J. Math. Soc. Japan
Vol. 43, No. 3, 1991

On cutting pseudo-foliations along incompressible surfaces

By Hiromichi NAKAYAMA

(Received April 5, 1990)
(Revised August 1, 1990)

A compact orientable irreducible 3-manifold containing a two-sided incompressible surface is called a *Haken manifold* ([2]). The Haken manifold has a hierarchy. I.e. we obtain cubes with handles by cutting along incompressible surfaces several times. We would like to investigate the relationship between the foliations without compact leaves and the hierarchies of the Haken manifolds.

In this paper, we introduce the notion of pseudo-foliations of a compact 3-manifold M and would like to give a condition for a pseudo-foliation $\mathcal{F}$ of M, on which there exists an incompressible surface S not parallel to ∂M such that the "foliation" obtained by cutting $\mathcal{F}$ along S is again a pseudo-foliation.

The notion of pseudo-foliations is used in [3] for classifying some transversely affine foliations $\mathcal{F}$ of some surface bundles M over S^1 with fiber Σ, where the main theorem of this paper is used for cutting $\mathcal{F}$ along incompressible surfaces into pseudo-foliations of cubes with handles. More precisely, $\mathcal{F}$ is first cut along some fiber into a pseudo-foliation of $\Sigma\times[0, 1]$ so that $\mathcal{F}|(\Sigma\times\{0\})$ is equal to $\mathcal{F}|(\Sigma\times\{1\})$ and $\mathcal{F}|(\Sigma\times[0, 1])$ has a transverse invariant measure by the conditions of M and $\mathcal{F}$. Next we cut $\mathcal{F}|(\Sigma\times[0, 1])$ along disjoint annuli into pseudo-foliations of cubes with handles. Since these pseudo-foliations with invariant measures can be classified, we obtain the classification of $\mathcal{F}$.

1. Definitions and results.

Let M be a compact orientable 3-manifold. A *pseudo-foliation* $\mathcal{F}$ of M is a transversely orientable C^2 foliation transverse to ∂M except at finitely many points $p_i\in\partial M$ $(i=1, 2, 3, \cdots, n)$ where p_i is a saddle singularity of $\mathcal{F}|\partial M$ and $\mathcal{F}|\partial M$ has no leaves connecting distinct saddle singularities. A simple closed curve on ∂M consisting of saddle singularities and leaves of $\mathcal{F}|\partial M$ is called a *cycle* of $\mathcal{F}|\partial M$ (A closed leaf is a cycle).

Let Σ be a compact connected orientable 2-manifold with boundary, and let ϕ, an embedding of Σ in M, where $\phi(\Sigma)\cap\partial M=\phi(\partial\Sigma)$. We put

$E(\phi)=\{\psi: \Sigma\to M \mid \psi$ is an embedding isotopic to ϕ, $\psi(\Sigma)\cap\partial M=\psi(\partial\Sigma)$ and $\psi(\partial\Sigma)$ contains no saddle singularities of $\mathcal{F}\mid\partial M\}$,

$E'(\phi)=\{\psi\in E(\phi)\mid\psi(\partial\Sigma)$ is tangent to $\mathcal{F}\mid\partial M$ at the least number of points among the elements of $E(\phi)\}$.

LEMMA 1 (Roussarie [6], [7]). *Let $\mathcal{F}$ be a pseudo-foliation of M without cycles on its boundary and let ψ be an element of $E'(\phi)$ for an embedding $\phi: \Sigma\to M$. Denote by q_i ($i=1, 2, 3, \cdots, m$) the points of $\psi(\partial\Sigma)$ where $\psi(\partial\Sigma)$ is tangent to $\mathcal{F}\mid\partial M$. Then there exist continuous maps $F_i: D^1\times[0, 1]\to\partial M$ ($i=1, 2, 3, \cdots, m$) satisfying the following conditions* (*Fig.* 1):

1) $F_i(D^1\times\{0\})=\{q_i\}$.

2) $F_i\mid(D^1\times\{t\})$ ($t\in(0, 1]$) *maps D^1 into a leaf of $\mathcal{F}\mid\partial M$ injectively.*

3) $F_i(\partial D^1\times(0, 1])$ *is contained in $\psi(\partial\Sigma)$ and transverse to $\mathcal{F}\mid\partial M$, $F_i\mid(\partial D^1\times(0, 1])$ is injective.*

4) $F_i(D^1\times\{1\})$ *consists of the unique saddle singularity r_i of $\mathcal{F}\mid\partial M$ and two arcs contained in leaves of $\mathcal{F}\mid\partial M$.*

5) $F_i(\partial D^1\times[0, 1))$*'s are mutually disjoint.*

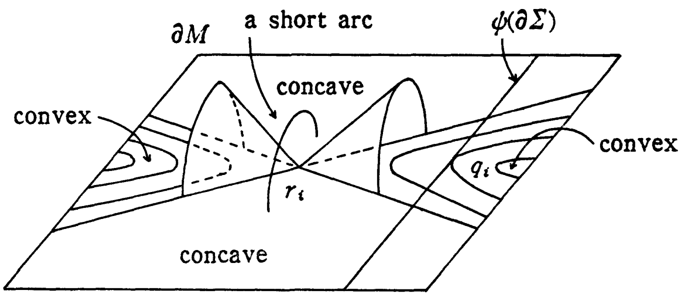

Fig. 1.

