J. Math. Soc. Japan
Vol. 43, No. 4, 1991

Interpolating sequences in the maximal ideal space of H^{∞}

By Keiji IZUCHI

(Received Oct. 1, 1990)

1. Introduction.

Let H^{∞} be the space of bounded analytic functions on the open unit disc D. H^{∞} becomes a Banach algebra with the supremum norm. We denote by $M(H^{\infty})$ the maximal ideal space of H^{∞} with the weak*-topology. We identify a function in H^{∞} with its Gelfand transform. For points x and y in $M(H^{\infty})$, the pseudo-hyperbolic distance is defined by

$$\rho(x, y)=\sup\{|h(x)|\,;\ h\in \text{ball }(H^{\infty}),\ h(y)=0\},$$

where ball (H^{∞}) stands for the unit closed ball of H^{∞}. For z and w in D, we have $\rho(z, w)=|z-w|/|1-\bar{z}w|$. A sequence $\{x_j\}_j$ in $M(H^{\infty})$ is called interpolating if for every bounded sequence $\{a_j\}_j$ there is a function f in H^{∞} such that $f(x_j)=a_j$ for every j. It is well known (see [2, p. 283]) that for a sequence $\{z_j\}_j$ in D, $\{z_j\}_j$ is interpolating if and only if

$$\inf_k \prod_{j\neq k}\rho(z_j, z_k)>0.$$

For a sequence $\{z_j\}_j$ in D with $\sum_{j=1}^{\infty}1-|z_j|<\infty$, a function

$$b(z)=\prod_{j=1}^{\infty}\frac{\bar{z}_j}{|z_j|}\frac{z_j-z}{1-\bar{z}_j z}\quad(z\in D)$$

is called a Blaschke product with zeros $\{z_j\}_j$, and $\{z_j\}_j$ is called the zero sequence of b. If $\{z_j\}_j$ is interpolating, we call b interpolating. For a function f in H^{∞}, put $Z(f)=\{x\in M(H^{\infty});\ f(x)=0\}$. For a subset E of $M(H^{\infty})$, we denote by cl E the weak*-closure of E in $M(H^{\infty})$.

For a point x in $M(H^{\infty})$, the set $P(x)=\{y\in M(H^{\infty});\ \rho(y, x)<1\}$ is called a Gleason part of x. If $P(x)\neq\{x\}$, $P(x)$ is called nontrivial. D is a typical nontrivial part. We set

$$G=\{x\in M(H^{\infty});\ x \text{ is nontrivial}\}.$$

Hoffman [5] proved that for a point x in G, there is an interpolating sequence $\{z_j\}_j$ such that x is contained in cl $\{z_j\}_j$, and there is a continuous map L_x from D onto $P(x)$ such that $f\circ L_x\in H^{\infty}$ for every $f\in H^{\infty}$, where L_x is given by $L_x(z)=\lim_\alpha(z_{j_\alpha}-z)/(1-\bar{z}_{j_\alpha}z)$ for a net $\{z_{j_\alpha}\}_\alpha$ in $\{z_j\}_j$ with $z_{j_\alpha}\to x$. When L_x is a homeomorphism, $P(x)$ is called a homeomorphic part.

Our problem is; if $\{x_j\}_j$ is an interpolating sequence in G, is there an interpolating Blaschke product b such that $Z(b)\supset\{x_j\}_j$? Generally the converse is not true. For, let b be an interpolating Blaschke product with zeros $\{z_n\}_n$ in D and let x be a cluster point of $\{z_n\}_n$. Put $\{x_j\}_j=\{z_n\}_n\cup\{x\}$. Then it is not difficult to see that $\{x_j\}_j$ is not interpolating and $Z(b)\supset\{x_j\}_j$. In [**3**] and [**6**], they independently proved that if P is a homeomorphic part and $\{x_j\}_j\subset P$, then $\{x_j\}_j$ is interpolating if and only if $\{x_j\}_j=Z(b)\cap P$ for an interpolating Blaschke product b. In this paper, we study an interpolating sequence whose elements are contained in distinct parts in G. Our theorem is the following.

THEOREM. *Let $\{x_j\}_j$ be a sequence in G such that $P(x_k)\cap\mathrm{cl}\{x_j\}_{j\neq k}=\phi$ for every k. Then the following conditions are equivalent.*

(i) *There is an interpolating Blaschke product b such that $Z(b)\supset\{x_j\}_j$.*

(ii) *$\{x_j\}_j$ is an interpolating sequence.*

The idea to prove our theorem is basically the same as in [**6**]. The difference between them is; let h be a function in H^∞ with $h(x_1)\neq 0$ and $h(x_j)=0$ for $j\geqq 2$ and let B be a Blaschke factor of h. If $\{x_j\}_j$ is contained in the same part, then $B(x_1)\neq 0$ and $B(x_j)=0$ for $j\geqq 2$, but under the assumption of our theorem we can not say anything about B. Previous paper's problem is how to construct an interpolating subproduct b of B such that $b(x_j)=0$ for $j\geqq 2$, but this paper's problem is how to construct an interpolating Blaschke product b such that $b(x_j)=0$ for $j\geqq 2$ using the function h. Therefore this paper is a little bit complicated more than the previous one. The main part of this paper is to prove (ii)$\Rightarrow$(i). In Section 2, we give eight lemmas. Using them, we prove our theorem in Section 3.

