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Three contributions to the homotopy theory of the exceptional Lie groups G_2 and F_4*

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1. Statement of results.

In this paper, we prove three theorems related to the homotopy theory of the exceptional Lie groups G_2 and F_4. These results will be useful in work of the first author with Bendersky and Mimura, which seeks to calculate v_1-periodic homotopy groups of all exceptional Lie groups.

Our first result, which will be proved in Section 2, should be useful in determining the homotopy groups of the homogeneous space F_4/G_2, and consequently in deducing information about $\pi_*(F_4)_{(2)}$ from information about $\pi_*(G_2)_{(2)}$.

THEOREM 1.1. *There is a 2-local fibration*

$$S^{15} \longrightarrow F_4/G_2 \longrightarrow S^{23}.$$

Such a fibration is known to exist localized at primes $\geqq 5$, ([**21**]) and to not exist at the prime 3. ([**7**])

Our second result is relevant to F_4 because of the equivalence $F_4/Spin(9)= \Pi$, where Π denotes the Cayley projective plane ([**6**]).

THEOREM 1.2. *There is a fibration*

$$S^7 \longrightarrow \Omega\Pi \longrightarrow \Omega S^{23}.$$

This result, which will be proved in Section 3, might allow one to extend the range of calculation of $\pi_*(\Pi)$ begun in [**20**]. In particular, it implies both upper- and lower-bounds for p-exponents of Π, which are defined by

$$\exp_p(\Pi) = \max\{e : \pi_*(\Pi) \text{ has an elements of order } p^e\}.$$

If $p \geqq 5$, then it is known (e.g., [**20**]) that the fibration of our Theorem 1.2 exists as a product, and so $\exp_p(\Pi)=\exp_p(S^{23})=11$, by [**10**]. Our theorem implies that

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$$\exp_p(\Pi) \leqq \exp_p(S^7)+\exp_p(S^{23}) \begin{cases} \leqq 22 & \text{if } p=2 \\ =14 & \text{if } p=3 \end{cases}$$

where we use [24] when $p=2$ and [10] when $p=3$. A lower bound, conjectured to be sharp, will be determined in the work with Bendersky and Mimura by using the exact sequence of v_1-periodic homotopy groups associated to the fibration of 1.2 to determine completely $v_1^{-1}\pi_*(\Pi)$, the maximal p-exponent of which is a lower bound for the p-exponent of the space.

The third result is the complete calculation of the 2-primary v_1-periodic homotopy groups of G_2. The definition of these groups is completely analogous to definitions given in [11] and [12]. $v_1^{-1}\pi_k(X)$ is a direct limit of $\pi_K(X)$ over values of $K\equiv k$ mod some specific 2-power. The periodic group $v_1^{-1}\pi_k(X)$ is a direct summand of $\pi_K(X)$ for K sufficiently large.

THEOREM 1.3. *The 2-primary v_1-periodic homotopy groups of G_2 are given by*

$$v_1^{-1}\pi_i(G_2) = \begin{cases} \boldsymbol{Z}/2^{\min(6,1+\nu(i-9))}\oplus \boldsymbol{Z}_2 & \textit{if } i\equiv 1 \bmod 8 \\ \boldsymbol{Z}/2^{\min(6,1+\nu(i-10))} & \textit{if } i\equiv 2 \bmod 8 \\ 0 & \textit{if } i\equiv 3,\ 4 \bmod 8 \\ \boldsymbol{Z}/8 & \textit{if } i\equiv 5 \bmod 8 \\ \boldsymbol{Z}/8\oplus \boldsymbol{Z}_2 & \textit{if } i\equiv 6 \bmod 8 \\ \boldsymbol{Z}_2\oplus \boldsymbol{Z}_2\oplus Z_2 & \textit{if } i\equiv 7,\ 8 \bmod 8 \end{cases}$$

where $\nu(-)$ denotes the exponent of 2.