We define a map $h_\psi: \{q_1, q_2, q_3, \cdots, q_m\}\to\{$the saddle singularities of $\mathcal{F}\mid\partial M\}$ by $h_\psi(q_i)=r_i$ for each $\psi\in E'(\phi)$. The four separatrices of $\mathcal{F}\mid\partial M$ passing through the saddle singularity r_i divide a small neighborhood of r_i into four *quadrants*. In these four quadrants, there are two quadrants connected by a short arc contained in a leaf of $\mathcal{F}$ (Fig. 1), which are called *concave*. The two other quadrants are called *convex*. If $F_i(D^1\times\{t\})$ approaches r_i from the side of a convex (resp. concave) quadrant as $t\to1$, then the tangent point q_i is called *convex* (resp. *concave*).

To state our main result, we need one more definition. Let N be a bundle over S^1 whose fiber is a compact connected orientable surface Σ with boundary. For the bundle foliation $\mathcal{G}$ of N whose leaves are the fibers of N, a foliation obtained by turbulizing $\mathcal{G}$ along ∂N toward the positive direction of S^1 is called a *bundle component*. The compact leaves of the bundle component are connected components of ∂N, and all the leaves contained in $\mathrm{int}\, N$ are diffeomorphic to $\mathrm{int}\, \Sigma$.

THE MAIN THEOREM. *Let $\mathcal{F}$ be a pseudo-foliation of a compact connected orientable 3-manifold M. Let Σ be a compact connected orientable 2-manifold whose Euler number k is smaller than 2, and ϕ, an embedding of Σ in M. Suppose that the pseudo-foliation $\mathcal{F}$ and the embedding $\phi: \Sigma \to M$ satisfy the following conditions:*

1) *$\phi(\Sigma)$ is an incompressible surface.*

2) *$\mathcal{F}$ does not contain a bundle component such that the Euler number of the fiber is greater than or equal to k.*

3) *If Σ has a boundary, then $\mathcal{F}|\partial M$ contains no cycles and there is an embedding $\psi \in E'(\phi)$ such that the inverse image of h_ψ for each saddle singularity of $\mathcal{F}|\partial M$ contains at most one convex tangent point.*

Then the following condition either 4) or 5) holds:

4) *There exists an embedding ϕ' isotopic to ϕ such that $\mathcal{F}|(\overline{M-\phi'(\Sigma)})$ is a pseudo-foliation. Here $\overline{M-\phi'(\Sigma)}$ is the completion of $M-\phi'(\Sigma)$.*

5) *$\mathcal{F}$ is a foliation of a Σ-bundle over S^1 without boundary such that both $\phi(\Sigma)$ and the leaves of $\mathcal{F}$ are isotopic to the fiber of M.*

When $\partial M = \emptyset$ or $\phi(\partial\Sigma)$ is either tangent or transverse to $\mathcal{F}|\partial M$, the results similar to the main theorem have already been known (Thurston's remark [**9**] and Levitt's comment [**5**]). However its detailed proof has not been seen yet. The proof of the main theorem includes that proof (See Section 2). In Section 3, we prove the main theorem ($\partial\Sigma \neq \emptyset$) by making use of Theorem 1 and its Corollary shown in Section 2.

The author wishes to thank Professor T. Tsuboi for his helpful suggestions and encouragement.

2. The isotopy of incompressible surfaces.

In this section, we show the following Theorem 1 and its Corollary. This Corollary will be used to show the main theorem in Section 3.

THEOREM 1. *Let Σ, M, $\mathcal{F}$ be as in the main theorem. Suppose that the pseudo-foliation $\mathcal{F}$ and the embedding $\phi: \Sigma \to M$ satisfy the conditions 1), 2) of the main theorem and the following condition 3').*

3') *If Σ has a boundary, then $\phi(\partial\Sigma)$ is transverse to $\mathcal{F}|\partial M$ except at saddle singularities of $\mathcal{F}|\partial M$ and $\phi(\partial\Sigma)$ crosses these singularities (i.e. the saddle singularity contained in $\phi(\partial\Sigma)$ has two separatrices on each side of $\phi(\partial\Sigma)$ (Fig. 2)).*

Then the conclusion of the main theorem holds.

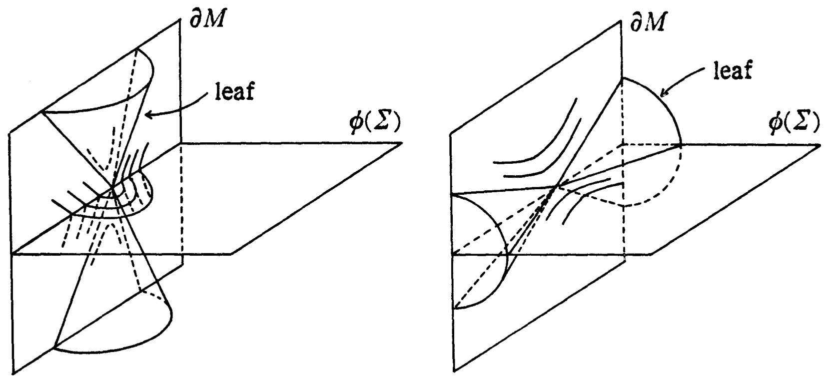

Fig. 2.