2. Blaschke subproducts.

For an interpolating Blaschke product b with zeros $\{z_j\}_j$, put

$$\delta(b)=\inf_k\prod_{j\neq k}\rho(z_j, z_k).$$

By Hoffman [**5**, p. 82], we have the following lemma.

LEMMA 1. *Let $x\in M(H^\infty)$ and let b be an interpolating Blaschke product with $b(x)=0$. If $0<\delta<1$, then there is a subproduct b_1 of b such that $b_1(x)=0$ and $\delta(b_1)>\delta$.*

We use the same idea to prove the following lemmas 2, 4 and 5, but these situations are different, so we shall give these detail proofs. Lemma 6 is a summary of these results. Let $\{x_j\}_j$ be an interpolating sequence. Then by the open mapping theorem, there is a universal constant M such that for every sequence $\{a_j\}_j$ with $|a_j|\leqq 1$ for every j, there is a function f in H^{∞} with $\|f\|\leqq M$ and $f(x_j)=a_j$ for every j. The constant M is called an interpolation constant for $\{x_j\}_j$.

LEMMA 2. *Let $\{x_j\}_j$ be a sequence in G and let $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let h be a function in ball (H^{∞}) with $Z(h)\cap D=\phi$ and $Z(h)\supset\{x_j\}_j$. If x is a point in $M(H^{\infty})$ with $h(x)\neq 0$, then for each r with $0<r<1$ there is a Blaschke product $\prod_{j=1}^{\infty}\psi_j$ such that*

(i) *ψ_j is a subproduct of b_j with $\psi_j(x_j)=0$; and*

(ii) $\left|\left(\prod_{j=1}^{\infty}\psi_j\right)(x)\right|>r$.

PROOF. Let $\{z_{j,k}\}_k$ be the zero sequence of b_j. Since $b_j(x_j)=0$, by [4, p. 205], $x_j\in \mathrm{cl}\{z_{j,k}\}_k$. Let M_j be an interpolation constant for $\{z_{j,k}\}_k$. Take a sequence $\{r_j\}_j$ such that

$$0<r_j<1 \quad \text{and} \quad \prod_{j=1}^{\infty} r_j > r .$$

Then take a sequence $\{\varepsilon_j\}_j$ such that

$$0<\varepsilon_j<1 \quad \text{and} \quad \prod_{j=1}^{\infty}\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j}>r . \tag{1}$$

Put

$$E=\{\zeta\in D;\ |h(\zeta)|>|h(x)|/2\} . \tag{2}$$

By the corona theorem (see [2, p. 318]), x is contained in $\mathrm{cl}\,E$.

Fix j arbitrary. Then there is a positive integer n, depending on j, such that

$$r_j^n<|h(x)|/2 . \tag{3}$$

Let ψ_j be a subproduct of b_j with zeros $F_j=\{z_{j,k};\ |h(z_{j,k})|<\varepsilon_j^n\}$. Since $h(x_j)=0$ and $x_j\in\mathrm{cl}\{z_{j,k}\}_k$, we have $x_j\in\mathrm{cl}\,F_j$, so that $\psi_j(x_j)=0$. Since $Z(h)\cap D=\phi$, we may consider that $h^{1/n}$ is a function in ball(H^{∞}). Since $|h^{1/n}|<\varepsilon_j$ on F_j and the interpolating sequence F_j has M_j as an interpolation constant, there is a function f in H^{∞} such that

$$\|f\|\leqq M_j\varepsilon_j \quad \text{and} \quad f(z_{j,k})=h^{1/n}(z_{j,k}) \quad \text{for every } z_{j,k}\in F_j .$$

Then there is a function g in H^{∞} such that

$$f-h^{1/n}=\psi_j g .$$

Here we have $\|g\|\leqq 1+M_j\varepsilon_j$. Consequently we get

$$(4) \qquad |h^{1/n}(z)|-M_j\varepsilon_j \leqq |(f-h^{1/n})(z)| \leqq (1+M_j\varepsilon_j)|\phi_j(z)|$$

for every $z\in D$. Therefore for $\zeta\in E$ we get

$$\begin{aligned} r_j &< (|h(x)|/2)^{1/n} && \text{by (3)} \\ &< |h^{1/n}(\zeta)| && \text{by (2)} \\ &\leqq (1+M_j\varepsilon_j)|\phi_j(\zeta)|+M_j\varepsilon_j && \text{by (4)}. \end{aligned}$$