This result generalizes work of [23]. In Section 4 we will prove this result and give a picture of it in an Adams spectral sequence-type chart. The proof of one of the differentials in this spectral sequence requires some delicate homotopy theory, which is the reason for splitting this result off from the work with Bendersky and Mimura mentioned above, which will be primarily algebraic.

Theorem 1.3 yields as an immediate corollary that $\exp_2(G_2)\geqq 6$, and we conjecture this to be sharp. The best upper bound easily derived is

$$\exp_2(G_2) \leqq \exp_2(S^3)+\exp_2(V_{7,2}) \leqq 14.$$

2. The fibration for F_4/G_2.

In this section, we prove Theorem 1.1. It was proved in [7] that $H^*(F_4/G_2;\ Z_2)$ is an exterior algebra on classes of degree 15 and 23. Thus, after localizing at 2, F_4/G_2 has the homotopy type of a complex $X=S^{15}\cup e^{23}\cup e^{38}$. Let α denote the attaching map in the quotient space

$$X/S^{15} = S^{23}\cup_\alpha S^{38}.$$

We will show that α is trivial, which implies that there is a pinch map $p: X/S^{15}$

$\to S^{23}$, and the fiber of the composite

$$F_4/G_2 \cong X \longrightarrow X/S^{15} \xrightarrow{p} S^{23}$$

must be a cohomology 15-sphere by the Serre spectral sequence. This is our desired fibration in the 2-local homotopy category.

The attaching map $\alpha \in \pi_{37}(S^{23})$ is in the stable range, and so it is the bottom attaching map in the S-dual of F_4/G_2. Now F_4/G_2 is a manifold, and by [**2**, 3.3] the S-dual of a manifold is the Thom spectrum of its stable normal bundle. By [**1**, 10.1], the bottom attaching map of the Thom spectrum of any stable vector bundle over a manifold M, classified by a map $M \xrightarrow{\theta} BO$, is $J(\theta | S^k)$. Here S^k is the bottom cell of M, and $J : \pi_k(BO) \to \pi_{k-1}^s(S^0)$ is the stable J-homomorphism. In our case, $k=15$, and since $\pi_{15}(BO)=0$, the map α must be trivial.

3. The fibration for $\Omega\Pi$.

In this section, we prove Theorem 1.2. We thank Fred Cohen for suggesting some of the ideas in this proof. We will prove

THEOREM 3.1. *There is a homotopy equivalence*

$$\Sigma\Omega\Pi \cong S^8 \cup_\alpha e^{23} \vee \bigvee_{i \geqq 1} (S^{22i+8} \vee S^{22i+23}),$$

where the attaching map $\alpha \in \pi_{22}(S^8)$ is an unstable element of order 24.

By "unstable", we mean a map which stabilizes to 0.

The map $f : \Omega\Pi \to \Omega S^{23}$ is obtained from the splitting of Theorem 3.1 by adjointing the collapse map $\Sigma\Omega\Pi \to S^{23}$. The Serre spectral sequence implies that the integral cohomology algebras satisfy

$$H^*(\Omega S^{23}) \approx \Gamma_{22}, \quad \text{and} \quad H^*(\Omega\Pi) \approx \Lambda(y_7) \otimes \Gamma_{22},$$

where $\Lambda(y_7)$ is an exterior algebra on a 7-dimensional generator, and Γ_{22} is a divided polynomial algebra, with basis $\{\gamma_i : i \geqq 0\}$ satisfying $\gamma_i\gamma_j = \binom{i+j}{i}\gamma_{i+j}$ and $|\gamma_i| = 22i$. (See, e.g. [**15**].) One readily verifies that f^* is bijective in degree 22, and hence the cup product structure implies that it is bijective in degree $22i$ for all i. Now the Serre spectral sequence of the fibration $F \to \Omega\Pi \to \Omega S^{23}$ implies that $H^*(F) \approx H^*(S^7)$, and hence F has the homotopy type of S^7.