In order to prove Theorem 1, we introduce a notion. Let $G(\phi)$ denote the set of embeddings $\psi: \Sigma \to M$ satisfying the following conditions:

- ψ is isotopic to ϕ with boundary fixed.
- $\psi|(\mathrm{int}\,\Sigma)$ is in general position with respect to $\mathcal{F}$.
  (I.e. $\psi(\mathrm{int}\,\Sigma)$ is transverse to $\mathcal{F}$ except at finitely many points and the points of $\psi(\mathrm{int}\,\Sigma)$ tangent to $\mathcal{F}$ are of Morse type with index 0 or 1. The tangent point whose index is equal to 0 (resp. 1) is called a *center* (resp. a *saddle singularity*).)
- Each singularity of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ is not contained in a leaf of $\mathcal{F}$ which contains the other singularities of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ or the saddle singularities of $\mathcal{F}|\partial M$.
- Each leaf of $\mathcal{F}$ containing singularities of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ has no holonomy ([**1**]).

By [**1**], [**4**], [**6**] and [**7**], $G(\phi)$ is not empty. Put $G'(\phi)=\{\psi\in G(\phi) \mid$ the number of singularities of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ is minimum among the elements of $G(\phi)\}$. Now Theorem 1 obviously follows from the following Theorem 2.

THEOREM 2. *Assume that M, $\mathcal{F}$, ϕ satisfy the conditions* 1), 2) *of the main theorem and* 3′) *of Theorem* 1. *If $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ contains a center for an element ψ of $G'(\phi)$, then $\mathcal{F}$ satisfies the condition* 5) *of the main theorem.*

For the proof of Theorem 2, we need several lemmas.

A pair (L, E) satisfying the following conditions is called a *patch*:

1) L is a compact connected orientable surface with boundary such that L is contained in a leaf of $\mathcal{F}$, $\partial L\subset\psi(\Sigma)$, L has no holonomy and ∂L contains no singularities of $\mathcal{F}|\psi(\Sigma)$.

2) E is a surface contained in $\psi(\Sigma)$ which is homeomorphic to L and satisfies that $E\cap L=\partial E=\partial L$.

3) $(\psi(\Sigma)-E)\cup L$ is isotopic to $\psi(\Sigma)$ with $\psi(\Sigma)-E$ fixed.

4) $\mathcal{F}|E$ has a unique center, and a transverse orientation of $\mathcal{F}|E$ points outwards on ∂E.

5) $\operatorname{int} L \cap \psi(\Sigma)=\emptyset$.

6) Each connected component of $\overline{\psi(\Sigma)-E}$ is not a disk.

The number of saddle singularities of $\mathcal{F}|E$ is called the *rank* of the patch (L, E).

If $\mathcal{F}|\psi(\Sigma)$ has a center q for $\psi \in G'(\phi)$, then there exists a patch of rank 0 near q because Σ is not a 2-sphere. By Lemmas 2$\sim$6, we will show the existence of patches of higher rank if $\psi(\Sigma)$ is isotopic to no interior compact leaves ($\partial\Sigma=\emptyset$), where we assume in Lemmas 2$\sim$6 that M, $\mathcal{F}$, ϕ satisfy 1), 2) of the main theorem and 3′) of Theorem 1 and $\psi \in G'(\phi)$.

REMARK. The following proof of Theorem 2 is similar to Roussarie's proof ([7]) which shows that the similar conclusion is true when Σ is an annulus.

LEMMA 2. *Let (L, E) be a patch. Let S be a surface homeomorphic to L, and suppose that $F: S\times[0, 1)\to M$ is an immersion satisfying the following conditions:*

1) *$F|(S\times\{t\})$ is an embedding in a leaf of $\mathcal{F}$ and $F(\partial S\times\{t\})\subset\psi(\Sigma)$ for $t\in [0, 1)$.*

2) *There is an embedding $f: \partial S\times[0, 1]\to\psi(\Sigma)$ which is the extension of $F|(\partial S\times[0, 1))$, and $f(\partial S\times\{1\})$ is a leaf of $\mathcal{F}|\psi(\Sigma)$ which contains no singularities of $\mathcal{F}|\psi(\Sigma)$.*

3) *There exists a Riemannian metric of M such that the trajectory $t\to F(x, t)$ $(0\leqq t<1)$ is normal to $\mathcal{F}$ for each $x\in S$.*

4) *$F(S\times\{0\})=L$ (In particular, $F(\operatorname{int} S\times\{0\})\cap\psi(\Sigma)=\emptyset$).*

Then F can be extended to an immersion of $S\times[0, 1]$ in M and $F|(S\times\{1\})$ for this extension is an embedding in a leaf of $\mathcal{F}$.

PROOF. We define $F_t: S\to M$ $(0\leqq t<1)$ by $F_t(x)=F(x, t)$ for each $x\in S$. By Thurston's lemma ([8]) and Roussarie's lemma 11 ([7]), there exists $\lim_{t\to 1}F_t(x)$ for each $x\in S$ and the map $g: S\to M$ defined by $g(x)=\lim_{t\to 1}F_t(x)$ is an immersion in a leaf of $\mathcal{F}$. Hence we need only prove that g is an embedding.