Hence

$$\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j} < |\phi_j(\zeta)| \qquad \text{for every} \quad \zeta\in E.$$

Consequently we have

$$\begin{aligned} r &< \prod_{j=1}^{\infty}\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j} && \text{by (1)} \\ &< \left|\left(\prod_{j=1}^{\infty}\phi_j\right)(\zeta)\right| && \text{for every} \quad \zeta\in E. \end{aligned}$$

Since $x\in \mathrm{cl}\, E$, we get $r < \left|\left(\prod_{j=1}^{\infty}\phi_j\right)(x)\right|$.

The following lemma comes from [**5**, Theorem 5.2].

LEMMA 3. *Let B be a Blaschke product with zeros $\{w_j\}_j$. Then there are subfactors B_1 and B_2 of B such that $B=B_1B_2$ and $B_1=B_2=0$ on $Z(B)\setminus \mathrm{cl}\{w_j\}_j$.*

LEMMA 4. *Let $\{x_j\}_j$ be a sequence in G and $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let B be a Blaschke product with zeros $\{w_k\}_k$ such that $Z(B)\supset\{x_j\}_j$ and $x_j\notin \mathrm{cl}\,\{w_k\}_k$ for every j. If x is a point in $M(H^\infty)$ with $B(x)\neq 0$, then for each r with $0<r<1$ there is a Blaschke product $\prod_{j=1}^{\infty}\phi_j$ such that*

(i) *ϕ_j is a subproduct of b_j with $\phi_j(x_j)=0$; and*

(ii) $\left|\left(\prod_{j=1}^{\infty}\phi_j\right)(x)\right| > r$.

PROOF. Take $\{z_{j,k}\}_k$, $\{M_j\}_j$, $\{r_j\}_j$, and $\{\varepsilon_j\}_j$ as in the proof of Lemma 2. Put

$$E=\{\zeta\in D;\ |B(\zeta)|>|B(x)|/2\}.$$

Then $x\in \mathrm{cl}\, E$. Fix j arbitrary. There is a positive integer n such that $r_j^n< |B(x)|/2$. Applying Lemma 3 succeedingly n-times for B and its subfactors, we get

$$B=B_1B_2\cdots B_n \quad \text{and} \quad B_i=0 \quad \text{on} \quad Z(B)\setminus \mathrm{cl}\,\{w_k\}_k$$

for every $1\leqq i\leqq n$. For each i, $1\leqq i\leqq n$, let $\phi_{j,i}$ be a subproduct of b_j with zeros

$$F_{j,i}=\{z_{j,k}\,;\ |B_i(z_{j,k})|<\varepsilon_j\}.$$

Since $B_i(x_j)=0$ and $x_j\in \mathrm{cl}\,\{z_{j,k}\}_k$, we have

$$\bigcap_{i=1}^{n} F_{j,i}\neq\phi \quad\text{and}\quad x_j\in \mathrm{cl}\bigcap_{i=1}^{n} F_{j,i}.$$

Let ψ_j be a subproduct of b_j with zeros $\bigcap_{i=1}^{n} F_{j,i}$. Then $\psi_j(x_j)=0$, $|\psi_{j,i}|\leqq|\psi_j|$ on D for every i, and $|B_i|<\varepsilon_j$ on $F_{j,i}$. Since the interpolating sequence $F_{j,i}$ has M_j as an interpolation constant, there is a function f_i in H^∞ such that

$$\|f_i\|\leqq M_j\varepsilon_j \quad\text{and}\quad f_i(z_{j,k})=B_i(z_{j,k}) \qquad \text{for} \quad z_{j,k}\in F_{j,i}.$$

Then there is a function g_i in H^∞ such that

$$f_i-B_i=\psi_{j,i}g_i.$$

Since $\|g_i\|\leqq 1+M_j\varepsilon_j$, we have

$$\frac{|B_i(z)|-M_j\varepsilon_j}{1+M_j\varepsilon_j}\leqq|\psi_{j,i}(z)| \quad \text{for} \quad z\in D \quad\text{and}\quad 1\leqq i\leqq n$$
$$\leqq|\psi_j(z)|.$$

Let $\zeta\in E$. Since

$$\prod_{i=1}^{n}|B_i(\zeta)|=|B(\zeta)|>|B(x)|/2>r_j{}^n,$$

we have $|B_i(\zeta)|>r_j$ for some i, where i depends on ζ. Hence

$$\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j}\leqq|\psi_j(\zeta)| \qquad \text{for every} \quad \zeta\in E.$$

Consequently for every $\zeta\in E$ we have

$$r<\prod_{j=1}^{\infty}\frac{r_j-M_j\varepsilon_j}{1+M_j\varepsilon_j}\leqq\left|\left(\prod_{j=1}^{\infty}\psi_j\right)(\zeta)\right|.$$

Since $x\in\mathrm{cl}\,E$, we get $r<|(\prod_{j=1}^{\infty}\psi_j)(x)|$.

