue $\Gamma_P$ is isomorphic to $Q(k)_0$ for some $k \geqq 1$. Then one of the following holds:*

(a) $k=1$, $|\mathcal{P}|=56$ *and* $G \cong L_3(4).2^2$,

(b) $k=1$, $|\mathcal{P}|=112$ *and* $G\cong 2\cdot L_3(4).2^2$,
(c) $k=1$, $|\mathcal{P}|=66$ *and* $G\cong M_{12}$,
(d) $k=1$, $|\mathcal{P}|=132$ *and* $G\cong 2\cdot M_{12}$ *or*
(e) $k=2$, $|\mathcal{P}|=2058$ *and* $G\cong He.2$.
*In each case* (a)-(e), *$\Gamma$ is uniquely determined.*

(1.2) THEOREM. *Let $\Gamma=(\mathcal{P}, \mathcal{L}, \mathcal{C}; *)$ be an extended generalized octagon with a flag-transitive group $G\leqq \mathrm{aut}(\Gamma)$. Suppose $\Gamma$ satisfies $(*)$ and that for each $P\in\mathcal{P}$, the residue $\Gamma_P$ is isomorphic to $\mathcal{O}(k)$ or $\mathcal{O}(k)^{\circ}$ for some $k\geqq 1$. Then $\Gamma_P\cong \mathcal{O}(1)$, $|\mathcal{P}|=4060$, $G\cong Ru$ and $\Gamma$ is uniquely determined.*

(1.3) THEOREM. *There are no flag-transitive extended generalized dodecagons satisfying $(*)$ for which the point-residues are isomorphic to $\mathcal{H}(k)_0$ for any $k$.*

For the sake of completeness (see [**6, 7**]), we include the following observation.

(1.4) THEOREM. *There is no flag-transitive geometry $\Gamma=(\mathcal{B}_1, \cdots, \mathcal{B}_4; *)$ having diagram*


*such that for $P\in\mathcal{B}_1$, the residue $\Gamma_P$ is isomorphic to one of the geometries classified in* (1.1) *or* (1.2).

The geometry (b) of (1.1) is the subgeometry of geometry (e) left pointwise fixed by an involution of type 2A of *He*. The group $G$ acts on $\mathcal{P}$ as a permutation group of rank 5 in cases (b) and (d), rank 4 in case (e) and rank 3 in the remaining cases. The rank 3 action of *Ru* on 4060 points is well known [**9**].

To prove (1.1)-(1.3), we start by observing in (2.2) below that a famous result of Suzuki [**12**] on the non-existence of certain one-point extensions implies that $k$ is bounded. Condition $(*)$ then serves two purposes. It is first applied in (2.4) to show that, in general, the stabilizer $G_P$ of a point $P$ acts faithfully on the residue $\Gamma_P$. This alone implies the non-existence of $\Gamma$ in the cases $\Gamma_P\cong\mathcal{O}(1)^{\circ}$ and $\Gamma_P=\mathcal{H}(1)_0$ in §6 and §7 below. In the remaining cases, condition $(*)$ provides extra cycles in the collineation graph on the points of $\Gamma$. These extra cycles are made to yield relations which do not hold in the universal cover of these geometries. The group $G$ is then determined by coset enumerations which have been carried out using CAYLEY.

It is perhaps of interest to note that the list of sporadic simple groups characterized in terms of towers of locally polar-spaces of extended generalized polygons now includes $M_{22}$, $M_{23}$, $M_{24}$, $Fi_{22}$, $Fi_{23}$, $Fi_{24}$, $McL$, $Suz$, $HS$, $J_2$, $Co_3$, $Co_1$,

$M_{12}$, $He$ and $Ru$.

Notation such as $2\cdot L_3(4).2^2$ and $A_6.2_2$ is taken from [**4**].

## 2. Preliminary observations.

Let $\Gamma=(\mathcal{P}, \mathcal{L}, \mathcal{C}; *)$ be an extended generalized $n$-gon with a flag-transitive group $G\leqq \mathrm{aut}(\Gamma)$. Suppose that for each $P\in\mathcal{P}$, the residue $\Gamma_P$ is isomorphic to $\mathcal{Q}(k)_0$, $\mathcal{O}(k)$, $\mathcal{O}(k)^\circ$ or $\mathcal{H}(k)_0$ for some $k\geqq 1$ (so $n=8$ in the first three cases and $n=12$ in the last). For each $P\in\mathcal{P}$, let $G_P^\circ$ denote the group induced by $G_P$ in $\Gamma_P$.

(2.1) LEMMA. *If $\Gamma_P\cong\mathcal{Q}(k)_0$, then either $G_P^\circ$ contains a normal subgroup isomorphic to $PSp_4(2^k)$ or $k=1$ and $G_P^\circ\cong A_6.2_2$ or $A_6.2_3$. If $\Gamma_P\cong\mathcal{O}(k)$ or $\mathcal{O}(k)^\circ$, then either $G_P^\circ$ contains a normal subgroup isomorphic to ${}^2F_4(2^k)$ or $k=1$ and $G_P^\circ\cong{}^2F_4(2)'$. If $\Gamma_P\cong\mathcal{H}(k)_0$, then $G_P^\circ$ contains a normal subgroup isomorphic to $G_2(3^k)$.*

PROOF. If $\Gamma_P\cong\mathcal{Q}(k)_0$, respectively $\mathcal{H}(k)_0$, then $G_P$ contains a subgroup of index two acting flag-transitively on the generalized $n$-gon $\mathcal{Q}(k)$, respectively $\mathcal{H}(k)$. The claim now follows from [**10**]. ■

Let $(P, l, \gamma)$ be a maximal flag of $\Gamma$. Then $\mathcal{P}^l=\mathcal{P}^{(l,\gamma)}$ since $\Gamma_l$ is a generalized 2-gon, so $|\mathcal{P}^l|=2$ since $\Gamma_\gamma$ is a geometry of type $\subset$. Also $|\mathcal{L}^{(P,\gamma)}|=s+1$, so $|\mathcal{P}^\gamma|=s+2$, where $s=2^k$ if $\Gamma_P\cong\mathcal{Q}(k)_0$ or $\mathcal{O}(k)$, $s=4^k$ if $\Gamma_P\cong\mathcal{O}(k)^\circ$ and $s=3^k$ if $\Gamma_P\cong\mathcal{H}(k)_0$.

Let $\Delta$ be the collinearity graph on $\mathcal{P}$. (This is the graph with vertex set $\mathcal{P}$ with two points joined by an edge whenever they lie on a line of $\Gamma$.) Since $\Gamma$ is connected, so is $\Delta$. Choose $P\in\mathcal{P}$. We define a map $\phi$ from $\mathcal{L}^P$ to $\Delta(P)$, the set of neighbors of $P$ in $\Delta$, by setting $\phi(l)=Q_l$ for each $l\in\mathcal{L}^P$, where $\mathcal{P}^l=\{P, Q_l\}$. The map $\phi$ is of course surjective. By (2.1), $G_P$ acts primitively on $\mathcal{L}^P$, so $\phi$ is in fact bijective. Thus, we can identify an element $x\in\mathcal{L}\cup\mathcal{C}$ with the set $\mathcal{P}^x$ and we will do so from now on. We denote by $G_\gamma^\circ$ the group induced by the stabilizer $G_\gamma$ on $\gamma$, i. e. on $\mathcal{P}^\gamma$, and $N_\gamma$ is the kernel of the action of $G_\gamma$ on $\gamma$.

(2.2) LEMMA. *$k\leqq 2$ and $G_\gamma^\circ\cong S_{s+2}$ if $\Gamma_P\cong\mathcal{Q}(k)_0$, $k=1$ and $G_\gamma^\circ\cong S_{s+2}$ if $\Gamma_P\cong\mathcal{O}(k)$ or $\mathcal{H}(k)_0$ and $k=1$ and $G_\gamma^\circ\cong PGL_2(5)$ if $\Gamma_P\cong\mathcal{O}(k)^\circ$.*

PROOF. By (2.1), $(G_\gamma^\circ)_P$ contains a normal subgroup isomorphic to $L_2(2^k)$ if $\Gamma_P\cong\mathcal{Q}(k)_0$ or $\mathcal{O}(k)$, to $Sz(2^k)$ if $\Gamma_P\cong\mathcal{O}(k)^\circ$ and to $PGL_2(3^k)$ if $\Gamma_P\cong\mathcal{H}(k)_0$. Since $G$ acts flag-transitively on $\Gamma$, the group $G_\gamma^\circ$ is a transitive extension of $(G_\gamma^\circ)_P$. By [**12**], it follows that $|\gamma|\leqq 6$. ■

(2.3) LEMMA. *Let $\Pi=(\mathcal{P}, \mathcal{L}; *)$ be a generalized $m$-gon and let $\Phi$ denote* *the bipartite graph* $(\mathcal{P}\cup\mathcal{L}, *)$. *Let* $K=\mathrm{aut}(\Phi)$ *and* $P\in\mathcal{P}$. *If* $\Pi=(\mathcal{P}, \mathcal{L}; *)=\mathcal{Q}(k)_0, \mathcal{O}(k)$ *or* $\mathcal{O}(k^\circ)$, *then* $K_P^{[2]}\cap K_T=1$ *for each* $T\in\mathcal{P}$ *at distance eight from* $P$ *in* $\Phi$ *and* $K_P^{[4]}=1$ *unless* $\Pi=\mathcal{O}(k)$, *in which case* $|K_P^{[4]}|=2^k$ *and* $K_P^{[4]}\cap K_U^{[4]}=1$ *for all vertices* $U$ *distinct from* $P$. *If* $\Pi=\mathcal{H}(1)_0$, *then* $K_P$ *acts irreducibly on* $K_P^{[4]}\cong 3^2$, $K_P^{[5]}=1$ *and* $K_P^{[4]}\cap K_U^{[4]}=1$ *for vertices* $U$ *at distance four from* $P$ *in* $\Pi$. *Here* $K_P^{[i]}$ *denotes the largest subgroup of* $K_P$ *fixing all the vertices of* $\Phi$ *at distance at most* $i$ *from* $P$ *in* $\Phi$.

PROOF. These facts are well known. ■

Choose $l=\{P, Q\}\in\mathcal{L}^P$. For $i=1, 2, 3$ and 4, let $X_i$ denote the set of elements of $\mathcal{L}^P$ at distance $2i$ from $l$ in the graph $\Pi_P=(\mathcal{L}^P\cup C^P, *)$ and let $Y_i=\{U\in\mathcal{P}\mid\{P, U\}\in X_i\}$. For each $U\in\mathcal{P}$, let $N_U$ denote the kernel of the action of $G_U$ on $\Delta(U)$.