PROOF OF THEOREM 3.1. Let X denote the 22-skeleton of $\Omega\Pi$. By [**20**], $X = S^7 \cup_\alpha e^{22}$, where $\alpha \in \pi_{21}(S^7)$ is an unstable element of order 24. The first 2-cell complex in Theorem 3.1 is ΣX. Let $f : \Sigma X \wedge X \cong X * X \to \Sigma\Omega\Pi$ be the map obtained by applying the Hopf construction to the restriction to $X \times X$ of the multiplication of $\Omega\Pi$.

The restriction of f to the bottom (15-) cell of $X*X$ is null homotopic. To see this, we first note that this map can be viewed as the composite

$$S^{15} \xrightarrow{H(\mu)} \Sigma\Omega S^8 \longrightarrow \Sigma\Omega\Pi,$$

where the first map is obtained by applying the Hopf construction to the map

$$\mu : S^7 \times S^7 \longrightarrow \Omega S^8 \times \Omega S^8 \xrightarrow{m} \Omega S^8,$$

where m is the loop multiplication, and the second map is obtained from the inclusion of the bottom cell of Π. Under the splitting

$$\Sigma\Omega S^8 \cong \bigvee_{i\geq 1} S^{7i+1},$$

the first two components of $H(\mu)$ are homotopic to $*$ and $1_{S^{15}}$, respectively. To see that the first component is null homotopic, we note that this map (which is *not* the map obtained by applying the Hopf construction to the Cayley multiplication of S^7) is

$$S^7 * S^7 \longrightarrow \Sigma S^7, \qquad [x, t, y] \longmapsto \begin{cases} [2t, x] & 0 \leqq t \leqq \frac{1}{2} \\ [2t-1, y] & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

and the null homotopy sends

$$([x, t, y], s) \longmapsto \begin{cases} [2ts, x] & 0 \leqq t \leqq \frac{1}{2} \\ [(2t-1)s, y] & \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

Alternatively, it is the restriction to the bottom cell of the composite in the fibration

$$\Omega S^8 * \Omega S^8 \longrightarrow \Sigma\Omega S^8 \longrightarrow S^8$$

of [3], which is of course trivial as is every composite $F \to E \to B$.

That the second component is $1_{S^{15}}$ follows from James' construction ([**16**]). This second component becomes irrelevant, however, since the 15-cell of $\Sigma\Omega S^8$ becomes a boundary in $\Sigma\Omega\Pi$. This can be seen from the Serre spectral sequence of either the fibration $\Omega\Pi \to * \to \Pi$ or $\Omega\Pi * \Omega\Pi \to \Sigma\Omega\Pi \to \Pi$.

Thus f factors through a map $f' : (\Sigma X \wedge X)/S^{15} \to \Sigma\Omega\Pi$. There is a splitting

$$(\Sigma X \wedge X)/S^{15} \cong S^{30} \vee S^{30} \vee S^{45}. \tag{3.2}$$

To see this, we note that the attaching map of the top cell is the 23-fold suspension of the element $\alpha \in \pi_{21}(S^7)$ which was the attaching map in $\Omega\Pi$. As already observed, the map α is unstable, and as $\pi_{44}(S^{30})$ is in the stable range, $\Sigma^{23}\alpha$ null. Indeed, α was described in [**20**, 7.1] to be $\sigma'\sigma_{14}$ at the prime 2, and $S^{-1}[[\iota_8, \iota_8], \iota_8]$ at the prime 3.

We note that, under the Pontryagin product, $H_*(\Omega\Pi; Z)$ is the tensor product of an exterior algebra on a class x_7 and a polynomial algebra on a class x_{22}. We let σ denote the homology suspension. Standard properties of the Hopf construction guarantee that $\sigma x_7 x_{22}$ and σx_{22}^2 are in the image of f'_*, and hence are spherical classes because of the splitting (3.2).