LEMMA 5. *Let $\{x_j\}_j$ be a sequence in G such that x_k is not contained in* $\mathrm{cl}\,\{x_j\}_{j\neq k}$ *for every k. Let $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let B be a Blaschke product with zeros $\{w_k\}_k$ such that $\{x_j\}_j\subset$* $\mathrm{cl}\,\{w_k\}_k$. *If x is a point in $M(H^\infty)$ such that $|B(x)|>\delta$, then there is a Blaschke product $\prod_{j=1}^{\infty}\psi_j$ such that*

(i) *ψ_j is a subproduct of b_j with $\psi_j(x_j)=0$; and*

(ii) $\left|\left(\prod_{j=1}^{\infty}\psi_j\right)(x)\right|>\delta.$

PROOF. Take $\{z_{j,k}\}_k$ and $\{M_j\}_j$ as in Lemma 2. Take σ as $\delta<\sigma<|B(x)|$. Take a sequence $\{\varepsilon_j\}_j$ such that $\varepsilon_j>0$ and

(5)
$$\sigma \prod_{j=1}^{\infty} \frac{1-M_j\varepsilon_j\sigma^{-1}}{1+M_j\varepsilon_j} > \delta .$$

Put

(6)
$$E = \{\zeta \in D ;\ |B(\zeta)| > \sigma\}.$$

Then $x \in \mathrm{cl}\, E$. By our assumption on $\{x_j\}_j$, there is a sequence of disjoint open subsets $\{U_j\}_j$ of $M(H^\infty)$ such that $x_j \in U_j$ for every j. Let B_j be the Blaschke product with zeros $\{w_k\}_k \cap U_j$. Then $\prod_{j=1}^{\infty} B_j$ is a subproduct of B and

(7)
$$|B_j| > \sigma \quad \text{on} \quad E .$$

Since $x_j \in \mathrm{cl}\, \{w_k\}_k$, $B_j(x_j)=0$.

Fix j arbitrary. Let ϕ_j be the subproduct of b_j with zeros $F_j = \{z_{j,k} ;\ |B_j(z_{j,k})| < \varepsilon_j\}$. Since $x_j \in \mathrm{cl}\, \{z_{j,k}\}_k$ and $B_j(x_j)=0$, we have $\phi_j(x_j)=0$. By the same way as Lemma 2 (replace $h^{1/n}$ by B_j), we have

(8)
$$\frac{|B_j(z)| - M_j\varepsilon_j}{1+M_j\varepsilon_j} \leqq |\phi_j(z)| \qquad \text{for every} \quad z \in D .$$

Therefore for $\zeta \in E$ we have

$$\begin{aligned} \left|\Big(\prod_{j=1}^{\infty} \phi_j\Big)(\zeta)\right| &= \prod_{j=1}^{\infty} |\phi_j(\zeta)| \\ &\geqq \prod_{j=1}^{\infty} |B_j(\zeta)| \prod_{j=1}^{\infty} \frac{1-M_j\varepsilon_j |B_j(\zeta)|^{-1}}{1+M_j\varepsilon_j} && \text{by (8)} \\ &\geqq |B(\zeta)| \prod_{j=1}^{\infty} \frac{1-M_j\varepsilon_j\sigma^{-1}}{1+M_j\varepsilon_j} && \text{by (7)} \\ &\geqq \sigma \prod_{j=1}^{\infty} \frac{1-M_j\varepsilon_j\sigma^{-1}}{1+M_j\varepsilon_j} && \text{by (6)} \\ &> \delta . && \text{by (5)} \end{aligned}$$

Since $x \in \mathrm{cl}\, E$, we get $|(\prod_{j=1}^{\infty} \phi_j)(x)| > \delta$.

The following lemma is a summary of Lemmas 2, 4 and 5.

LEMMA 6. *Let $\{x_j\}_j$ be a sequence in G such that x_k is not contained in $\mathrm{cl}\, \{x_j\}_{j \neq k}$ for every k. Let $\{b_j\}_j$ be a sequence of interpolating Blaschke products with $b_j(x_j)=0$. Let $x \in M(H^\infty)$. If $|f(x)| > \delta$ for some function f in* ball(H^∞) *with $Z(f) \supset \{x_j\}_j$, then there is a Blaschke product $\prod_{j=1}^{\infty} \phi_j$ such that*

(i) *ϕ_j is a subproduct of b_j with $\phi_j(x_j)=0$; and*

(ii) $\left|\Big(\prod_{j=1}^{\infty} \phi_j\Big)(x)\right| > \delta$.