(2.4) LEMMA. *If condition* $(*)$ *is satisfied, then either* $G_P$ *acts faithfully on* $\Delta(P)$ *or* $\Gamma_P\cong\mathcal{O}(1)$, $|N_P|=2$ *and* $Y_2\subseteqq\Delta(Q)$ *but* $Y_m\cap\Delta(Q)=\emptyset$ *for* $m=3$ *and* 4.

PROOF. Suppose first that $\Gamma_P\cong\mathcal{Q}(1)_0, \mathcal{Q}(2)_0$ or $\mathcal{O}(1)^\circ$. We claim that $N_Q$ acts trivially on $\Delta(P)$. By condition $(*)$, $\Delta(Q)\cap Y_m\neq\emptyset$ for some $m\in\{2, 3, 4\}$. If $H$ denotes the pointwise stabilizer of $Y_1$ in $G_{P,Q}$, then $H\cap G_T\leqq N_P$ for each $T\in Y_4$ by (2.3). Thus we can assume that $\Delta(Q)\cap Y_4=\emptyset$, so $m=2$ or 3. By (2.1), $G_{P,Q}$ acts transitively on both $Y_2$ and $Y_3$, so $N_Q$ acts trivially on $Y_m$. Thus (2.3) implies that $N_Q\leqq N_P$ as claimed. Since $|N_Q|=|N_P|$, we have $N_Q=N_P$, and so $N_P$ is normal in $\langle G_P, G_l\rangle$. Since $\Delta$ is connected, $\langle G_P, G_l\rangle$ acts transitively on $\mathcal{P}$. Thus $N_P=1$.

If $\Gamma_P\cong\mathcal{O}(1)$ or $\mathcal{H}(1)_0$, then $N_P=1$ follows by a similar argument unless $\Delta(Q)\cap Y_2\neq\emptyset$ and $\Delta(Q)\cap Y_m=\emptyset$ for $m\geqq 3$. In this case, (2.3) implies that $N_Q\cap N_R\leqq N_P$ for $R\in Y_2$, so $N_Q\cap N_R\leqq N_Q\cap N_P$ and $N_Q\cap N_R\leqq N_R\cap N_P$. Since these three groups are conjugate in $G$, we conclude that

$$N_P\cap N_Q = N_P\cap N_R = N_Q\cap N_R.$$

Thus $N_P\cap N_Q$ is normal in $\langle G_{P,Q}, G_{P,R}, G_l\rangle=\langle G_P, G_l\rangle=G$, so $N_P\cap N_Q=1$. By (2.3), we have either $N_Q\leqq N_P\cap N_Q=1$ or $\Gamma_P\cong\mathcal{O}(1)$ and $|N_Q|=|N_Q/N_Q\cap N_P|=2$ or $\Gamma_P\cong\mathcal{H}(1)_0$ and $|N_Q|=|N_Q/N_Q\cap N_P|=9$. Suppose we are in this last case.

Let $M_\gamma=\langle N_U \mid U\in\gamma\rangle$. Since $N_P\cap N_Q=1$, we have $|M_\gamma|\geqq|N_PN_Q|=3^4$. Since $|N_\gamma|=|N_\gamma/N_P|\cdot|N_P|=3^5\cdot 3^2$, it follows that $|N_\gamma/M_\gamma|\leqq 3^3$. On the other hand, $M_\gamma$ acts trivially on the set of vertices of the graph $\Pi_P=(\mathcal{L}^P\cup C^P, *)$ at distance three from $\gamma$, so $G_{P,\gamma}$ acts non-trivially on $N_\gamma/M_\gamma$. Thus $G_\gamma$ acts non-trivially on $N_\gamma/M_\gamma$. Since $G_\gamma^\circ\cong S_5$ and $|N_\gamma/M_\gamma|\leqq 3^3$, this is impossible. ■

(2.5) LEMMA. *If* $\Gamma_P\cong\mathcal{Q}(1)_0, \mathcal{Q}(2)_0$ *or* $\mathcal{O}(1)$, *then* $G_l$ *acts trivially on the set of* $G_{P,Q}$*-orbits in* $\Delta(P)\cap\Delta(Q)$.

PROOF. Let $v \in G_l \setminus G_{P,Q}$. By (2.1), the group $G_P$ contains elements exchanging any given pair of elements of $\Delta(P)$ (i. e. all the suborbits of $G_P$ in $\Delta(P)$ are self-paired). It follows that if $U \in \Delta(P) \cap \Delta(Q)$, then $G_U$ contains elements exchanging $P$ and $Q$. If $g$ is such an element, then $gv \in G_{P,Q}$, so $U$ and $U^{gv}=U^v$ lie in the same $G_{P,Q}$-orbit. ■

## 3. The proof of (1.1): $k=1$.

Let $\Gamma$ and $G$ fulfill the hypotheses of (1.1). By (2.2), we have $k \leqq 2$. Suppose first that $k=1$. By (2.1) and (2.4), $G_P \cong A_6.2_2$, $A_6.2_3$ or $A_6.2^2$. If $G_P \cong A_6.2_i$, then there exist generators $a$, $b$, $c$, $d$ and $u$ of $G_P$ such that

(3.1)
$$a^2=[a, b]=1, \quad [a, c]=b, \ a^r=d \ \text{ and } \ b^r=c \ \text{ for } r=ada, \qquad b^u=d \ \text{ and } \ c^u=c$$

and either

(3.2/2)
$$u^2=(a(ru)^2)^4=1 \quad \text{if} \quad i=2 \quad \text{or}$$

(3.2/3)
$$u^2=c \quad \text{and} \quad (ru)^8=1 \quad \text{if} \quad i=3.$$

If $G_P \cong A_6.2^2$, then there exists an additional generator $t$ such that (3.1) and

(3.3)
$$t^2=[t, a]=[t, b]=[t, c]=[t, d]=1, \qquad [t, u]=c \quad \text{and} \quad (tru)^8=u^2=1$$

hold. If we set $x_0=abt$, $x_1=bt$, $x_2=t$, $x_3=tc$, $x_4=tcd$ and $x_5=x_0^u$, then

$$x_i^\rho=x_{4-i} \text{ for } \rho=x_0^{x_4} \text{ and } 0 \leqq i \leqq 4, \ x_i^\sigma=x_{6-i} \text{ for } \sigma=x_1^{x_5} \text{ and } 1 \leqq i \leqq 5,$$
$$x_0^2=x_1^2=[x_0, x_1]=[x_0, x_2]=1, \quad [x_0, x_3]=x_1x_2 \text{ and } (\rho\sigma)^4=1.$$

These are just the Steinberg relations for $PSp_4(2)$, from which (3.1)-(3.3) are easily deduced. By the definition of $\mathcal{Q}(1)_0$, there exists a point $Q \in \Delta(P)$ and a circle $\gamma \in \mathcal{C}^P \cap \mathcal{C}^Q$ such that $G_{P,\gamma}=\langle a, b, c, d, t\rangle$ and $G_{P,Q}=\langle b, c, d, u, t\rangle$, where the element $t$ is to be ignored unless $G_P \cong A_6.2^2$. If $N_\gamma$ denotes the kernel of the action of $G_\gamma$ on $\gamma$, then $N_\gamma=\langle b, c, t\rangle$. Since $G_\gamma^\circ \cong S_4$, there exists an element $v \in G_\gamma$ inducing the transposition $(P, Q)$ on $\gamma$. Then

$$v^2 \equiv (va)^3 \equiv [v, d] \equiv 1 \qquad (\text{mod } N_\gamma).$$

We have in fact

(3.4) LEMMA. *The element $v$ can be chosen so that $v^2=(va)^3=[v, c]=1$, $[v, u]=b^ic^jd^i$ and either*

(A) $[v, t]=1$, $[v, b]=c$ *and* $[v, d]=c^{i+1}$ *or*
(B) $[v, t]=1$, $[v, b]=tc$ *and* $[v, d]=tc^i$ *or*
(C) $[v, t]=c$, $[v, b]=c$ *and* $[v, d]=c^{i+1}$

*for $i, j=0$ or 1. If $G_P \cong A_6.2_2$ or $A_6.2_3$, then, in fact,* (A) *holds (without the relation $[v, t]=1$).*

PROOF. The element $v$ can be chosen conjugate to $a$ in $N_\gamma$, so that $v^2=1$. Suppose that $G_P \cong A_6.2_2$ or $A_6.2_3$. Since $G_\gamma$ acts on $N_\gamma \cong 2^2$, it follows that $[vd, N_\gamma]=1$. In particular, $va$ acts fixed point freely on $N_\gamma$, so $(va)^3=1$. Since $[v, d] \in N_\gamma$ and $d^2=1$, we have $[v, d]=c^x$ for $x=0$ or 1. The element $v$ normalizes both $\langle b, c, d\rangle$ and $G_{P,Q}=\langle b, c, d, u\rangle$, so $[v, u]=b^i c^j d^k$ for $i, j, k=0$ or 1. Since $u^2=1$, we have $k=i$. Then $u^{vd}=(b^i c^j d^i u)^d = b^{i+1}c^{j+1}d^{i+1}u$ and $u^{dv}=(bcdu)^v = b^{i+1}c^{x+i+j}d^{i+1}u$. Since $[u, [v, d]]=1$, it follows that $x \equiv i+1 \pmod 2$.

Now suppose that $G_P \cong A_6.2^2$, so $N_\gamma \cong 2^3$. With respect to the basis $b, t, c$ of $N_\gamma$, the elements $a$ and $d$ are represented by the matrices

$$A=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

where we interpret matrices as acting on column vectors by left multiplication. If $V \in GL_3(2)$ satifies $V^2=(AV)^3=[V, D]=1$, then

$$V=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

We will call these three possibilities case (A), case (B) and case (C). Since $[v, d] \in N_\gamma$ and $d^2=1$, we have $[v, d]=c^x t^y$. In cases (A) and (B), $C_{\langle b,t,c\rangle}(va)=\langle t\rangle$. Replacing $v$ by $vt$ if necessary, we can assume that $(va)^3=1$. In case (C), we arrange that $(va)^3=1$ by substituting $vbt$ for $v$ if necessary. The element $v$ normalizes $\langle b, c, d, t\rangle$ and $G_{P,Q}=\langle b, c, d, t, u\rangle$ and $u^2=1$, from which it follows (as above) that $[v, u]=b^i c^j d^i$ for $i, j=0$ or 1. Then $u=(u^v)^v=(b^i c^j d^i u)^v = c^{ix}t^{iy}u$, so $xi=yi=0$ in cases (A) and (C). By a similar calculation, we find that $xi=0$ and $yi=i$ in case (B). We then calculate that $c^y u = u^{c^x t^y} = u^{[v,d]} = (b^{i+1}c^{j+1}d^{i+1}u)^{vd} = c^{1+i+x}t^y u$ and therefore $y=0$ and $x \equiv i+1 \pmod 2$ (using $xi=yi=0$) in cases (A) and (C). By a similar calculation, we find that $y=1$ and $x=i$ in case (B). ∎

We now apply condition (*). Recall the definition of $Y_i$ from § 2 above.