Assume by induction that we have constructed a map $f_i : S^{22i+1} \to \Sigma\Omega\Pi$ such that $\sigma x_{22}^i \in \mathrm{im}(f_{i*})$. Then the composite

$$c_{i+1} : S^{22i+8} \vee S^{22(i+1)+1} \cong S^{22i+1} \wedge X \longrightarrow \Sigma\Omega\Pi \wedge \Omega\Pi \longrightarrow \Sigma\Omega\Pi$$

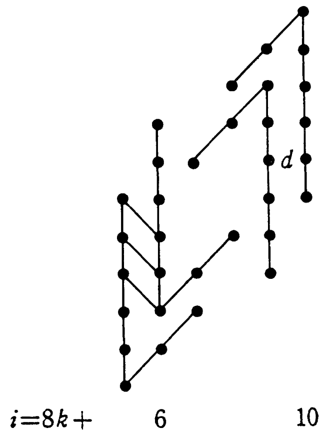
extends the induction and shows that $\sigma x_7 x_{22}^i$ is also spherical. Here the first splitting is due again to the fact that the attaching map α is unstable, the middle map is the product of f_i with the inclusion of X, and the last map is the Hopf construction on the multiplication of $\Omega\Pi$. The map of Theorem 3.1 is obtained as the wedge of all these maps c_i together with the inclusion of the bottom two cells.

4. 2-primary v_1-periodic homotopy groups of G_2.

In this section, we prove Theorem 1.3. The proof is very similar to the calculations of [**12**], except for one delicate calculation of a differential. In particular, we use charts which are not, strictly speaking, Adams spectral sequence charts, but have the same form. Dots represent nonzero elements, vertical lines multiplication by 2, positively sloping lines multiplication by $\eta \in \pi_1(S^0)$, and negatively sloping lines differentials, or, more properly, boundary morphisms in exact sequences. At any rate, elements connected by negatively sloping lines do not yield homotopy classes. The group $v_1^{-1}\pi_i(-)$ appears in horizontal coordinate i, and we use the term d_r-differential for one going nontrivially from position (i, s) to $(i-1, s+r)$. We hope the reader will find the following restatement of Theorem 1.3 more illuminating.

THEOREM 4.1. *The 2-primary v_1-periodic homotopy groups of G_2 are given by the following chart, with*

$$d = \begin{cases} d_2 & \text{if } k \text{ is odd} \\ d_3 & \text{if } k \equiv 2 \bmod 4 \\ 0 & \text{if } k \equiv 0 \bmod 4 \end{cases}$$



We calculate $v_1^{-1}\pi_*(G_2)$ by using the exact sequences in $v_1^{-1}\pi_*(-)$ of the following three fibrations.

$$S^3 \longrightarrow G_2 \longrightarrow V_{7,2}$$

$$S^5 \longrightarrow V_{7,2} \longrightarrow S^6 \tag{4.2}$$

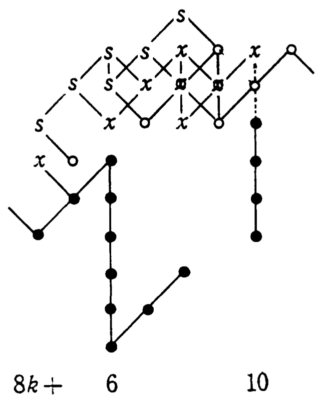
$$S^5 \longrightarrow \Omega S^6 \longrightarrow \Omega S^{11} \tag{4.3}$$