PROOF. Let $f = Bh$, where B is a Blaschke factor of f and $Z(h) \cap D = \phi$. Let $\{w_k\}_k$ be a zero sequence of B. Put

$$\{x_{1,j}\}_j=\{x_i\,;\ B(x_i)=0 \quad \text{and} \quad x_i\in \mathrm{cl}\,\{w_k\}_k\}\,;$$

$$\{x_{2,j}\}_j=\{x_i\,;\ B(x_i)=0 \quad \text{and} \quad x_i\notin \mathrm{cl}\,\{w_k\}_k\}\,; \quad \text{and}$$

$$\{x_{3,j}\}_j=\{x_i\}_i\setminus(E_1\cup E_2)=\{x_i\,;\ B(x_i)\neq 0\}\,.$$

Note that $|B(x)|>\delta$ and $h(x)\neq 0$. We devide $\{b_j\}_j$ into three parts $\{b_{1,j}\}_j$, $\{b_{2,j}\}_j$ and $\{b_{3,j}\}_j$ such that

$$b_{k,j}(x_{k,j})=0 \quad \text{for} \quad k=1, 2, 3 \quad \text{and} \quad j=1, 2, \cdots.$$

Take δ_1 such that $\delta<\delta_1<|f(x)|$, and take r such that

$$0<r<1 \quad \text{and} \quad \delta<\delta_1 r^2\,.$$

We apply Lemma 5 for $\{x_{1,j}\}_j$ and $\{b_{1,j}\}_j$. Then there is a subproduct $\psi_{1,j}$ of $b_{1,j}$ such that $\psi_{1,j}(x_{1,j})=0$ and $|(\prod_{j=1}^{\infty}\psi_{1,j})(x)|>\delta_1$. We apply Lemma 4 for $\{x_{2,j}\}_j$ and $\{b_{2,j}\}_j$. Then there is a subproduct $\psi_{2,j}$ of $b_{2,j}$ such that $\psi_{2,j}(x_{2,j})=0$ and $|(\prod_{j=1}^{\infty}\psi_{2,j})(x)|>r$. Since $Z(h)\supset\{x_{3,j}\}_j$, we can apply Lemma 2 for $\{x_{3,j}\}_j$ and $\{b_{3,j}\}_j$. Then there is a subproduct $\psi_{3,j}$ of $b_{3,j}$ such that $\psi_{3,j}(x_{3,j})=0$ and $|(\prod_{j=1}^{\infty}\psi_{3,j})(x)|>r$. Consequently, we have a desired Blaschke product $\prod_{i=1}^{3}\prod_{j=1}^{\infty}\psi_{i,j}$.

LEMMA 7. *Let $x\in G$ and let b be an interpolating Blaschke product with $b(x)=0$. If b_1 and b_2 are subproducts of b with $b_1(x)=b_2(x)=0$, then x is contained in the closure of the intersection of zero sequences of b_1 and b_2.*

PROOF. Suppose not. Let $\{z_j\}_j$ and $\{w_j\}_j$ be the zero sequences of b_1 and b_2 respectively. Put $W=\{z_j\}_j\cap\{w_j\}_j$. Then $x\notin \mathrm{cl}\,W$, so that $x\in \mathrm{cl}\,(\{z_j\}_j\setminus W)$ and $x\in \mathrm{cl}\,(\{w_j\}_j\setminus W)$. Since disjoint subsets in an interpolating sequence have disjoint closures, we get a contradiction.

LEMMA 8. *Let $x\in G$ and let E be a closed subset of $M(H^{\infty})$ with $P(x)\cap E=\phi$. If b is an interpolating Blaschke product with $b(x)=0$ and $0<r<1$, then there is a subproduct ψ of b such that $\psi(x)=0$ and $|\psi|>r$ on E.*

PROOF. For each $y\in E$, since $\rho(x, y)=1$ there is a function h_y in ball(H^{∞}) such that $h_y(x)=0$ and $|h_y(y)|>r$. As a special case of Lemma 6, there is a subproduct b_y of b such that $b_y(x)=0$ and $|b_y(y)|>r$. Put

$$U_y=\{\zeta\in M(H^{\infty})\,;\ |b_y(\zeta)|>r\}\,.$$

Then $\cup\{U_y\,;\ y\in E\}\supset E$. Hence there is a finite set $\{y_1, y_2, \cdots, y_n\}$ in E such that $\cup\{U_{y_i}\,;\ 1\leqq i\leqq n\}\supset E$. Let ψ be an interpolating Blaschke product with zeros $\bigcap_{i=1}^{n}Z(b_{y_i})\cap D$. By Lemma 7, we have $\psi(x)=0$. Since $|\psi|\geqq|b_{y_i}|$ on D, we have

$$|\psi(y)|\geqq \max\{|b_{y_i}(y)|\,;\ 1\leqq i\leqq n\}>r$$

for every $y\in E$.