Assume that g is not an embedding. Then there are points $x\in\operatorname{int} S$ and $y\in\partial S$ satisfying $g(x)=g(y)$ because g is an immersion in a leaf of $\mathcal{F}$ and $g|\partial S$ is an embedding. $F_0(x)$, $F_0(y)$ and $g(x)$ are contained in the same orbit of the normal vector field of $\mathcal{F}$. If $F_0(x)$ is located between $g(x)$ and $F_0(y)$ in this orbit, then $F_0(x)$ is contained in $\psi(\Sigma)$. This contradicts the assumption that $F(\operatorname{int} S\times\{0\})\cap\psi(\Sigma)=\emptyset$. Hence the orbit starting from $F_0(x)$ passes through $F_0(y)$ and reaches $g(x)$. That is, there is $s_0(>0)$ such that $F_{s_0}(x)=F_0(y)$.

There is a monotone increasing continuous function $\mu: [s_0, 1]\to[0, 1]$ such that $\mu(s_0)=0$ and $F_t(x)=F_{\mu(t)}(y)$ for $t\geqq s_0$. Then μ has no fixed points, and

$\mu(1)=1$ because $z=\lim_{t\to1}F_t(x)=\lim_{t\to1}F_{\mu(t)}(y)\in F(\partial S, \mu(1))$. Let s_i ($i\geqq1$) denote $\mu^{-i}(s_0)$. Then $\lim_{i\to\infty}s_i=1$ and $F(S\times\{s_i\})\cap F(S\times\{s_{i+1}\})\neq\emptyset$. Since $F(\text{int}\, S\times\{0\})\cap\phi(\Sigma)=\emptyset$, $F(S\times\{s_i\})\subset F(S\times\{s_{i+1}\})$ ([7]).

For each s_i, there exists a minimal element of $\{t>s_i \mid F(S\times\{t\})\supset F(S\times\{s_i\})\}$, denoted by s_i'. Then $F|(S\times[s_i, s_i'))$ is injective, $F(S\times\{s_i\})\subset F(S\times\{s_i'\})$, and $\lim_{i\to\infty}s_i'=1$.

Next we show that each connected component of $\overline{F(S\times\{s_i'\})-F(S\times\{s_i\})}$ is not a disk. If so, there exists a connected component of $F(\partial S\times\{s_i\})$ which is contractible in M, hence a connected component γ of ∂E is contractible in M. Since $\phi(\Sigma)$ is incompressible, γ bounds a 2-disk contained in $\phi(\Sigma)$. By 6) of the definition of the patch, E is contained in this disk, and again by 6) E coincides with the disk. Since S is also a disk, $\overline{F(S\times\{s_i'\})-F(S\times\{s_i\})}$ is an annulus. Thus each connected component of $\overline{F(S\times\{s_i'\})-F(S\times\{s_i\})}$ is not a disk.

By the Mayer-Vietoris sequence, $\chi(\overline{F(S\times\{s_i'\})-F(S\times\{s_i\})})=0$. Since each connected component of $\overline{F(S\times\{s_i'\})-F(S\times\{s_i\})}$ is not a disk, $\overline{F(S\times\{s_i'\})-F(S\times\{s_i\})}$ is a disjoint union of annuli. Thus $F(S\times[s_i, s_i'))$ is a surface bundle over S^1 whose fiber is homeomorphic to S.

By Roussarie's proof of Lemma 11 of [7], $\overline{F(S\times[0, 1))}$ is a bundle component whose fiber is homeomorphic to S. Since $\chi(\phi(\Sigma))-\chi(E)=\chi(\overline{\phi(\Sigma)-E})\leqq0$, $\chi(S)\geqq\chi(\phi(\Sigma))\geqq k$.

This contradicts the condition 2) of the main theorem. Thus g is an embedding. ∎

REMARK. The condition that $F(\text{int}\, S\times\{0\})\cap\phi(\Sigma)=\emptyset$ is essential for Lemma 2. If $F(\text{int}\, S\times\{0\})\cap\phi(\Sigma)\neq\emptyset$, then $F(S\times\{s_{i+1}\})$ does not always contain $F(S\times\{s_i\})$.

By Lemma 2, the following Lemmas 3 and 4 are proved in a way similar to Roussarie's proof of his Lemmas 12 and 13 ([7]).

LEMMA 3. *Let (L, E) be a patch and let S be a surface homeomorphic to L. Then there exists an immersion $F: S\times[0, 1)\to M$ satisfying the following conditions:*

1) $F|(S\times\{t\})$ $(t\in[0, 1))$ *is an embedding in a leaf of* $\mathcal{F}$.

2) $F|(\partial S\times[0, 1))$ *is an embedding in* $\overline{\phi(\Sigma)-E}$.

3) $F(S\times\{0\})=L$.

4) *For a small* $\varepsilon>0$, $F(\partial S\times(0, \varepsilon))\cap E=\emptyset$.

5) *F is a maximal element among all the immersions satisfying the above conditions* 1), 2), 3) *and* 4) *with respect to the inclusion relation.*

6) *There exists* $\lim_{t\to1}F(x, t)$ *for each* $x\in S$ *and, by defining* $F: S\times\{1\}\to M$ *by* $F(x, 1)=\lim_{t\to1}F(x, t)$, $F|(\text{int}\, S\times\{1\})$ *is an embedding in a leaf of* $\mathcal{F}$.

Furthermore, the immersion F is one of the following eight types.

(1) $F(\partial S\times\{1\})$ *contains no singularities of* $\mathcal{F}|\phi(\Sigma)$.