(3.5) LEMMA. *If $\Delta(Q) \cap Y_2 \neq \emptyset$, then $G_P \cong A_6.2^2$,* (3.4.B) *holds, $|\mathcal{P}|=66$, $G \cong M_{12}$ and $\Gamma$ is uniquely determined.*

PROOF. Let $R=Q^{rur}$. Then $R \in Y_2$ and $G_{P,Q,R}=\langle b, t\rangle$. If $\Delta(Q) \cap Y_2 \neq \emptyset$, then $Y_2 \subseteq \Delta(Q)$. Thus $R \in \Delta(P) \cap \Delta(Q)$, and so $R^{vd} \in \Delta(P) \cap \Delta(Q)$ as well. We have $G_{P,Q,R^{vd}}=G_{P,Q,R}$ in cases (A) and (B) and $G_{P,Q,R^{vd}}=\langle b, tc\rangle$ in case (C). The only fixed points of $b$ in $\Delta(P)$ not co-circular with $\{P, Q\}$ are $R$ and $R^c$. Since $tc$ fixes neither $R$ nor $R^c$, it follows that we are in case (A) or (B) and that $vc^k d \in G_{P,Q,R}$ for $k=0$ or 1. Substituting $vc$ for $v$ if necessary, we can assume that $k=0$. Thus $vd\cdot u^r$ induces a 3-cycle on $\{P, Q, R\}$. In case (A), we have $C_{G_{P,Q,R}}(vdu^r)=\langle b\rangle$ and $t^{vdu^r}=tb$, from which we conclude that $G_P \not\cong A_6.2^2$. Let $\tilde{G}_{i,j,m,n}$ be the group generated by elements $a, b, c, d, u$ and $v$ defined by the relations (3.1), (3.2/m), (3.4.A) (without $[v, t]=1$) and $(vdu^r)^3=b^n$. By coset enumeration, we find that $\tilde{G}_{i,j,m,n}=1$ for all $i, j, m$ and $n$. Thus $G_P \cong A_6.2^2$, we are in case (B) and $C_{G_{P,Q,R}}(vdu^r)=1$. This time let $\tilde{G}_{i,j}$ be the group generated by elements $a, b, c, d, u, t$ and $v$ defined by the relations (3.1), (3.3), (3.4.B) and $(vdu^r)^3=1$. By coset enumeration, we find that $\tilde{G}_{i,j}=1$ unless $i=0$ and $j=1$, in which case the index of $\langle a, b, c, d, t, u\rangle \cong A_6.2^2$ in this group is 66. Thus $|G_{0,1}|=|M_{12}|$. Let $\tilde{G}=\tilde{G}_{0,1}$. Since $M_{12}$ contains subgroups $H_1 \cong A_6.2^2$, $H_2 \cong [64]$ and $H_3 \cong 2^3.S_4$ such that $H_1 \cap H_2 \cong [32]$, $H_1 \cap H_3 \cong 2^3.S_3$ and $H_2 \cap H_3 \cong 2^3.2^2$ (see [**4**]), it follows that $M_{12}$ actually acts on an extended generalized octagon with point residues isomorphic to $\mathcal{Q}(1)_0$; it is easily seen that this geometry satisfies condition (*). (In fact, $Y_m \subseteq \Delta(Q)$ for $m=2$ and 4.) We conclude that $G \cong \tilde{G} \cong M_{12}$ and $\Gamma$ is isomorphic to

$$\Gamma(\tilde{G}\,;\ \tilde{H}_1, \tilde{H}_2, \tilde{H}_3),$$

the geometry with points the left-cosets of $\tilde{H}_1=\langle a, b, c, d, t, u\rangle$, lines the left-cosets of $\tilde{H}_2=\langle b, c, d, t, u, v\rangle$ and circles the left-cosets of $\tilde{H}_3=\langle a, b, c, d, t, v\rangle$ in $\tilde{G}$, in which two of these left-cosets are incident whenever their intersection contains a left-coset of the intersection of the corresponding subgroups. ■

(3.6) LEMMA. *If $\Delta(Q) \cap Y_3 \neq \emptyset$, then either*

(i) $|\mathcal{P}|=56$ *and* $G \cong L_3(4).2_1$, $L_3(4).2_3$ *or* $L_3(4).2^2$ *or*

(ii) $|\mathcal{P}|=112$ *and* $G \cong 2\cdot L_3(4).2_1$, $2\cdot L_3(4).2_3$ *or* $2\cdot L_3(4).2^2$ (*where $G'$ is perfect*). *In both cases, $\Gamma$ is uniquely determined. If $G_P \cong A_6.2^2$, then* (3.4.C) *holds.*

PROOF. If $S=Q^{r^{u r}}$, then $S \in Y_3$. If $\Delta(Q) \cap Y_3 \neq \emptyset$, then $\Delta(Q) \subseteq Y_3$, so $S \in \Delta(Q)$. Then $vg \in G_S$ for some $g \in G_{P,Q}$ by (2.5). Suppose first that $G_P \cong A_6.2^2$, so $G_{P,Q,S}=\langle bt\rangle$. Thus $\langle bt\rangle$ is normalized by $vg$. This implies that (3.4.C) holds, and we can assume that $g \in \langle c, t\rangle$. Replacing $v$ by $vc$ if necessary, we can assume that in fact $g \in c\langle t\rangle$. Let $\tilde{G}_{i,j,m,n}$ be the group generated by elements $a, b, c, d, t, u$ and $v$ defined by the relations (3.1), (3.3), (3.4.C) and $(vct^m\cdot r^{ur})^3=(bt)^n$ for $i, j, m, n=0$ or 1. By coset enumeration, we find that $\tilde{G}_{i,j,m,n}=1$ unless $i=j=1$ and $m=n=0$ and that the index of $\langle a, b, c, d, t, u\rangle$ in $\tilde{G}_{1,1,0,0}$ is 112. Let $\tilde{G}=\tilde{G}_{1,1,0,0}$. Then $|\tilde{G}|=2\cdot|L_3(4)|\cdot 2^2$. Since $2\cdot L_3(4).2^2$ has the right configuration of subgroups (see [**4**]), it follows that either $G \cong 2\cdot L_3(4).2^2$ and $\Gamma \cong \Gamma(\tilde{G}\,;\ \tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$ with $\tilde{H}_1=\langle a, b, c, d, t, u\rangle$, $\tilde{H}_2=\langle b, c, d, t, u, v\rangle$ and $\tilde{H}_3=\langle a, b, c, d, t, v\rangle$ or $G \cong L_3(4).2^2$ and $\Gamma$ is isomorphic to the quotient of $\Gamma(\tilde{G}\,;\ \tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$ obtained by identifying vertices at distance 4 in $\Delta$.

Suppose that $G_P \cong A_6.2_i$ for $i=2$ or 3, so $G_{P,Q,S}=1$. In this case, we know only that $g \in G_{P,Q}$. By replacing $v$ by $vc$ if necessary, we can assume that in fact $g=b^l cd^m u^n$ for $l$, $m$, $n=0$ or 1. Let $\tilde{G}_{i,j,k,l,m,n}$ be the group generated by elements $a$, $b$, $c$, $d$, $u$ and $v$ defined by the relations (3.1), (3.2/k), (3.4.A) (without the relation $[v, t]=1$) and $(vb^l cd^m u^n \cdot r^{ur})^3=1$. By coset enumeration, we find that $\tilde{G}_{i,j,k,l,m,n}=1$ unless $l=m=n=0$, $i=1$ and either $j=1$ and $k=2$ or $j=0$ and $k=3$. It is easily checked, however, however, that $\tilde{G}_{1,1,2,0,0,0}$ and $\tilde{G}_{1,0,3,0,0,0}$ are both isomorphic to subgroups of index 2 in the group $\tilde{G}$ defined in the previous paragraph. ■

(3.7) LEMMA. *If $\Delta(Q) \cap Y_4 \neq \emptyset$, then either $G$ and $\Gamma$ are as in* (3.5) *or* (3.6.i) *or $G_P \cong A_6.2^2$,* (3.4.B) *holds, $|\mathcal{P}|=132$, $G \cong 2 \cdot M_{12}$ and $\Gamma$ is uniquely determined.*

PROOF. If $\Delta(Q) \cap Y_4 \neq \emptyset$, then either $Y_4 \subseteq \Delta(Q)$ or $G_P \cong A_6.2_2$, $Y_4$ consists of two self-paired $G_{P,Q}$-orbits and $\Delta(Q) \cap Y_4$ is one of them. Suppose first that $G_P \cong A_6.2^2$. Let $e=(bdu)^{rur}$ and $T=Q^e$. Then $T \in Y_4$ and $G_{P,Q,T}=\langle u \rangle$. If $T \in \Delta(Q)$, then $vg \in G_T$ for some $g \in \langle b, c, d, t \rangle$ by (2.5). Since $vg$ normalizes $G_{P,Q,T}$, we must have $g \in d^i t^{i+j} \langle bdt \rangle$. Replacing $v$ by $vc$ if necessary, we can assume that $g=d^i t^{i+j}(bdt)^m$ for $m=0$ or 1. Let $\tilde{G}_{i,j,X,m,n}$ be the group generated by elements $a$, $b$, $c$, $d$, $u$, $t$ and $v$ defined by the relations (3.1), (3.3), (3.4.X) and $(vd^i t^{i+j}(bdt)^m \cdot e)^3=u^n$ for $i$, $j$, $m$, $n=0$ or 1 and X=A, B or C. By coset enumeration, we find that $\tilde{G}_{i,j,X,m,n}=1$ unless $m=0$, $j=n=1$ and either $i=0$ and X=B or $i=1$ and X=C. The index of the subgroup $\langle a, b, c, d, t, u \rangle$ in $\tilde{G}_{0,1,B,0,1}$ is 132. Adding the relation $(vdu^r)^3=1$ to the definition of $\tilde{G}_{0,1,B,0,1}$ yields the group called $\tilde{G} \cong M_{12}$ in the proof of (3.5). The index of the subgroup $\langle a, b, c, d, t, u \rangle$ in $\tilde{G}_{1,1,C,0,1}$ is 56. By coset enumeration, we check that the relation $(vcr^{ur})^3=1$ holds in $G_{1,1,C,0,1}$, so this group is a homomorphic image of the group called $\tilde{G} \cong 2 \cdot L_3(4).2^2$ in the proof of (3.6).