3. Proof of Theorem.

PROOF. (i)$\Rightarrow$(ii) Let b be an interpolating Blaschke product with zeros $\{z_k\}_k$ such that $Z(b)\supset\{x_j\}_j$. Since $x_j\notin \mathrm{cl}\,\{x_k\}_{k\neq j}$ for every j, there is a sequence of disjoint open subsets $\{U_j\}_j$ of $M(H^\infty)$ such that $x_j\in U_j$. Since x_j is a cluster point of $\{z_k\}_k$, $\{z_k\}_k\cap U_j$ is an infinite set for each j. For a bounded sequence $\{a_j\}_j$, there is a function h in H^∞ such that $h(z_i)=a_j$ for every $z_i\in\{z_k\}_k\cap U_j$. Since $x_j\in \mathrm{cl}\,\{z_k\}_k\cap U_j$, we have $h(x_j)=a_j$ for every j. Therefore $\{x_j\}_j$ is an interpolating sequence.

(ii)$\Rightarrow$(i) Suppose that $\{x_j\}_j$ is an interpolating sequence. Since $x_j\in G$, there is an interpolating Blaschke product b_j such that $b_j(x_j)=0$. By the open mapping theorem, there is a positive number δ such that

(#) $\inf_k \sup\{|h(x_k)|\,;\ h\in \mathrm{ball}\,(H^\infty),\ h(x_j)=0 \text{ for } j\neq k\}>\delta$.

Let h_1 be a function in ball (H^∞) such that $|h_1(x_1)|>\delta$ and $h_1(x_j)=0$ for $j\neq 1$. By Lemma 6 (consider as $x=x_1$ and $f=h_1$), there is a Blaschke product $B_1=\prod_{j=2}^\infty b_{1,j}$ such that $|B_1(x_1)|>\delta$ and $b_{1,j}$ is a subproduct of b_j with $b_{1,j}(x_j)=0$ for $j\geqq 2$.

Let $\{r_j\}_j$ be a sequence of numbers such that

$$0<r_j<1 \quad\text{and}\quad \prod_{j=1}^\infty r_j>\delta .$$

By Lemma 8 (consider as $x=x_1$, $b=b_1$ and $E=\mathrm{cl}\,\{x_i\}_{i\neq 1}$), there is an interpolating Blaschke subproduct ψ_1 of b_1 such that $\psi_1(x_1)=0$ and $|\psi_1(x_i)|>r_1$ for $i\neq 1$. By Lemma 1, we may assume that $\delta(\psi_1)>\delta$. Since $|B_1(x_1)|>\delta$, there is a subsequence $\{z_{1,i}\}_i$ of the zero sequence of ψ_1 such that $|B_1(z_{1,i})|>\delta$ for every i. Then $x_1\in \mathrm{cl}\,\{z_{1,i}\}_i$. Let ϕ_1 be the interpolating Blaschke product with zeros $\{z_{1,i}\}_i$. Then ϕ_1 is a subproduct of b_1, $\delta(\phi_1)>\delta$, $\phi_1(x_1)=0$, and $|\phi_1(x_i)|>r_1$ for $i\neq 1$.

By induction, we shall construct a sequence of Blaschke products $\{B_j\}_{j\geqq 2}$ and sequences of interpolating Blaschke products $\{\phi_j\}_{j\geqq 2}$ and $\{b_{j,t}\}_{t>j}$ such that:

(a) $B_j=\prod_{t=j+1}^\infty b_{j,t}$ is a subproduct of $B_{j-1}=\prod_{t=j}^\infty b_{j-1,t}$ such that $|B_j(x_j)|>\delta$;

(b) $b_{j,t}$ is an interpolating Blaschke subproduct of $b_{j-1,t}$ such that $b_{j,t}(x_t)=0$ for $t\geqq j+1$;

(c) ϕ_j is a subproduct of $b_{j-1,j}$ with zeros $\{z_{j,i}\}_i$ and $\delta(\phi_j)>\delta$;

(d) $|B_j(z_{j,i})|>\delta$ for every i;

(e) $\phi_j(x_j)=0$ and $|\phi_j(x_i)|>r_j$ for $i\neq j$; and

(f) $|\phi_s(z_{j,i})|>r_s$ for every $s<j$ and i.

Our induction works on k. If we put $b_{0,t}=b_t$, then B_1, ϕ_1 and $\{b_{1,t}\}_{t>1}$ satisfy all conditions $(a-f)$ for $k=1$.