For the first, see [7]. The second could be thought of as

$$SO(6)/SO(5) \longrightarrow SO(7)/SO(5) \longrightarrow SO(7)/SO(6),$$

and the third yields the EHP sequence. In [**13**], the calculation of $v_1^{-1}\pi_*(S^{2n+1})$ as $v_1^{-1}J_*(\Sigma^{2n+1}P^{2n})$ was discussed, and this method was applied in [**12**]. These $J_*(-)$-charts consist of a *bo*-part and a *bsp*-part, which equals the *bo*-part shifted by $(-1, -2)$ units. There is a boundary morphism from the *bo*-part to the *bsp*-part, which is represented by a differential in the chart. We will use the above fibrations to combine the charts for S^3, two S^5's, and S^{11}, suitably positioned, to obtain a chart for G_2. Boundary morphisms in the exact sequences will be represented as differentials in the chart. For simplicity, our first combining will include only the *bo*-part of the relevant charts. The differentials within the *bsp*-parts are exactly the same as those within the *bo*-parts. We will then study the differentials from the *bo*-part to the *bsp*-part.

In the chart below, classes from S^3 are represented by s's, those from the S^5 which maps into $V_{7,2}$ by x's, those from the S^5 which maps into ΩS^6 by ○'s, and those from S^{11} by ●'s.



The differentials from ○ to x are due to the fact that the bottom cells of $V_{7,2}$ are attached by $\cdot 2$. The differentials from x to s are due to the fact that in G_2 the 5-cell is attached to the 3-cell by η. The differentials from ○ to s are a consequence of the above differentials and the fact that if one formed the chart for $V_{7,2}$, there would be an η-extension from the ○ in $8k+5$ to the x in $8k+6$ and a $\cdot 2$ from the ○ in $8k+7$ to the x in $8k+7$. These extensions are derivable by easy Toda bracket relations similar to those used in [**12**]. The lower of the two extensions in $8k+10$ indicated by dashed lines is implied by the argument involving [**12**, 2.2]; if α and β denote the classes involved, then $\delta(\langle\alpha, 2, \eta\rangle)=\beta\eta$, and this implies $2\alpha=\beta$. The upper of the two extensions follows from the lowest differential from $8k+9$ to $8k+8$ by [**22**, 2.1]; see [**5**, §3] for a similar application.

After these differentials and extensions are taken into account, and the *bsp*-part is inserted, the chart of Theorem 4.1 is obtained. It remains to justify the differentials in this chart. The d_1's from $8k+6$ to $8k+5$ are present because they are present in the chart for S^{11}; in effect, it is because $\nu(8k+6-11+1)=2$. The d_2 from $8k+10$ to $8k+9$ when k is odd involves classes in S^6, where the differential was established in [**12**, §4] by comparison of the chart for S^6 obtained from (4.3) with that obtained from the fibration

$$S^6 \longrightarrow \Omega S^7 \longrightarrow \Omega S^{13}. \tag{4.4}$$

It remains to determine whether there is a d_3 when k is even; this is where the argument is somewhat more delicate.

We will prove

PROPOSITION 4.5. *The composite*

$$v_1^{-1}\pi_{43}(\Omega S^{13}) \xrightarrow{P} v_1^{-1}\pi_{42}(S^6) \xrightarrow{\partial} v_1^{-1}\pi_{41}(S^5)$$

is zero, where P is the boundary map in the exact sequence of (4.4), *and ∂ is the boundary map in the exact sequence of* (4.2).

Because of the comparison of the two ways of computing $v_1^{-1}\pi_*(S^6)$ (from (4.3) and (4.4)), this proposition implies that the differential from 42 to 41 in Theorem 4.1 is zero. That it is zero in all $32k+10$ follows from this by application of the period-32 Adams map of the mod 64 Moore space ([**1**]). Equivalently, one may use Toda brackets to promulgate elements of order 64 with period 32. See [**23**] or [**12**, § 2] for a similar argument. That the differential is nonzero on $32k+26$ now follows since, by [**18**], the generator of the height-6 tower in $32k+25$ is obtained from the generator of the height-6 tower in $32k+10$ by composing with the generator ρ_{15} of the image of the stable J-homomorphism in the 15-stem, and since $32\rho_{15}=0$, the top element of the tower in $32k+25$ must be killed by a differential.