1) $F(S\times\{1\})$ *has holonomy* (*Type* A).

2) $F(\partial S\times\{1\})\cap F(\partial S\times\{0\})\neq\emptyset$ (*Type* B).

3) $F(\partial S\times\{1\})\cap\phi(\partial\Sigma)\neq\emptyset$ (*Type* C).

(2) $F(\partial S\times\{1\})$ *contains a center of* $\mathcal{F}|\phi(\Sigma)$ (*Type* D).

(3) $F(\partial S\times\{1\})$ *contains a saddle singularity* p *of* $\mathcal{F}|\phi(\Sigma)$.

1) $F(\partial S\times\{t\})$ *approaches* p *from a unique quadrant as* $t\to 1$.

i) $F(S\times\{t\})$ *contains no bridges of* p *for sufficiently large* $t\in[0, 1)$ (*Type* E), *where the bridge of* p *is a short arc joining two opposite quadrants of* p *and contained in a leaf of* $\mathcal{F}$ ([8]) (*Fig.* 3).

ii) $F(S\times\{t\})$ *contains a bridge of* p *for sufficiently large* $t\in[0, 1)$ (*Type* F).

2) $F(\partial S\times\{t\})$ *approaches* p *from two opposite quadrants as* $t\to 1$,

i) $F(S\times\{t\})$ *contains no bridges of* p *for sufficiently large* $t\in[0, 1)$ (*Type* G).

ii) $F(S\times\{t\})$ *contains a bridge of* p *for sufficiently large* $t\in[0, 1)$ (*Type* H).

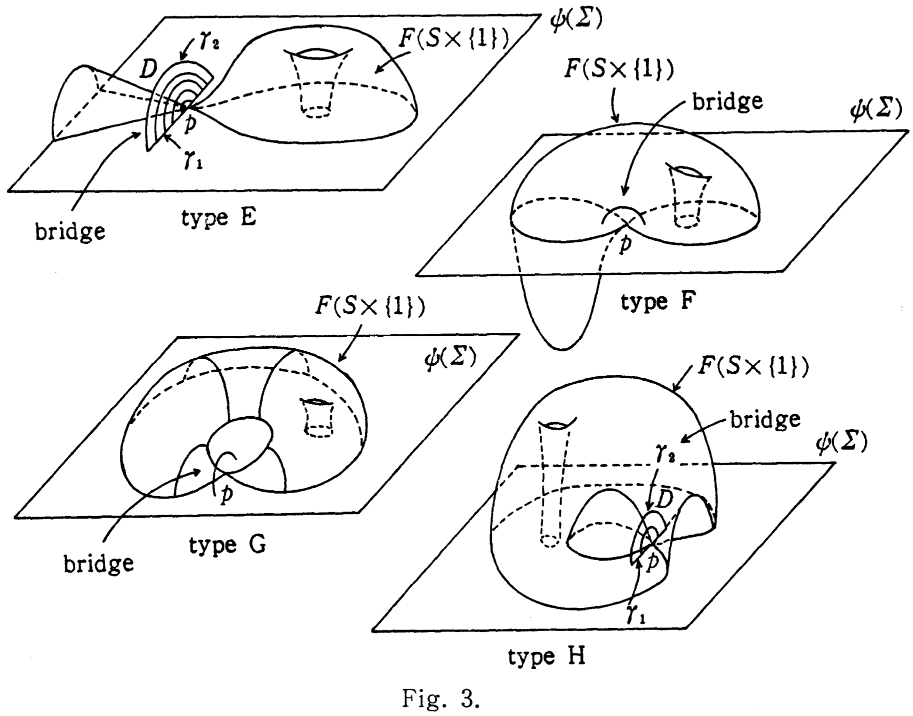

Fig. 3.

The immersion F: $S\times[0, 1]\to M$ constructed in Lemma 3 is called a *maximal coordinate* of (L, E).

LEMMA 4. *Let (L, E) be a patch of rank k $(k \geqq 0)$ and let F be a maximal coordinate of (L, E). Then there exists an embedding $\psi' \in G'(\phi)$, a patch (L', E') of rank k with respect to ψ' and a maximal coordinate F' of (L', E') satisfying $F'(S \times \{t\}) \cap F'(S \times \{t'\}) = \emptyset$ for $t, t' \in [0, 1]$ $(t \neq t')$. (In particular, $F' | (S \times [0, 1))$ is an embedding.)*

LEMMA 5. *Let (L, E) be a patch of rank k and let F be a maximal coordinate of (L, E) such that $F(S \times \{t\}) \cap F(S \times \{t'\}) = \emptyset$ for $t, t' \in [0, 1]$ $(t \neq t')$. Then the maximal coordinate F is not of types A, B, C, D, E, F and H. If the maximal coordinate F is of type G and $\psi(\Sigma)$ is isotopic to no interior compact leaves of $\mathcal{F}$, then there exists an embedding $\psi' \in G'(\phi)$ which has a patch of rank $k+1$.*

PROOF. We will consider the maximal coordinates F of type A$\sim$H, respectively. By the definition of the patch and the situation of $\mathcal{F}$ in the neighborhood of $\psi(\partial\Sigma)$ (Fig. 2), the maximal coordinate F is not of type B, C and D.
(Type E) Let F be a maximal coordinate of type E. There exists a small disk D satisfying the following conditions (Fig. 3):

- ∂D consists of two arcs γ_1, γ_2 such that $p \in \gamma_1$, $D \cap \psi(\Sigma) = \gamma_1$ and γ_2 is contained in a leaf of $\mathcal{F}$.
- D is transverse to $\mathcal{F}$, and $\mathcal{F}|D$ has a unique singularity p, which is a half of the center.