Suppose that $\{B_j\}_{j<k}$, $\{\phi_j\}_{j<k}$ and $\{b_{j,t}\}_{t>j}(j<k)$ are already chosen. By

(#) and Lemma 6 (consider as $x=x_k$ and $\{b_j\}_j=\{b_{k-1,t}\}_{t\geq k+1}$), there is a subproduct $B_k=\prod_{t=k+1}^{\infty}b_{k,t}$ of B_{k-1} such that $|B_k(x_k)|>\delta$ and $b_{k,t}$ is an interpolating Blaschke subproduct of $b_{k-1,t}$ such that $b_{k,t}(x_t)=0$ for $t\geqq k+1$. Thus we get (a) and (b).

By Lemma 8 (consider as $x=x_k$, $b=b_{k-1,k}$ and $E=\mathrm{cl}\,\{x_j\}_{j\neq k}$), there is an interpolating Blaschke subproduct ψ_k of $b_{k-1,k}$ such that $\psi_k(x_k)=0$ and $|\psi_k(x_i)|>r_k$ for $i\neq k$. By Lemma 1, we may assume that $\delta(\psi_k)>\delta$. Since $|B_k(x_k)|>\delta$, there is a subsequence $\{z_{k,i}\}_i$ of the zero sequence of ψ_k such that $|B_k(z_{k,i})|>\delta$ for every i. Then we get (d) and $x_k\in\mathrm{cl}\,\{z_{k,i}\}_i$.

Let ϕ_k be the interpolating Blaschke product with zeros $\{z_{k,i}\}_i$. Then $\phi_k(x_k)=0$. Since ϕ_k is a subproduct of ψ_k, we get (c) and (e).

Since $|\phi_s(x_k)|>r_s$ for $s<k$ by (e), moreover we may assume that $\{z_{k,i}\}_i$ satisfies $|\phi_s(z_{k,i})|>r_s$ for every $s<k$ and i. Thus we get (f). This completes the induction.

Put $b=\prod_{k=1}^{\infty}\phi_k$. By (e), we have $Z(b)\supset\{x_j\}_j$. We shall prove that b is an interpolating Blaschke product. We note that $\{z_{k,j}\}_{k,j}$ is the zero sequence of b. We have

$$\begin{aligned}
&\inf_{(k,i)}\prod_{(t,s)\neq(k,i)}\rho(z_{t,s},z_{k,i})\\
&\quad=\inf_{(k,i)}\left[\prod_{t\neq k}\prod_{s=1}^{\infty}\rho(z_{t,s},z_{k,i})\right]\left[\prod_{s\neq i}\rho(z_{k,s},z_{k,i})\right]\\
&\quad\geqq\inf_{(k,i)}\left[\prod_{t\neq k}|\phi_t(z_{k,i})|\right]\delta(\phi_k) && \text{by (c)}\\
&\quad\geqq\delta\inf_{(k,i)}\left[\prod_{t<k}|\phi_t(z_{k,i})|\right]\left[\prod_{t>k}|\phi_t(z_{k,i})|\right] && \text{by (c)}\\
&\quad\geqq\delta\inf_{(k,i)}\left[\prod_{t<k}r_t\right]|B_k(z_{k,i})| && \text{by (a), (b), (c), (f)}\\
&\quad\geqq\delta^2\prod_{t=1}^{\infty}r_t && \text{by (d)}\\
&\quad\geqq\delta^3.
\end{aligned}$$

Hence b is an interpolating Blaschke product. This completes the proof.

REMARK. By the proof of (i)$\Rightarrow$(ii), for a sequence $\{x_k\}_k$ such that $Z(b)\supset\{x_k\}_k$ for some interpolating Blaschke product b, $\{x_k\}_k$ is interpolating if and only if $x_j\notin\mathrm{cl}\,\{x_k\}_{k\neq j}$ for every j.

4. Comments.

A closed subset E of $M(H^\infty)$ is called an interpolation set for H^∞ if for every continuous function f on E there is a function g in H^∞ such that $g|_E=f$. In [7], Lingenberg proved that if E is an interpolation set such that $E\subset G$ then there is an interpolating Blaschke product b such that $Z(b)\supset E$. If E is an interpolation set, then E is ρ-separating, that is,

$$\inf\{\rho(x, y);\ x, y\in E, x\neq y\} > 0.$$

Recently Lingenberg and the author showed that if E is a closed ρ-separating subset of $M(H^\infty)$ with $E\subset G$, E is an interpolation set. Since every closed subset of $Z(b)$, where b is an interpolating Blaschke product, is ρ-separating, the following conditions for closed subsets E of $M(H^\infty)$ are equivalent:

(i) E is an interpolation set and $E\subset G$;

(ii) E is ρ-separating and $E\subset G$; and

(iii) there is an interpolating Blaschke product b such that $Z(b)\supset E$.

The closedness of E is an unremovable condition in the above assertion.

Now let $\{x_n\}_n$ be an interpolating sequence in $M(H^\infty)$. If $\{x_n\}_n$ is contained in D, then $\mathrm{cl}\,\{x_n\}_n\subset G$ by [**5**]. We have a following conjecture.