We complete the argument by proving Proposition 4.5. We will prove the following result at the end of this section.

PROPOSITION 4.6. *Let $(S^5)_K$ denote the K-theory localization as constructed in* [**19**]. *There is a map $\Omega^\infty(\Sigma^5 P^4\wedge J)\to(S^5)_K$ which induces an isomorphism in $\pi_i(-)$ for $i\geqq 10$.*

The composite in Proposition 4.5 may be thought of as the morphism in $v_1^{-1}\pi_{41}(-)$ induced by a certain map $\Omega^3S^{13}\to S^5$. By Proposition 4.6, the composition of this map followed by $S^5\to(S^5)_K$ lifts to a map

$$\Omega^3S^{13}\longrightarrow \Omega^\infty\Sigma^5P^4\wedge J. \tag{4.7}$$

For $i\geqq 10$, there is an isomorphism $v_1^{-1}\pi_i(S^5)\approx\pi_i(\Omega^\infty\Sigma^5P^4\wedge J)$, and so Proposition 4.5 will follow from showing that the morphism in $\pi_{41}(-)$ induced by (4.7) is 0 on a $Z/2^6$-summand which localizes isomorphically to $v_1^{-1}\pi_{41}(\Omega^3S^{13})$.

After adjointing (4.7), we obtain a question of *stable* homotopy theory. There is a stable splitting through dimension 42 ([**25**])

$$\Sigma^\infty\Omega^3S^{13}\cong\Sigma^\infty(S^{10}\vee K_2\vee K_3\vee K_4),$$

where

$$K_i=(S^{10i}\vee S^{10i+1})\cup_{\eta,2}e^{10i+2}. \tag{4.8}$$

The key part of the argument—the part that distinguishes the differential in 41 from that in 25—is the following result.

PROPOSITION 4.9. *If $G\in\pi_{41}(\Omega^3S^{13})$ passes to a generator of $v_1^{-1}\pi_{41}(\Omega^3S^{13})$, then the component of the composite below which passes through $\pi_{41}^s(S^{10})$ sends G to* 0.

$$\pi_{41}(\Omega^3S^{13})\longrightarrow\pi_{41}^s(\Omega^3S^{13})\approx\pi_{41}^s(S^{10})\oplus\bigoplus\pi_{41}^s(K_i)\longrightarrow\pi_{41}(\Sigma^5P^4\wedge J)$$

PROOF. The $Z/2^6$ in $\pi_{41}(\Omega^3S^{13})$ injects into $Z/2^7$ in $\pi_{41}^s(S^{10})$, and so the image of G is a multiple of 2. Since $\pi_{41}(\Sigma^5P^4\wedge J)\approx Z_2\oplus Z_2$, the image of G

through the $\pi^s_{41}(S^{10})$-component is 0. This calculation of $\pi_*(\Sigma^5 P^4\wedge J)$ is done, e. g., by the method of [**17**]. A chart appears at the end of this section. ■

Proposition 4.5 is a consequence of 4.9 and the following result, which implies that the components through the K_i-summands are 0.

PROPOSITION 4.10. *There is an element of $\pi_{41}(\Omega^3 S^{13})$ of Adams filtration* 12 *which passes to a generator of $v_1^{-1}\pi_{41}(\Omega^3 S^{13})$. For $2\leqq i\leqq 4$, $\pi^s_{41}(K_i)$ consists entirely of elements of Adams filtration less than* 12.

The first part of this proposition is read off from the chart for the unstable Adams spectral sequence of S^{13} in [**4**]. A chart for the stable Adams spectral sequence (ASS) of K_i is formed by combining charts of the ASS's for stable homotopy groups of spheres, suspended by $10i$, $10i+1$, and $10i+2$ dimensions, and inserting differentials to correspond to the attaching maps. In particular, all elements of $\pi_{41}(K_i)$ will be represented by those elements of the ASS for S^0 in stems $39-10i$, $40-10i$, and $41-10i$ which are not involved in differentials. Since the first positive-stem element of the ASS of S^0 in filtration $\geqq 12$ occurs in the 23-stem, the proposition is proved.