Since we assumed that each leaf of $\mathcal{F}$ containing singularities of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ has no holonomy, $F(S \times \{1\})$ has no holonomy. Hence there is a sufficiently small number $\varepsilon(>0)$ such that the embedding $F: S \times [0, 1] \to M$ is extended to $S \times [0, 1+\varepsilon]$ as follows:

- $F(\partial S \times \{t\}) \subset D \cup \psi(\Sigma)$ $(0 \leqq t \leqq 1+\varepsilon)$.
- Each $F(S \times \{t\})$ $(0 \leqq t \leqq 1+\varepsilon)$ is contained in a leaf of $\mathcal{F}$.
- $F|(S \times [0, 1+\varepsilon])$ is injective.

We choose a closed collar neighborhood U of ∂S in S and a sufficiently small number $\delta(>0)$ so that $V \cap (\psi(\Sigma) \cup D) = F(\partial S \times [0, 1+\varepsilon])$ where $V = F((U \times (\delta, 1+\varepsilon]) \cup (S \times [0, \delta]))$. Then there exists an ambient isotopy which takes each $F(S \times \{t\}) - V$ $(t \in [\delta, 1+\varepsilon])$ to $F(S \times \{s\}) - V$ for some $s \in [1+\varepsilon/2, 1+\varepsilon]$ and whose support is contained in $F(\mathrm{int}\, S \times (0, 1+\varepsilon))$. We change ψ by this isotopy. Then $\psi(\Sigma)$ does not intersect $F(\mathrm{int}\, S \times [0, 1+\varepsilon/2])$ and $\mathcal{F}|\psi(\Sigma)$ does not change.

Let $\psi': \Sigma \to M$ be an embedding satisfying the following conditions:

- $\psi' \in G(\phi)$ and ψ' is isotopic to ψ with $\Sigma - \psi^{-1}(E \cup F(\partial S \times [0, 1+\varepsilon/2]))$ fixed.
- E' is contained in $F(S \times [1, 1+\varepsilon/2])$ and $p \in E'$, where $\overline{\psi'(\Sigma) \cap F(\mathrm{int}\, S \times [0, 1+\varepsilon/2])}$ is denoted by E'. The connected component of $\partial E'$ containing p is tangent to $\mathcal{F}$ at p and another point q $(\in F(S \times \{1+\varepsilon/2\}))$, and all the singularities of $\mathcal{F}|(\mathrm{int}\, E')$ are saddle

singularities.

- $\mathcal{F}|E'$ is illustrated below. (Since E' is normalized as in Fig. 4, this figure can be regarded as a contour map and ϕ' is constructed according to this map.)

Since p and q are not singularities of $\mathcal{F}|\phi'(\Sigma)$, the number of singularities of $\mathcal{F}|\phi'(\Sigma)$ is less than that of $\mathcal{F}|\phi(\Sigma)$. This contradicts the minimality of the number of singularities of $\mathcal{F}|\phi(\mathrm{int}\,\Sigma)$. Thus the maximal coordinate of type E does not exist.

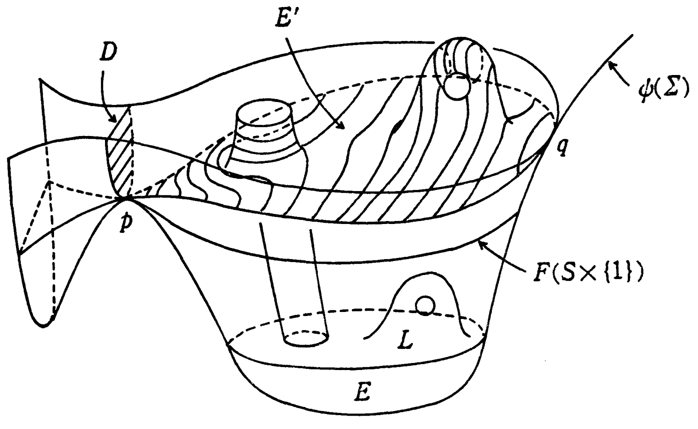

Fig. 4.

(Type G) Let F be a maximal coordinate of type G. There are surfaces L' and E' satisfying the following conditions:

- (L', E') satisfies the conditions 1), 2), 3), 4) of the patch.
- L' is sufficiently close to $F(S\times\{1\})$, and E' contains $E\cup F(\partial S\times[0, 1])$ in its interior.
- The number of saddle singularities of $\mathcal{F}|E'$ is one more than that of $\mathcal{F}|E$.

Changing ϕ by an isotopy as in the above proof of the case of type E, we can assume that $\mathrm{int}\,L'\cap\phi(\Sigma)=\emptyset$.

If each connected component of $\overline{\phi(\Sigma)-E'}$ is not a disk, then (L', E') is a patch of rank $k+1$. Hence we have only to show that $\phi(\Sigma)$ is isotopic to an interior compact leaf of $\mathcal{F}$ if some connected component of $\overline{\phi(\Sigma)-E'}$ is a disk.