Now suppose that $G_P \cong A_6.2_2$. Let $e=(bdu)^{rur}$ (as above), $f=(cu)^{rur}$, $T_0=Q^e$ and $T_1=Q^f$. Then $T_0$ and $T_1$ are representatives of the two $G_{P,Q}$-orbits in $Y_4$ and $G_{P,Q,T_k}=\langle c^k u \rangle$ for $k=0$ and 1. If $T_k \in \Delta(Q)$, then again $vg$ fixes $T_k$ for some $g \in \langle b, c, d \rangle$ by (2.5). Since $vg$ normalizes $G_{P,Q,T_k}$, it follows that $d^i g \in \langle c \rangle$ and $i=j$. Replacing $v$ by $vc$ if necessary, we can assume that in fact $g=c^k d^i$. Let $\tilde{G}_{i,k,n}$ denote the group generated by elements $a$, $b$, $c$, $d$, $u$ and $v$ defined by the relations (3.1), (3.2/2), (3.4.A) with $i=j$, but without the relation $[v, t]=1$, and either $(vd^i e)^3=u^n$ if $k=0$ or $(vcd^i f)^3=(cu)^n$ if $k=1$. By coset enumeration, we find that $\tilde{G}_{i,k,n}=1$ unless $i=n=1$ and that $(vcdf)^3=cu$ and $(vcr^{ur})^3=1$ both hold in $\tilde{G}_{1,0,1}$. Thus both $\tilde{G}_{1,0,1}$ and $\tilde{G}_{1,1,1}$ are homomorphic images of the group called $\tilde{G}_{1,1,2,0,0,0} \cong 2 \cdot L_3(4).2$ in the proof of (3.6).

Suppose, finally, that $G_P \cong A_6.2_3$. This time, let $T=u^{rur}$. Then $T \in Y_4$ and $G_{P,Q,T}=1$. Again we have $vg \in G_T$ for some $g \in G_{P,Q}$. Replacing $v$ by $vc$ if necessary, we can assume that in fact $g=b^k cd^m u^n$ for $k$, $m$, $n=0$ or 1. Let

$\tilde{G}_{i,j,k,m,n}$ denote the group generated by elements $a$, $b$, $c$, $d$, $u$ and $v$ defined by the relations (3.1), (3.2/3), (3.4.A) (without the relation $[v, t]=1$) and $(vg\cdot u^{rur})^3=1$. By coset enumeration, we find that $\tilde{G}_{i,j,k,m,n}=1$ unless $i=n=1$ and $j=k=m=0$ and that $(vc\cdot r^{ur})^3=1$ holds in $\tilde{G}_{1,0,0,0,1}$, so that this group is a homomorphic image of the group called $\tilde{G}_{1,0,3,0,0,0}\cong 2\cdot L_3(4).2$ in the proof of (3.6). ∎

## 4. The proof of (1.1): Conclusion.

To conclude the proof of (1.1), it remains only to consider the case $k=2$. By (2.1) and (2.4), we have $G_P\cong PSp_4(4).4$, so there are elements $x_0$, $x_1$, $x_2$, $x_3$, $x_4$, $c$, $f$ and $u$ generating $G_P$ which satisfy the following relations:

(4.1)
$$x_i^r=x_{4-i} \text{ for } r=x_0^{x_4} \text{ and } 0\leqq i\leqq 2,\quad x_i^u=x_{5-i} \text{ for } 1\leqq i\leqq 2,\quad (ur)^8=1,$$
$$x_0^2=x_1^2=[x_0, x_1]=[x_0, x_2]=1,\quad [x_0, x_3]=x_1x_2,\ c^3=f^3=[c, f]=1,$$
$$c^r=c^{-1},\ f^r=fc^{-1},\ c^u=cf^{-1},\ t^2=1 \text{ where } t=u^2,$$
$$c^t=c^{-1} \text{ and } [t, r]=[t, x_0]=[c, x_2]=(cx_0)^3=(cx_0x_4)^5=1.$$

These are just the Steinberg relations for $PSp_4(4)$ together with a graph automorphism $u$; the element $t=u^2$ is a field automorphism centralizing $\langle x_0, \cdots, x_4, x_0^u\rangle\cong PSp_4(2)$ and normalizing $\langle c, f\rangle\cong 3^2$, a Cartan subgroup of $(G_P)'$. There exists a point $Q\in\Delta(P)$ and a circle $\gamma\in C^P\cap C^Q$ such that $G_{P,\gamma}=\langle x_0, \cdots, x_4, c, f, t\rangle$ and $G_{P,Q}=\langle x_1, \cdots, x_4, c, f, u\rangle$. The elements $x_0$, $x_4$ and $c$ satisfy the following defining relations for $A_5$:

$$c^3=x_0^2=x_4^2=(x_0x_4)^3=(cx_0)^3=(cx_0x_4)^5=(cr)^2=1.$$

Since $(G_\gamma^\circ)'\cong A_6$, there exists an element $v\in G_\gamma$ exchanging $P$ and $Q$ such that

$$v^2\equiv(vx_0)^3\equiv[v, x_4]\equiv(vc)^2\equiv 1 \pmod{N_\gamma}.$$

Let $y_i=x_i^c$ for $i=1$ and 3 and $y_2=x_2^f$. Then $O_2(N_\gamma)=\langle x_1, y_1, x_2, y_2, x_3, y_3\rangle\cong 2^6$ and $N_\gamma=O_2(N_\gamma)\cdot\langle cf\rangle$.

(4.2) LEMMA. *The element $v$ can be chosen so that $v^2=(vx_0)^3=[v, x_1]=[v, x_4]=1$, $x_2^v=x_3$, $c^v=cf^{-1}$ and $u^v=u^{-1}$.*

PROOF. We have $\langle cf\rangle\in \mathrm{Syl}_3(N_\gamma)$ and $C_{G_{P,\gamma}}(cf)N_\gamma/N_\gamma\cong A_5$, so $C_{G_\gamma}(cf)N_\gamma/N_\gamma\cong A_6$. Thus we can choose $v\in C_{G_P}(cf)$ conjugate to $x_0$ so that $v^2=1$. Let $M=O_2(N_\gamma)$. Since $x_0$, $x_4$ and $v$ are all involutions, both $(vx_0)^3$ and $[v, x_4]$ are inverted by $v$. This implies that both $(vx_0)^3$ and $[v, x_4]$ lie in $M$ and hence in $C_M(cf)=1$. Since $C_M(cf)=1$, we can consider $M$ as a $GF(4)$-module for $C_{G_P}(cf)$. With respect to the basis $x_1$, $x_2$, $x_3$, the elements $x_0$, $x_4$ and $c$ are represented by the matrices

$$X_0=\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix},\quad X_4=\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix},\quad C=\begin{pmatrix}\lambda&0&0\\0&1&0\\0&0&\mu\end{pmatrix},$$

where $\lambda$ and $\mu$ are the two elements of $GF(4)$ different from 0 and 1, and the element $cf$ by

$$D=\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}$$

since $x_1^c=x_1^{cf}$. If $V\in GL_3(4)$ satisfies $V^2=(X_0V)^3=[V, X_4]=1$ and $(CV)^2\in\langle D\rangle$, then

$$V=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Thus $[v, x_1]=1$ and $x_2^v=x_3$. Since $(CV)^2=D^{-1}$, we have $(cv)^2cf\in C_{G_\Gamma}(N_\Gamma)=1$, so $c^v=c^{-1}(cf)^{-1}=cf^{-1}$. Since $v$ normalizes both $\langle x_1, \cdots, x_4, c, f\rangle$ and $G_{P,Q}$, we have $[u, v]\in\langle x_1, \cdots, x_4, c, f, t\rangle$. Since the element $[u, v]$ inverts the Cartan subgroup $\langle c, f\rangle$, it follows that $[u, v]\in t\langle c, f\rangle$. Since $[u, v]$ also centralizes $x_4$, in fact $[u, v]=t(cf)^i$ for some $i$. Then $u=(u^v)^v=(u^{-1}c^if^i)^v=uc^i$ implies that $u^v=u^{-1}$. ∎

Recall that $\Delta(Q)\cap Y_m\neq\emptyset$ for $m=2$, 3 or 4 by condition (*).

(4.3) LEMMA. *$\Delta(Q)\cap Y_m=\emptyset$ for $m=2$ and 4.*

PROOF. Let $R=Q^{rur}$. Then $R\in Y_2$ and $G_{P,Q,R}=\langle x_1, x_2, c, f, t\rangle$. If $\Delta(Q)\cap Y_2\neq\emptyset$, then $R\in\Delta(Q)$ and hence $vg\in G_R$ for some $g\in G_{P,Q}$. Since, however, $G_{P,Q,R}^v=\langle x_1, x_3, c, f, t\rangle$ is not conjugate to $G_{P,Q,R}$ in $G_{P,Q}$, we conclude that $\Delta(Q)\cap Y_2=\emptyset$.