CONJECTURE. *If* $\{x_n\}_n$ *is an interpolating sequence in* G, *then* $\mathrm{cl}\,\{x_n\}_n\subset G$.

If this conjecture is affirmative, we may discuss as follows. Let $\{y_n\}_n$ be a sequence in $M(H^\infty)$. We put

$$\{y_{1,n}\}_n = \{y_n\}_n\cap M(L^\infty);$$

$$\{y_{2,n}\}_n = \{y_n\}_n\cap[M(H^\infty)\setminus(M(L^\infty)\cup G)];\quad \text{and}$$

$$\{y_{3,n}\}_n = \{y_n\}_n\cap G.$$

If $\{y_n\}_n$ is interpolating, then each $\{y_{k,n}\}_n$ is interpolating. We see the converse assertion is also true. Since $M(L^\infty)$ is closed, $\mathrm{cl}\,\{y_{1,n}\}_n\subset M(L^\infty)$. Since $\{y_{2,n}, y_{3,n}\}_n$ is a countable subset of $M(H^\infty)\setminus M(L^\infty)$, by [8] we have $\mathrm{cl}\,\{y_{2,n}, y_{3,n}\}_n\cap M(L^\infty)=\phi$. Since G is an open subset of $M(H^\infty)$ [**5**], $\mathrm{cl}\,\{y_{2,n}\}_n\subset M(H^\infty)\setminus G$. Suppose that each $\{y_{k,n}\}_n$ is interpolating. Then $\mathrm{cl}\,\{y_{3,n}\}_n\subset G$ (if our conjecture is true), and $\mathrm{cl}\,\{y_{k,n}\}_n$, $k=1, 2, 3$, become mutually disjoint interpolation sets. Moreover

$$\rho(\mathrm{cl}\{y_{k,n}\}_n, \mathrm{cl}\,\{y_{j,n}\}_n) = 1 \qquad \text{for} \quad k\neq j.$$

Hence by [**9**], $\bigcup_{k=1}^{3}\mathrm{cl}\,\{y_{k,n}\}_n$ is an interpolation set. Then $\{y_n\}_n=\bigcup_{k=1}^{3}\{y_{k,n}\}_n$ becomes an interpolating sequence.

Hence to determine whether $\{y_n\}_n$ is interpolating or not it is sufficient to study three sequences independently. Hoffman (unpublished note) proved that $\{y_{1,n}\}_n$ is interpolating if and only if $y_j\notin\mathrm{cl}\,\{y_{1,n}\}_{n\neq j}$ for every j. If $\{y_{3,n}\}_n$ is interpolating, then $\mathrm{cl}\,\{y_{3,n}\}_n$ is an interpolation set with $\mathrm{cl}\,\{y_{3,n}\}_n\subset G$ (if our conjecture is true) and $y_j\notin\mathrm{cl}\,\{y_{3,n}\}_{n\neq j}$ for every j. The converse is also true. For, by the first paragraph, there is an interpolating Blaschke product b such that $Z(b)\supset\{y_{3,n}\}_n$. By the remark in Section 3, $\{y_{3,n}\}_n$ is interpolating.

But we do not know anything when $\{y_{2,n}\}_n$ is interpolating.

References

[1] S. Axler and P. Gorkin, Sequences in the maximal ideal space of H^∞, Proc. Amer. Math. Soc., **108** (1990), 731-740.

[2] J. Garnett, Bounded analytic functions, Academic Press, New York-London, 1981.

[3] P. Gorkin, H.-M. Lingenberg and R. Mortini, Homeomorphic disks in the spectrum of H^∞, Indiana Univ. Math. J., **39** (1990), 961-983.

[4] K. Hoffman, Banach spaces of analytic functions, Prentice Hall, Englewood Cliffs, New Jersey, 1962.

[5] K. Hoffman, Bounded analytic functions and Gleason parts, Ann. of Math., **86** (1967), 74-111.

[6] K. Izuchi, Interpolating sequences in a homeomorphic part of H^∞, Proc. Amer. Math. Soc., **111** (1991), 1057-1065.

[7] H.-M. Lingenberg, Interpolationsmengen im raum der maximalen ideale der algebra H^∞, Thesis, Univ. Karlsruhe, 1989.

[8] C. Sundberg, A note on algebras between L^∞ and H^∞, Rocky Mountain J. Math., **11** (1981), 333-336.

[9] N. Th. Varopoulos, Sur la reunion de deux ensembles d'interpolation d'une algèbre uniforme, C. R. Acad. Sci. Paris, Sér. A, **272** (1971), 950-952.

Keiji IZUCHI
Department of Mathematics
Kanagawa University
Rokkakubashi, Yokohama 221
Japan