Suppose that some connected component D' of $\overline{\phi(\Sigma)-E'}$ is a disk. If $\mathcal{F}|D'$ has a saddle singularity, then there exists a good couple of tangent points defined in [**6**], where it is shown that some singularities of $\mathcal{F}|D'$ can be canceled. Since this contradicts the minimality of the number of singularities of $\mathcal{F}|\phi(\mathrm{int}\,\Sigma)$, $\mathcal{F}|D'$ consists of one center q and leaves homeomorphic to S^1. By Novikov's theorem ([4]), we construct a continuous map $F': D^2\times[0, 1]\to M$ satisfying the following conditions:

- $F'|(D^2\times\{t\})$ $(t\in(0, 1])$ is an embedding in a leaf of $\mathcal{F}$, $F'|(D^2\times(0, 1])$ is an immersion and $F'(\partial D^2\times\{t\})\subset D'$.

• $F'(D^2\times\{0\})=\{q\}$ and $F'(\partial D^2\times\{1\})=\partial D'$.

Since L' is not contained in $F'(D^2\times\{1\})$, $L'\cap F'(\mathrm{int}\,D^2\times\{1\})=\emptyset$.

If $\partial L'$ is not connected, then ϕ is isotopic to an embedding $\phi'\in G(\phi)$ with $\Sigma-\phi^{-1}(D'\cup E')$ fixed such that $\mathcal{F}|\phi'(\phi^{-1}(D'\cup E'))$ has a unique center (Fig. 5). The number of singularities of $\mathcal{F}|\phi'(\Sigma)$ is less than that of $\mathcal{F}|\phi(\Sigma)$ because $\mathcal{F}|\phi(\Sigma)$ has two centers in $E'\cup D'$. This contradicts the minimality of the number of singularities of $\mathcal{F}|\phi(\mathrm{int}\,\Sigma)$. Therefore $\partial L'$ is connected.

Thus $\phi(\Sigma)$ $(=D'\cup E')$ is isotopic to an interior compact leaf $L'\cup F'(D^2\times\{1\})$.

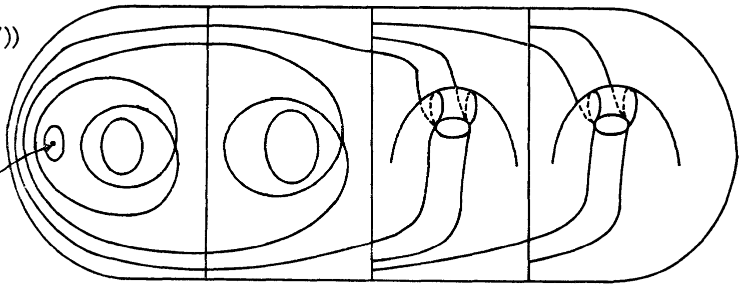

Fig. 5.

(Type H) Next we consider the case where the maximal coordinate F is of type H. Let D, γ_1, γ_2 be defined in the case of type E where D is contained in $F(S\times[0, 1])$ (Fig. 3). $F(S\times\{t\})$ containing γ_2 is denoted by $F(S\times\{\tau\})$ and $(F|S\times\{\tau\})^{-1}(\gamma_2)$ is denoted by l. Since $F(l\times\{0\})$ is an arc of $F(S\times\{0\})$, there exists an arc γ on E isotopic to $F(l\times\{0\})$ with boundary fixed. Let $c=\gamma_1\cup F(\partial l\times[0, \tau])\cup\gamma$. Since c is a simple closed curve on $\phi(\Sigma)$ null-homotopic in M, c bounds a disk B in $\phi(\Sigma)$. Two separatrices of $\mathcal{F}|\phi(\Sigma)$ passing through p and contained in B make a null-homotopic simple closed curve, which is contained in $F(\partial S\times\{1\})$. Hence there exists a good couple of tangent points. Thus the maximal coordinate of type H cannot exist.

(Type F) Next we consider the case of type F. Let q denote $F^{-1}(p)$ and the foliation of $S\times[0, 1]$ whose leaves are $S\times\{t\}$ $(t\in[0, 1])$ is denoted by $\mathcal{G}$. We denote by R the connected component of $F^{-1}(\phi(\Sigma))$ containing q and let $W=\overline{R-\partial S\times[0, 1]}$ (Fig. 6). Since $\mathcal{G}|W$ has a center, there exists a patch of rank 0 with respect to $\mathcal{G}$ and W, denoted by (L_0, E_0). Let F' be a maximal coordinate of (L_0, E_0) with respect to $\mathcal{G}$ and W. By the above consideration, the maximal coordinate F' is not of types B, D, E and H. Since $\mathcal{G}$ has no holonomy, F' is not of type A. If F' is of type C (i.e. $F'(D^2\times\{1\})\subset S\times\{1\}$), then F' is also a maximal coordinate of type E. Thus the maximal coordinate F' is of type F or G.