Let $T=Q^{u^{rur}}$. Then $T\in Y_4$ and $G_{P,Q,T}=\langle c, f, u\rangle$. If $\Delta(Q)\cap Y_4\neq\emptyset$, then $T\in\Delta(Q)$ and hence $vg\in G_T$ for some $g\in G_{P,Q}$. Since $v$ normalizes $G_{P,Q,T}$ and $T$ is the only fixed point of $G_{P,Q,T}$ in $Y_4$, we conclude that in fact $v\in G_T$. Hence $(v\cdot u^{rur})^3\in\langle c, f, u\rangle$. By coset enumeration, we find that the only group generated by elements $x_0, \cdots, x_4, c, f, u$ and $v$ satisfying the relations (4.1), (4.2) and $(v\cdot u^{rur})^3=h$ for any given element $h\in\langle c, f, u\rangle$ is the trivial group. ∎

(4.4) LEMMA. *If $\Delta(Q)\cap Y_3\neq\emptyset$, then $|\mathcal{P}|=2058$, $G\cong He.2$ and $\Gamma$ is uniquely determined.*

PROOF. Let $S=Q^{r^{ur}}$. Then $S\in Y_3$ and $G_{P,Q,S}=\langle x_1, c, f, t\rangle$. If $\Delta(Q)\cap Y_3\neq\emptyset$, then $S\in\Delta(Q)$ and so $vg\in G_{P,Q,S}$ for some $g\in G_{P,Q}$. Since $v$ normalizes $\langle x_1, c, f, t\rangle$ and $S$ is the only fixed point of this subgroup in $Y_3$, we must have, in fact, $v\in G_{P,Q,S}$. Thus $(v\cdot r^{ur})^3\in G_{P,Q,S}$. Since $C_{\langle x_1,c,f,t\rangle}(v\cdot r^{ur})=\langle x_1, c, t\rangle$

and $v$ inverts $(v\cdot r^{ur})^3$, it follows that either

$$(v\cdot r^{ur})^3 = x_1^i f^j \tag{4.5}$$

for $i=0$ or 1 and $j=1$ or 2 or

$$(v\cdot r^{ur})^3 = x_1^i t^j \tag{4.6}$$

for $i, j=0$ or 1. Let $\tilde{G}_{i,j}$ be the group generated by elements $x_0, \cdots, x_4, c, f, u$ and $v$ defined by the relations (4.1), (4.2) and (4.5). By coset enumeration, we find that $\tilde{G}_{i,j}=1$ for all $i$ and $j$. Replacing (4.5) by (4.6) in the definition of $\tilde{G}_{i,j}$, we then find that $\tilde{G}_{i,j}=1$ unless $i=j=0$ and that the index of $\langle x_0, \cdots, x_4, c, f, u\rangle$ in $\tilde{G}_{0,0}$ is 2058, so $|\tilde{G}_{0,0}|=|He|\cdot 2$. Since $He.2$ has the right configuration of subgroups, we conclude that $G\cong He.2$ and $\Gamma$ is uniquely determined. ∎

## 5. The proof of (1.2).

Let $G$ and $\Gamma$ fulfill the hypotheses of (1.2). By (2.1) and (2.4), $G_P^\circ\cong {}^2F_4(2)'$ or ${}^2F_4(2)$ and $|N_P|\leqq 2$. If $N_P\neq 1$, we let $z_U$ denote the unique involution in $N_U$ for each $U\in\mathcal{P}$. The Schur multiplier of ${}^2F_4(2)'$ is trivial, so $(G_P)'\cong{}^2F_4(2)'$. Thus, there exist generators $t_0, \cdots, t_9, a, b, c,$ and $d$ of $(G_P)'$ satisfying the following defining relations for ${}^2F_4(2)'$:

$$\begin{gathered} t_i^r = t_{8-i} \text{ for } r = t_8^{t_0} \text{ and } 0 \leqq i \leqq 8,\ t_i^s=t_{10-i} \text{ for } s=t_9^{t_3 b^{-1} a^{-1}} \text{ and } 1\leqq i\leqq 9, \\ a^s=d,\quad b^s=c,\quad a^r=c,\quad [b, r]=t_4,\quad t_0^2=t_1^2=(rs)^8=1, \\ a^2=t_1t_2t_3,\ [t_0, t_i]=1 \text{ for } 1\leqq i\leqq 4,\quad [t_0, t_5]=t_3,\quad [t_0, t_6]=t_2t_4, \\ [t_0, t_7]=t_1t_2t_3t_4bt_6,\quad [t_1, t_3]=[t_1, t_5]=1,\ [t_1, t_7]=t_3t_4t_5,\quad [a, t_4]=t_3,\quad [a, b]=t_2t_3t_4, \\ [a, t_6]=t_3t_4,\quad [a, c]=t_2t_3t_4bt_6, \quad\text{and}\quad [a, t_8]=t_2t_3c^{-1}. \end{gathered} \tag{5.1}$$

If $G_P^\circ\cong{}^2F_4(2)$, then there is an additional generator $f$ with $f^2\equiv t_5 \pmod{N_P}$ whose action on $(G_P)'$ is given by the following relations:

$$\begin{gathered} [t_0, f] = t_1t_2t_3,\quad [t_i, f] = 1 \quad \text{for}\quad i = 1 \text{ and } 3, \\ [t_2, f] = t_3,\quad [a, f] = [b, f] = t_4,\quad [t_8, f] = t_7 \quad\text{and}\quad [f, s] = 1. \end{gathered} \tag{5.2}$$

If we set $u_i=t_i$ for $i$ even, $u_5=f$, $u_3=bf^{-1}$, $u_1=au_3^{-1}$ and $u_i=u_{10-i}^s$ for $i=7$ and 9, then the relations (5.1) and (5.2) are easily deduced from the following defining relations for ${}^2F_4(2)$ taken from [**13**]:

$$\begin{gathered} u_i^2 = 1 \quad \text{for } i \text{ even},\quad u_i^4 = 1 \quad \text{for } i \text{ odd}, \\ u_i^s=u_{10-i} \text{ for } s=(u_9^2)^{u_1^{-1}} \text{ and } 1\leqq i\leqq 9,\ u_i^r=u_{8-i} \text{ for } r=u_8^{u_0} \text{ and } 0\leqq i\leqq 8, \\ [u_1, u_2]=[u_1, u_5]=[u_2, u_4]=[u_2, u_6]=1,\ [u_1, u_3]=u_2,\ [u_1, u_4]=u_3^2, \end{gathered} \tag{5.3}$$

$$[u_2, u_8]=u_4u_6,\ [u_1, u_6]=u_3^2u_4u_5^2,\ [u_1, u_7^{-1}]=u_2u_3^{-1}u_5,$$
$$[u_1, u_8]=u_2u_3^2u_4u_5^{-1}u_6u_7 \text{ and } (rs)^8=1.$$

In this section, we suppose that $\Gamma_P\cong\mathcal{O}(1)$, so there exist $Q\in\Delta(P)$ and $\gamma\in C^P\cap C^Q$ such that

$$G_{P,\gamma}=\langle t_0, \cdots, t_8, a, b, c, f, z_P\rangle \quad\text{and}\quad G_{P,Q}=\langle t_1, \cdots, t_9, a, b, c, d, f, z_P\rangle,$$

where the element $f$ is to be ignored if $G_P^\circ\cong{}^2F_4(2)'$ and the element $z_P$ is to be ignored if $N_P=1$. Let $h=t_3b^{-1}a^{-1}$, $w=ht_7dc$ and $W=O_2(C_{G_l}(w))$, where $l=\{P, Q\}$. Then $|w|=5$, $|h|=4$ and $w^h=w^3$. Since $\langle w\rangle\in\mathrm{Syl}_5(G_{P,Q})$, we can choose $v\in W\cap G_\gamma$ exchanging $P$ and $Q$. Then

$$[v, h]\in W\cap G_{P,Q}=\begin{cases}\langle t_5, z_P\rangle & \text{if } G_P^\circ\cong{}^2F_4(2)',\\ \langle f, z_P\rangle & \text{if } G_P^\circ\cong{}^2F_4(2).\end{cases}$$

(5.4) LEMMA. *If $N_P\neq1$, then $[v, z_P]=t_5$ and if $G_P^\circ\cong{}^2F_4(2)$, then $f^2=t_5$. In particular, $G_P\cong2\times{}^2F_4(2)$ if both $G_P^\circ\cong{}^2F_4(2)$ and $N_P\neq1$.*

PROOF. If $G_P^\circ\cong{}^2F_4(2)$, then $f^2\equiv t_5 \pmod{N_P}$. Since $v$ normalizes $W\cap G_{P,Q}$, we have $[v, f]\in\langle t_5, z_P\rangle$ and therefore $[v, f^2]=1$. If $N_P\neq1$, then $z_P\neq z_Q$ (since otherwise $\langle z_P\rangle\trianglelefteq\langle G_P, G_Q\rangle=G$ which would imply that $z_P=1$), so $z_P^v=z_Q$ implies $[v, z_P]=t_5$ and $f^2=t_5$ since $f^2\in C_{\langle t_5, z_P\rangle}(v)$. ■

Recall that $\Delta(Q)\cap Y_m\neq\emptyset$ for $m=2$, 3 or 4 by condition (*).

(5.5) LEMMA. *If $Y_4\cap\Delta(Q)\neq\emptyset$ and $G_P\cong{}^2F_4(2)$, then $[v, h]\in\langle t_5\rangle$ and $[v, u_1]=1$, where $u_1=au_3^{-1}=afb^{-1}$ as above.*

PROOF. Let $T=Q^{srsr}$. Then $T\in Y_4$, $O_5(G_{P,Q,T})=\langle w\rangle$ and $w$ has exactly four fixed points in $Y_4$. Let $X$ be the set of these four fixed point together with $P$ and $Q$. If $Y_4\cap\Delta(Q)\neq\emptyset$, then $Y_4\subseteq\Delta(Q)$ since $G_{P,Q}$ acts transitively on $Y_4$. Also $N_P=1$ by (2.4). Let $G_X^\circ$ denote the permutation group induced on $X$ by the stabilizer $G_X$. Then $(G_X^\circ)_P\cong5:4$. Since $[v, w]=1$ and $v$ exchanges $P$ and $Q$, it follows by (2.5) that $v\in G_X$ (so $G_X^\circ$ is a transitive extension of $(G_X^\circ)_P$ and hence $G_X^\circ\cong PGL_2(5)$). Let $N_X$ denote the kernel of the action of $G_X$ on $X$. Now suppose that $G_P\cong{}^2F_4(2)$. Then $N_X=\langle w, u_1\rangle$. Since $[v, h]\in\langle f\rangle$, $h=u_1^{-1}f^{-1}$ and $[v, f]\in\langle t_5\rangle$, we have $[v, u_1]\in\langle f\rangle$. It follows that $[v, u_1]\in N_X\cap\langle f\rangle=1$ and hence $[v, h]=[v, u_1^{-1}f^{-1}]\in\langle t_5\rangle$. ■

(5.6) LEMMA. *If condition (*) holds, then $[v, h]\notin\langle t_5, z_P\rangle$.*