We note, for possible future generalization and application, that we have also determined the v_1-periodic homotopy groups of $V_{7,2}$.

THEOREM 4.11. *There is an isomorphism*

$$v_1^{-1}\pi_*(V_{7,2}) \approx v_1^{-1}\pi_*(G_2)\oplus v_1^{-1}\pi_{*-1}(S^3),$$

where $v_1^{-1}\pi_*(G_2)$ is as in Theorem 1.3 *or Theorem* 4.1, *and $v_1^{-1}\pi_*(S^3)$ is as described in* [**13**, 4.4, 4.1].

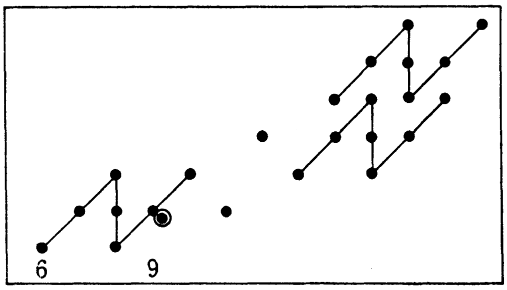
We close by proving Proposition 4.6. We recall that it was shown in [**19**] that the universal cover of the fiber G of the Snaith map ([**25**])

$$(QS^5)_K \longrightarrow (Q\Sigma^5 P_5)_K$$

serves as the localization $(S^5)_K$. Here $Q(-)=\Omega^\infty\Sigma^\infty(-)$, and the localizations $(QX)_K$ are as described in [**9**]. Our desired map will be obtained by constructing a commutative diagram of fiber sequences as below, taking the induced map of the indicated fibers, and lifting it to the universal cover of G, which is $(S^5)_K$.

$$\begin{array}{ccccc} \Omega^\infty(\Sigma^5 P^4\wedge J) & \longrightarrow & \Omega^\infty(\Sigma^5 P\wedge J) & \longrightarrow & \Omega^\infty(\Sigma^5 P_5\wedge J) \\ & & \downarrow & & \downarrow \\ G & \longrightarrow & (QS^5)_K & \longrightarrow & (Q\Sigma^5 P_5)_K \end{array} \tag{4.12}$$

By [**19**] and [**14**], $\pi_*((S^5)_K)$ agrees, for $*>5$, with $v_1^{-1}\pi_*(\Sigma^5 P^4\wedge J)$, which agrees with $\pi_*(\Sigma^5 P^4\wedge J)$ for $*\geqq 10$. Indeed, $\pi_*(\Sigma^5 P^4\wedge J)$ begins as in the chart below, and $v_1^{-1}\pi_*(\Sigma^5 P^4\wedge J)$ is the periodified version of this chart, without the circled dot.



The diagram (4.12) is obtained by first noting that, by [9], there is a map H from the bottom 2/3 row of (4.12) into

$$\Omega^{\infty}(\Sigma^{\infty}S^{5})_{K} \longrightarrow \Omega^{\infty}(\Sigma^{\infty}\Sigma^{5}P_{5})_{K} \tag{4.13}$$

which induces isomorphisms in $\pi_i(-)$ for $i>2$. Here of course we mean the underlying infinite loop space of the K-localization of the suspension spectrum, as defined in [8]. Then we note from [14] that there is a map F from the top 2/3 row of (4.12) to (4.13) which induces a surjection in $\pi_*(-)$ for $*\geqq 10$. This is because it is shown in [14] that the mapping telescope of v_1^4-maps of stunted real projective spaces is equivalent to the telescope obtained after applying $\wedge J$ to all spectra, and these are K_*-local. Elementary obstruction theory yields the lifting over H of the map F to obtain the diagram (4.12).

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