PROOF. We have $[v, t_5]=1$. If $[v, h]\in\langle t_5, z_P\rangle$, then $h^2=t_1t_5$ and $[h, t_5]=[h, z_P]=1$ imply that $[v, t_1]=1$. Let $M=O_2(G_{P,Q})$. Then

$$M=\langle t_2, \cdots, t_8, b, c, f, z_P\rangle \quad\text{and}\quad M'=\langle t_3, \cdots, t_7\rangle$$

and $v$ acts on $M/\langle M', f, z_P\rangle \cong 2^4$. Since $w$ acts non-trivially on $M/\langle M', f, z_P\rangle$ and commutes with the 2-element $v$, it follows that $v$ acts trivially on $M/\langle M', f, z_P\rangle$. Thus

$$[v, b^{-1}] = t_3^{\alpha} t_4^{\beta} t_5^{\theta} t_6^{\delta} t_7^{\varepsilon} f^{\lambda} z_P^{\mu}$$

for $\alpha, \cdots, \mu=0$ or 1. Let $N=O_2(G_\gamma)$. Then

$$N = \langle t_1, \cdots, t_7, a, b, c, f, z_P\rangle \quad \text{and} \quad N' = \langle t_2, \cdots, t_6, b\rangle.$$

Since $v$ normalizes $N'$, we have $\varepsilon=\lambda=\mu=0$. Since $[b, t_1]=[v, t_1]=1$ and $[t_1, t_6] \neq 1$, it follows that $\delta=0$. Then $b^2=t_3t_4t_5$, $[v, t_5]=1$ and $(b^2)^v=(t_3^{\alpha}t_4^{\beta}t_5^{\theta}b)^2=b^2$ imply that $[v, t_3t_4]=1$. Since $s=(t_5\cdot(h^{-1}w)^2)^h$, we have $[v, s]=1$. This implies that also $[v, t_6t_7]=1$. Let $U=\langle x^2, N' \mid x\in N\rangle$. Then $U=\langle t_1, t_7, N'\rangle$ and $U/N'\cong 2^2$. Since $[v, t_6t_7]=[v, t_1]=1$, the element $v$ acts trivially on $U/N'$. Since $G_\gamma/C_{G_\gamma}(U/N')\cong S_3$, we have $C_{G_\gamma}(U/N')=O_2(G_\gamma)$. Thus $v\in O_2(G_\gamma)$ (and $v$ acts as the product of two transpositions on $\gamma$). Then $G_\gamma/C_{G_\gamma}(\langle t_3, t_5\rangle)\cong S_3$, so $C_{G_\gamma}(\langle t_3, t_5\rangle)=O_2(G_\gamma)$ as well. It follows that $v$ also acts trivially on $Z(N)\cap N'=\langle t_3, t_5\rangle$. Thus $[v, t_3t_4]=1$ implies $[v, t_4]=1$ and hence $[v, t_6]=1$ (since $[v, s]=1$). Conjugating $[h, b]=t_2t_4$ by $v$, we find that $[v, t_2]=t_3^{\beta}$. Then $c^v=t_7^{\alpha}t_6^{\beta}t_5^{\theta}c$ implies that $[h, c]^v=[h, t_7^{\alpha}t_6^{\beta}c]=t_3^{\alpha}t_4^{\alpha+\beta}t_6^{\alpha}\cdot t_3b$. Since $(t_3b)^v=t_3^{\alpha}t_4^{\beta}t_5^{\theta}\cdot t_3b$, it follows that $\alpha=\theta=0$. Since $a=h^{-1}b^{-1}t_3$, we have $[v, a]\in t_4^{\beta}\langle t_5, z_P\rangle$.

The group $G_P$ acts as a rank 5 permutation group on $\Delta(P)$, so condition (*) implies that $Y_m\subseteqq\Delta(Q)$ for $m=2$, 3 or 4. Suppose first that $Y_2\subseteqq\Delta(Q)$. Let $R=Q^{s^r}$. Then $R\in Y_2$ and

$$G_{P,Q,R} = \langle t_1, \cdots, t_5, a, b, f, z_P\rangle.$$

By (2.5), there exists an element $g\in G_{P,Q}$ such that $vg\in G_R$ and hence $(vg\cdot s^r)^3 \in G_{P,Q,R}$. Since $v$ normalizes $\langle t_1, \cdots, t_5, a, b, f, z_P\rangle$, so must $g$. Thus, we can assume that $g=t_6^k$ for $k=0$ or 1. Then $t_1^{(vt_6^k s^r)^3}=(t_3^k t_5)^{(vt_6^k s^r)^2}=\cdots=t_5$. This contradicts the fact that $t_1$ and $t_5$ are not conjugate in $\langle t_1, \cdots, t_5, a, b, f, z_P\rangle$.

Suppose that $Y_3\subseteqq\Delta(Q)$, so $N_P=1$ by (2.4). If $S=Q^{r^{sr}}$, then $S\in Y_3$ and $G_{P,Q,S}=\langle t_1, t_2, t_3, a\rangle$ or $\langle t_1, t_2, t_3, a, bf^{-1}\rangle$. Thus $G_{P,Q,S}^v\cap(G_P)'=\langle t_1, t_2, t_3, at_4^{\beta}t_5^{j}\rangle$ for $j=0$ or 1. We have $vg\in G_S$ for some $g\in G_{P,Q}$ by (2.5). Since $\langle t_1, t_2, t_3, a\rangle =G_{P,Q,S}\cap(G_P)'=(G_{P,Q,S}^v\cap(G_P)')^g=\langle t_1, t_2, t_3, at_4^{\beta}t_5^{j}\rangle^g$, it follows that $j=0$ and $g\in t_6^{\beta}\langle t_1, \cdots, t_5, a, bf^{-1}, bt_6\rangle$. Note that $[t_1, g]\neq 1$ since otherwise $t_1^{(vgr^{sr})^3}=t_3$, which contradicts the fact that $t_1$ and $t_3$ are not conjugate in $G_{P,Q,S}$. Suppose that $G_P\cong {}^2F_4(2)$. Substituting $vf$ for $v$ if necessary, we can assume that $\beta=0$. Then $(bf^{-1})^{vg}\in G_{P,Q,S}$ implies that $g\in\langle t_1, \cdots, t_5, a, bf^{-1}\rangle$ which contradicts the fact that $[t_1, g]\neq 1$. Hence $G_P\cong {}^2F_4(2)'$ and $g\in t_6^{\beta}(bt_6)^{\lambda}t_4^{\mu}\langle t_1, t_2, t_3, t_5, a\rangle$ for $\lambda, \mu=0$ or 1. This time $[t_1, g]\neq 1$ implies that $\lambda\equiv\beta+1 \pmod 2$. Then $a^{(vgr^{sr})^3}=a(t_1t_2t_3)^{\beta}$, so $\beta=0$ since $a$ and $at_1t_2t_3$ are not conjugate in $G_{P,Q,S}$. Thus $g=bt_6x$ for some $x\in\langle t_1, \cdots, t_5, a\rangle$. Since $r^{sr}$ is an involution in $G_P$ exchanging $Q$ and

$S$ and $G_{P,Q,S}$ acts transitively on $\{P, Q, S\}$, we can assume that $(vg)^2=1$. Thus $x\in\langle t_1, \cdots, t_5\rangle$, since otherwise $(vg)^2\equiv a^2\equiv t_1 \pmod{\langle t_2, \cdots, t_5\rangle}$. But then $(vg)^2\equiv t_4 \pmod{\langle t_3, t_5\rangle}$.

Let $T=Q^{s^{rsr}}$ and suppose that $Y_4\subseteq\Delta(Q)$, so again $N_P=1$ by (2.4). Then $T\in\Delta(Q)$. Let $X$ be as in the proof of (5.5). Suppose first that $G_P\cong{}^2F_4(2)$. Substituting $vf$ for $v$ if necessary, we can assume that $\beta=0$. We have $vf^m\in G_T$ for some $m$ since $\langle f\rangle$ acts transitively on $X\backslash\{P, Q\}$. By (5.5), both $v$ and $f$ centralize $G_{P,Q,T}=\langle w, u_1\rangle$, so $(vf^m)^2\in Z(G_{P,Q,T})=1$. Thus $vf^m$ inverts $vf^m\cdot s^{rsr}$ and commutes with $w$, so $(vf^m\cdot s^{rsr})^3=1$. Since $[v, t_2]=1$ and $v^2\in\langle f\rangle$, we have $v^2=t_5^i$ for $i=0$ or 1. Since $(vf^m)^2=1$, we have $i\equiv m(j+1) \pmod 2$, where $[v, f]=t_5^j$. By coset enumeration, however, we find that the group generated by elements $t_0, \cdots, t_9, a, b, c, d, f$ and $v$ defined by (5.1), (5.2), $f^2=t_5$, $v^2=t_5^{m(j+1)}$, $[v, w]=[v, b]=1$, $[v, f]=[v, h]=t_5^j$ and $(vf^m\cdot s^{rsr})^3=1$ is trivial for all values of $j$ and $m$. Thus $G_P\cong{}^2F_4(2)'$, so $C_{G_X}(w)$ induces a group isomorphic to $L_2(5)$ on $X$ and therefore $vt_5^k\in G_T$ for $k=0$ or 1. Since $(vt_5^k)^2\in G_{P,Q,T}\cap W=1$, the element $(vt_5^k\cdot s^{rsr})^3$ of $G_{P,Q,T}$ commutes with $w$ and is inverted by $vt_5^k$, so $(vt_5^k\cdot s^{rsr})^3=1$. Let $\tilde{G}_{\beta,j,k}$ be the group generated by elements $t_0, \cdots, t_9, a, b, c, d$ and $v$ defined by (5.1), $v^2=[v, t_5]=[v, w]=1$, $[v, b^{-1}]=t_4^\beta$, $[v, h]=t_5^j$ and $(vt_5^k\cdot s^{rsr})^3=1$. By coset enumeration, we find that $\tilde{G}_{\beta,j,k}=1$ for all $\beta$, $j$ and $k$. ■

By (5.6), we can assume that $[v, h]\in\langle f, z_P\rangle\backslash\langle t_5, z_P\rangle$. In particular, $G_P^\circ\cong{}^2F_4(2)$. As above, we let $u_i=t_i$ for $i$ even, $u_5=f$, $u_3=bf^{-1}$, $u_1=au_3^{-1}$ and $u_i=u_{10-i}^s$ for $i=7$ and 9. By (5.4), $u_i^2=t_i$ for $i$ odd, and $G_P$ is generated by the elements $u_0, \cdots, u_9$ satisfying (5.3) and, if $N_P\neq1$, also $z_P$ with

(5.7) $$[v, z_P]=t_5,\quad [z_P, u_i]=1 \quad \text{for} \quad 0\leqq i\leqq 9 \quad \text{and} \quad z_P^2=1.$$

In particular, $w=u_1^{-1}u_9$ and $h=u_1^{-1}u_5^{-1}$. We will continue, sometimes, to write $t_i$ in place of $u_i$ for $i$ even.

(5.8) LEMMA. *The element $v$ can be chosen so that $[v, u_1^{-1}]=[v, u_9^{-1}]=u_5t_5^iz_P^n$, $u_3^v=u_3^{-1}t_5^n$, $[v, u_5]=t_5^{n+1}$, $v^2=t_5^j$, and*

$$(u_0u_8v)^3=t_1u_5u_6t_7\cdot t_3^\rho u_4^{i+n+1}t_5^\tau$$

*for $i, j, n, \rho, \tau=0$ or 1.*

PROOF. Since $v$ normalizes $W\cap G_{P,Q}=\langle u_5, z_P\rangle$, we have $u_5^v=u_5t_5^kz_P^m$ for $k$, $m=0$ or 1. Since $[v, h]\in\langle u_5, z_P\rangle\backslash\langle t_5, z_P\rangle$, it follows that $[v, u_1^{-1}]=u_5t_5^iz_P^n$ for $i, n=0$ or 1. Then $v^2\in W\cap G_{P,Q}=\langle u_5, z_P\rangle$ and $(u_1^v)^v=(u_1u_5t_5^iz_P^n)^v=u_1t_5^{k+n+1}z_P^m$ imply that $m=0$ and $k\equiv n+1 \pmod 2$. Since $[v, w]=1$ and $w=u_1^{-1}u_9$, also $[v, u_9^{-1}]=u_5t_5^iz_P^n$ and thus $s^v=(t_9^{u_1^{-1}})^v=t_5s$. As in the proof of (5.6), $v$ acts trivially on $M/\langle M', u_5, z_P\rangle\cong2^4$, so

$$[v, u_3^{-1}]=t_3^\alpha u_4^\beta u_5^\delta t_5^\varepsilon u_6^\lambda t_7^\mu z_P^\theta$$

for $\alpha, \cdots, \theta=0$ or 1. Since $v$ normalizes $N'=\langle t_2, \cdots, t_6, u_3u_5\rangle$ (where $N=O_2(G_\gamma)$ as above) and $[v, u_5]=t_5^{n+1}$, it follows that $\delta=\mu=\theta=0$. Then $t_3^v=(u_3^v)^2=t_3t_5^\lambda$ and, since $s^v=t_5s$, also $[v, t_7]\in\langle t_5\rangle$, so $v$ acts trivially on $U/N'=\langle t_1, t_7, N'\rangle/N'\cong 2^2$, where $U$ is as in the proof of (5.6). This implies that $v$ induces the product of two transpositions on $\gamma$. It follows that $v$ acts trivially on $Z(N)\cap N'$, so $\lambda=0$. Replacing $v$ by $vu_5$ if necessary, we can assume that $\beta=0$. Then $u_4^v=[u_3^v, u_5^v]=[u_3, u_5]=u_4$ and $(u_3^v)^v=u_3$. This second equation implies that $v^2\in C_{\langle u_5, z_P\rangle}(\langle v, u_3\rangle)=\langle t_5\rangle$. Then $u_2^v=[u_1^v, u_3^v]=u_2t_3u_4$.

The element $u_0u_8v$ induces a 3-cycle on $\gamma$, so $(u_0u_8v)^3\in N$. Let $K=\langle x\in N\,|\,x^2\in N'\rangle$. Then $K=\langle t_1, u_5, t_7, z_P, N'\rangle$ and $u_0u_8v$ acts fixed point freely on $N/K=\langle u_1, u_7\rangle K/K\cong 2^2$. Thus $(u_0u_8v)^3\in K$. We have $Z(N')=\langle t_3, u_4, t_5\rangle$,

$$K/Z(N') = \langle t_1, u_2, u_3, u_5, u_6, t_7, z_P\rangle Z(N')/Z(N')\cong 2^6 \text{ or } 2^7$$

and

$$C_{K/Z(N')}(u_0u_8v) = \langle t_1u_5u_6t_7,\ u_2u_3u_5u_6,\ z_P\rangle Z(N')/Z(N'),$$

so

$$(u_0u_8v)^3 = t_1^A u_2^B u_3^B u_5^{A+B} u_6^{A+B} t_7^A z_P^C \cdot t_3^\rho u_4^\sigma t_5^\tau$$

for $A, B, C, \rho, \sigma, \tau=0$ or 1. Replacing $v$ by $vz_P^C$, we can assume that $C=0$. Next we calculate that

$$u_5^{(u_0u_8v)^3} = (t_1u_3t_3^\alpha t_5^{\varepsilon+1})^{(u_0u_8v)^2} = (u_3t_3^{\varepsilon+1}t_5^{\alpha+\varepsilon+1}u_6t_7)^{(u_0u_8v)} = t_3^{\alpha+1}t_5^{\varepsilon+n}u_5,$$

whereas $[t_1^A u_2^B u_3^B u_5^{A+B} u_6^{A+B} t_7^A t_3^\rho u_4^\sigma t_5^\tau, u_5^{-1}]=t_3^B u_4^B$. Thus $\alpha=1$, $B=0$ and $\varepsilon=n$. Calculating $u_3^{(u_0u_8v)^3}$ in two different ways, we conclude that $A=1$. Calculating $u_7^{(u_0u_8v)^3}$ in two different ways, we conclude that $\sigma\equiv i+n+1 \pmod 2$. ■

(5.9) LEMMA. *If $Y_2\cap\Delta(Q)\neq\emptyset$, then $G\cong Ru$ and $\Gamma$ is uniquely determined.*

PROOF. Let $R=Q^{rsr}$. Then $Q\in Y_2$ and $G_{P,Q,R}=\langle u_1, \cdots, u_5, z_P\rangle$. If $Y_2\cap\Delta(Q)\neq\emptyset$, then $Y_2\subseteq\Delta(Q)$. By (2.5), there exists an element $g\in G_{P,Q}$ such that $vg\in G_R$. By (5.8), $v$ normalizes $\langle u_1, \cdots, u_5, z_P\rangle$. This group has only two fixed points in $Y_2$, namely $R$ and $R^{u_6}$, so we can assume that $g=u_6^\omega$ for $\omega=0$ or 1. Thus $(vu_6^\omega\cdot s^r)^3=u_1^\alpha h u_5^\nu$ for $\alpha, \nu=0, 1, 2$ or 3 and $h\in\langle u_2, u_3, u_4, z_P\rangle$. Then $(u_1^\alpha h u_5^\nu)^{vu_6^\omega s^r}\equiv u_1^{\alpha+\nu}u_5^\alpha \pmod{\langle t_1, u_2, u_3, u_4, z_P\rangle}$. Since $vu_6^\omega s^r$ must centralize a power of itself, we conclude that $\alpha=\nu$ and that $\alpha$ is even. Replacing $\alpha$ by $\alpha/2$, we have therefore $(vu_6^\omega s^r)^3=t_1^\alpha u_2^\beta u_3^\delta t_3^\varepsilon u_4^\mu t_5^\alpha z_P^\theta$ for $\alpha, \cdots, \theta=0$ or 1. Conjugating with $vu_6^\omega s^r$, we obtain $t_1^{\delta\omega+\delta n+\theta}u_2^{\beta+\mu}u_3^\delta t_3^{\delta+\alpha\omega+\beta+\varepsilon}u_4^\beta t_5^\alpha z_P^\theta$, and so $\beta=\mu=0$, $\delta=\alpha\omega$ and $\theta\equiv\alpha(\omega+\omega n+1) \pmod 2$. Thus $(vu_6^\omega s^r)^3=t_1^\alpha u_3^{\alpha\omega}t_3^\varepsilon t_5^\alpha z_P^{\alpha(\omega+\omega n+1)}$. Next we calculate that $u_5^{(vu_6^\omega s^r)^3}=t_3^{i\omega}u_4^\omega t_5^{n+i+\omega+1}u_5$, whereas $[t_1^\alpha u_3^\delta t_3^\varepsilon t_5^\alpha z_P^{\alpha(\omega+\omega n+1)}, u_5^{-1}]=u_4^\delta$. Thus $\theta\equiv\omega+\alpha \pmod 2$, $\omega\equiv i+n+1 \pmod 2$ and $\delta=\omega$. Replacing $v$ by $vt_5$ if necessary, we can assume that $\varepsilon=0$. Hence

$$(vu_6^{i+n+1}s^r)^3 = t_1^\alpha u_3^{i+n+1}t_5^\alpha z_P^{i+n+1+\alpha}. \tag{5.10}$$

Let $\tilde{G}_{i,j,\alpha,\rho,\tau,n}$ be the group generated by elements $u_0, \cdots, u_9, z_P, v$ defined by the relations (5.3), (5.7), (5.8) and (5.10). By coset enumeration, we find that $\tilde{G}_{i,j,\alpha,\rho,\tau,n}=1$ for all $i, j, \alpha, \rho, \tau$ and $n$. If follows that $N_P=1$, so we omit the element $z_P$, the parameter $n$ and the relations (5.7) from the definition of $\tilde{G}_{i,j,\alpha,\rho,\tau,n}$. It then turns out that $\tilde{G}_{i,j,\alpha,\rho,\tau}=1$ unless $i=0$ and $j=\alpha=\rho=\tau=1$ and the index of the subgroup $\langle u_0, \cdots, u_9\rangle$ in $\tilde{G}_{0,1,1,1,1}$ is 4060. Thus $|\tilde{G}_0{}'_{1,1,1,1}| =|Ru|$. Since $Ru$ has the right configuration of subgroups [**9**], we conclude that $G\cong Ru$ and that $\Gamma$ is uniquely determined. ■

(5.11) LEMMA. *If $Y_3\cap\Delta(Q)\neq\emptyset$, then $G$ and $\Gamma$ are as in* (5.9).

PROOF. By (2.4), we can ignore $z_P$. Let $S=Q^{r^{s r}}$. Then $S\in Y_3$ and $G_{P,Q,S} =\langle u_1, u_2, u_3\rangle$. If $Y_3\cap\Delta(Q)\neq\emptyset$, then $Y_3\subseteq\Delta(Q)$. By (2.5), there exists an element $g\in G_{P,Q}$ such that $vg\in G_S$ and hence $(vg\cdot r^{sr})^3=u_1^A u_2^B u_3^C$ for some $A$, $B$, $C$. We have $(G_{P,Q,S})^v=\langle u_1u_5t_5^i, u_2u_4, u_3\rangle$. This group is conjugate to $\langle u_1, u_2, u_3\rangle$ in $G_{P,Q}$ only if $i=0$. Since $vg$ normalizes $G_{P,Q,S}$, we can assume that $g=u_4^\lambda t_5^\mu u_7$ for $\lambda$, $\mu=0$ or 1. Replacing $v$ by $vt_5$ if necessary, we can assume that $\mu=1$. Next we calculate $u_1^A u_2^B u_3^C=(u_1^A u_2^B u_3^C)^{(vg r^{sr})}=(u_1t_1^\lambda u_3)^A u_2^B(u_1t_1)^C$ and thus $A=C=0$, so

$$(vu_4^\lambda t_5 u_7 r^{sr})^3 = u_2^B. \tag{5.12}$$

Let $\tilde{G}_{j,\rho,\tau,\lambda,B}$ be the group generated by elements $u_0, \cdots, u_9$ and $v$ defined by the relations (5.3), (5.8) with $i=n=0$ and (5.12). By coset enumeration, we find that $\tilde{G}_{j,\rho,\tau,\lambda,B}=1$ unless $j=\rho=\tau=\lambda=B=1$, that $|\tilde{G}_{1,1,1,1,1}|=|Ru|$ and that $(vu_6s^r)^3 =t_1u_3t_5$ holds in $\tilde{G}_{1,1,1,1,1}$. ■

With (5.5) and (5.6), this concludes the proof of (1.2) in the case $\Gamma_P\cong\mathcal{O}(1)$.

## 6. The proof of (1.2): Conclusion.

We assume now that $\Gamma_P\cong\mathcal{O}(1)^\circ$, so that $N_P=1$, and there exist $Q\in\Delta(P)$ and $\gamma\in C^P\cap C^Q$ such that

$$G_{P,Q} = \langle t_0, \cdots, t_8, a, b, c, f\rangle \quad\text{and}\quad G_{P,\gamma}=\langle t_1, \cdots, t_8, a, b, c, d, f\rangle,$$

where $t_0, \cdots, t_8, a, \cdots, d, f$ are as in (5.1) and (5.2) with $f^2=t_5$; as usual, the element $f$ is to be ignored if $G_P\cong{}^2F_4(2)'$. Let $M=O_2(G_\gamma)$. Then $M=\langle t_2, \cdots, t_8, b, c, f\rangle$, $Z(M)=\langle t_5\rangle$ and $M'=\langle t_3, \cdots, t_7\rangle$. Let $X=M/M'$. As above, we let $h=t_3b^{-1}a^{-1}$ and $w=ht_7dc$. Then $|h|=4$, $|w|=5$ and $w^h=w^3$. The elements $h$ and $w$ are represented on $X\cong 2^4$ or $2^5$ with respect to the basis $t_2M'$, $bM'$, $cM'$, $t_8M'$ and, only if $G_P\cong{}^2F_4(2)$, $fM'$ by the matrices

$$H=\begin{pmatrix}1&1&0&1&0\\0&1&1&1&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \quad \text{and} \quad W=\begin{pmatrix}1&1&0&1&0\\1&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\end{pmatrix}$$

if $G_P\cong {}^2F_4(2)$ or by $H_0$ and $W_0$ otherwise, where $E_0$ denotes the $4\times 4$ matrix obtained from any given $5\times 5$ matrix $E$ by deleting the last row and column. Since $G_\gamma^\circ\cong PGL_2(5)$, there is an element $v$ in $G_\gamma$ such that

$$v^2\equiv(hv)^2\equiv(vh^w)^3\equiv 1 \qquad (\text{mod } M).$$

If $V\in GL_5(2)$ satisfies $V^2=(VH)^2=(H^WV)^3=1$ (where $H^W=WHW^{-1}$ is the matrix representing $h^w=w^{-1}hw$), then $V$ equals either

$$A=\begin{pmatrix}1&0&0&0&0\\0&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \quad \text{or} \quad B=\begin{pmatrix}1&1&0&0&i\\0&1&0&0&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&j&1\end{pmatrix},$$

where $ij\neq 1$. Let $U=M'/Z(M)$. Since the elements $h$ and $w$ are represented on $U$ with respect to the basis $t_3Z(M)$, $t_4Z(M)$, $t_6Z(M)$, $t_7Z(M)$ by the matrices $H_0$ and $W_0$, the element $v$ is represented on $U$ by $A_0$ or $B_0$.

Suppose that the element $v$ is represented by $A$ or $A_0$ on $X$. Then $b^v\equiv b$ (mod $M'$). Since $[M, M']=Z(M)$ and $M'$ is elementary abelian, $(b^2)^v\equiv b^2$ (mod $Z(M)$) follows. Since $b^2=t_3t_4t_5$ and $t_3^v\equiv t_3$ (mod $Z(M)$), it follows that $t_4^v\equiv t_4$ (mod $Z(M)$). This implies that $v$ is represented on $U$ by $A_0$. Hence $(t_4t_6)^v\equiv t_6$ (mod $Z(M)$). On the other hand, $[t_2, t_8]^v\equiv[t_2, t_8]$ (mod $Z(M)$) since $v$ is represented by $A$ or $A_0$ on $X$ and $[M, M']=Z(M)$. This contradicts the fact that $[t_2, t_8]=t_4t_6$.

We conclude that the element $v$ is represented by $B$ with $ij\neq 1$ or $B_0$ on $X$. Then $t_8^v\equiv ct_8f^j$ (mod $M'$). Since $[M, M']=Z(M)$, $t_8^2=1$ and $M'$ is elementary abelian, it follows that $(ct_8f^j)^2\in Z(M)$. By (5.1), however, we have $(ct_8)^2=t_5t_6$ and $(ct_8f)^2=t_7$. This completes the proof of (1.2).

## 7. The proofs of (1.3) and (1.4).

Now suppose that $\Gamma_P\cong\mathcal{H}(k)_0$. By (2.2), we have $k=1$ and by (2.4), $N_P=1$, so $G_P$ acts faithfully on $\Gamma_P$. In particular, $N_\gamma/Z(N_\gamma)\cong 3^2$. Since $G_\gamma^\circ\cong S_5$, it follows that $G_\gamma$ acts trivially on $N_\gamma/Z(N_\gamma)$. But $G_{P,\gamma}$ acts non-trivially on $N_\gamma/Z(N_\gamma)$. This concludes the proof of (1.3).

Let $\Gamma=(\mathcal{B}_1, \cdots, \mathcal{B}_4; *)$ be a geometry fulfilling the hypotheses of (1.4), let

$G \leqq \mathrm{aut}(\Gamma)$ be a group acting flag-transitively on $\Gamma$ and let $\Delta$ be the collineation graph of $\Gamma$ on $\mathcal{B}_1$. As in §2 above, it is easily seen that we can identify each element $x \in \mathcal{B}_i$ for $i \geqq 2$ with the subset $\mathcal{B}_1^x$. Since the point-stabilizers act faithfully on the point-residues for each of the geometries characterized in (1.1)-(1.2), it follows that $G_P$ acts faithfully on $\Gamma_P$ for $P \in \mathcal{B}_1$. Choose $P \in \mathcal{B}_1$, $Q \in \Delta(P)$ and $\gamma \in \mathcal{B}_4^P \cap \mathcal{B}_4^Q$.

Suppose first that $\Gamma_P$ is isomorphic to one of the geometries (a)-(d) of (1.1). Then $N_\gamma \cong 2^2$ or $2^3$ and $G_{P,\gamma}$ acts non-trivially on $N_\gamma$. On the other hand, $G_\gamma^\circ \cong S_5$, which implies that $G_\gamma$ acts trivially on $N_\gamma$. Suppose $\Gamma_P$ is isomorphic to the geometry (e) of (1.1). Let $V = O_2(N_\gamma)$ and $H \in \mathrm{Syl}_3(N_\gamma)$. Then $C_{G_{P,\gamma}}(H)V/V$ acts faithfully on $V$. Since, however, $C_{G_\gamma}(H)N_\gamma/N_\gamma \cong S_7$ and $A_7$ is not involved in $L_3(4)$, we have a contradiction. If $\Gamma_P$ is isomorphic to the extended generalized octagon of (1.2), then $Z(N_\gamma) \cong 2^2$ and $G_{P,\gamma}$ acts non-trivially on $Z(N_\gamma)$. Since $G_\gamma^\circ \cong S_5$, however, $G_\gamma$ cannot possibly act non-trivially on $Z(N_\gamma)$. This concludes the proof of (1.4).

## References

[1] F. Buekenhout, Diagrams for geometries and groups, J. Combin. Theory Ser. A, **27** (1979), 121-151.

[2] F. Buekenhout and X. Hubaut, Locally polar spaces and related rank 3 groups, J. Algebra, **45** (1977), 391-434.

[3] R. W. Carter, Simple Groups of Lie Type, John Wiley & Sons, London, New York, 1972.

[4] J. H. Conway et al., Atlas of Finite Groups, Clarendon Press, Oxford, 1985.

[5] A. Del Fra, D. Ghinelli, T. Meixner and A. Pasini, Flag-transitive extensions of $C_n$-geometries, Geom. Dedicata, to appear.

[6] T. Meixner, Some polar towers, preprint.

[7] T. Meixner, Two geometries related to the groups $Co_3$ and $Co_1$, preprint.

[8] M. Ronan, Coverings of certain finite geometries, Finite Geometries and Designs, (eds P. Cameron, J. Hirschfeld and D. Hughes), London Math. Soc. Lecture Notes, **49** (1981), 316-331.

[9] A. Rudvalis, A rank 3 simple group of order $2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$ I, J. Algebra, **86** (1984), 181-218.

[10] G. Seitz, Flag-transitive subgroups of Chevalley groups, Ann. Math., **97** (1974), 27-56.

[11] M. Suzuki, A finite simple group of order 448, 345, 497, 600, Theory of Finite Groups, (eds. R. Brauer and C. Sah), Benjamin, New York, Amsterdam, 1969, pp. 113-119.

[12] M. Suzuki, Transitive extensions of a class of doubly transitive groups, Nagoya Math. J., **27** (1966), 159-169.

[13] J. Tits, Algebraic and abstract simple groups, Ann. Math., **80** (1964), 313-329.

[14] R. Weiss and S. Yoshiara, A geometric characterization of the groups *Suz* and *HS*, J. Algebra, **133** (1990), 251-282.

[15] R. Weiss, Extended generalized hexagons, Math. Proc. Cambridge Philos. Soc., **108** (1990), 7-19.

[16] R. Weiss, A geometric characterization of the groups $McL$ and $Co_3$, J. London Math. Soc., to appear.
[17] S. Yoshiara, A classification of flag-transitive classical $c.C_2$-geometries by means of generators and relations, preprint.

Richard WEISS
Department of Mathematics
Tufts University
Medford, MA 02155
U.S.A.