Suppose that F' is of type G. We change ϕ by an isotopy so that $F'(\mathrm{int}\,D^2\times(0, 1])\cap F^{-1}(\phi(\Sigma))=\emptyset$. There is a patch of rank 1 with respect to

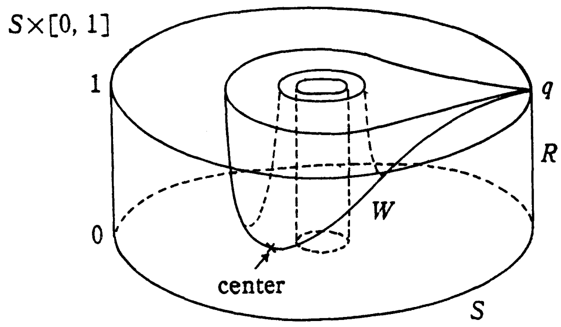

Fig. 6.

$\mathcal{G}$ and W. By the above argument, its maximal coordinate is also of type F or G. Since the number of singularities of $\mathcal{G}|W$ is finite, there exists a maximal coordinate of type F contained in $\operatorname{int} S\times(0, 1)$.

For the maximal coordinate of type F contained in $\operatorname{int} S\times(0, 1)$, there exists another maximal coordinate of type F in its interior by the above consideration. Therefore $\mathcal{G}|F^{-1}(\phi(\Sigma))$ contains infinitely many singularities. Since this contradicts the assumption, the maximal coordinate F of type F does not exist.

(Type A) Finally, we show that the maximal coordinate F is not of type A. Since $F(S\times\{1\})$ has holonomy in this case, we cannot remove the intersection of $F(\operatorname{int} S\times(0, 1])$ and $\phi(\Sigma)$ in the same way as in the case of type E. For this reason, we consider the case of type A at the end of the proof of Lemma 5.

First we show that there exists a patch (L', E') such that L' has holonomy (that is, $\operatorname{int} L'\cap\phi(\Sigma)=\emptyset$). We use the induction on the number m of the connected components of $F(\operatorname{int} S\times(0, 1])\cap\phi(\Sigma)$. Suppose that $m\neq 0$. Let R denote one of the connected components of $(\operatorname{int} S\times(0, 1])\cap F^{-1}(\phi(\Sigma))$ and let $\mathcal{G}=\{S\times\{t\}\,;\, t\in[0, 1]\}$. Since $\mathcal{G}|R$ has at least one center, there exists a patch of rank 0. The maximal coordinate of this patch with respect to $\mathcal{G}$ and R is of type C or G. If this is of type G, then there exists a patch of rank 1 whose maximal coordinate is of type C or G again by changing ϕ by an isotopy. Thus there is a maximal coordinate $F'\colon S'\times[0, 1]\to S\times[0, 1]$ of type C with respect to $\mathcal{G}$ and R, where $F'(S'\times\{1\})$ is contained in $S\times\{1\}$. If $F(F'(S'\times\{1\}))$ has holonomy, then there exists a patch with holonomy by the induction because the number of the connected components of $F(F'(\operatorname{int} S^1\times(0, 1]))\cap\phi(\Sigma)$ is less than m. If $F(F'(S'\times\{1\}))$ has no holonomy, then we remove $F(R)$ by changing ϕ by an isotopy as in the proof of the case of type E. Since the number of the connected components of $F(\operatorname{int} S\times(0, 1])\cap\phi(\Sigma)$ decreases, there exists a patch with holonomy by the induction. Thus there is a patch (L', E') such that L' has holonomy.

Let n denote the number of boundaries of L' and let g denote the genus of L'. Then there are arcs $l_1, l_2, l_3, \cdots, l_{n+g-2}$ such that the manifolds obtained by cutting L' along these arcs are annuli $E_1, E_2, E_3, \cdots, E_{n-1}$ and one-punctured tori $H_1, H_2, H_3, \cdots, H_g$. Let γ_i be a simple closed curve on E_i which generates $\pi_1(E_i)$ $(1 \leqq i \leqq n-1)$ and let α_j and β_j be simple closed curves on H_j which generate $\pi_1(H_j)$ and intersect at one point $(1 \leqq j \leqq g)$ (Fig. 7). Since L' has holonomy, one of the simple closed curves $\gamma_i, \alpha_j, \beta_j$ has holonomy.

First assume that γ_1 has holonomy. There exists a sufficiently small block B_i containing l_i (Fig. 7). Here a *block* B_i means a 3-cell satisfying the following conditions:

- $B_i \cap L'$ is a rectangle containing l_i, denoted by A_i, and $\phi(\Sigma) \cap B_i$ consists of two disjoint disks contained in ∂B_i.
- $\mathcal{F} | (\partial B_i - \phi(\Sigma) - L')$ is a singular foliation with the unique saddle singularity q_i.
- B_i is on the opposite side of E' with respect to L'.

Denote by N_i $(1 \leqq i \leqq n+g-1)$ the closure of the connected component of $L' - \bigcup_{i=1}^{n+g-2} A_i$ lying between A_{i-1} and A_i. Since we may change the height of B_i, we can assume that there is a saturated product neighborhood $N_i \times [0, 1]$ $(2 \leqq i \leqq n+g-1)$ of $\mathcal{F}$ satisfying the following conditions if N_i has no holonomy.

- $N_i \times \{0\} = N_i$ and $q_{i-1} \in N_i \times \{1\}$.
- $\partial N_i \times [0, 1] \subset \phi(\Sigma) \cup \partial B_{i-1} \cup \partial B_i$.
- $\mathcal{F} | (N_i \times [0, 1]) = \{N_i \times \{t\}\, ;\, t \in [0, 1]\}$.

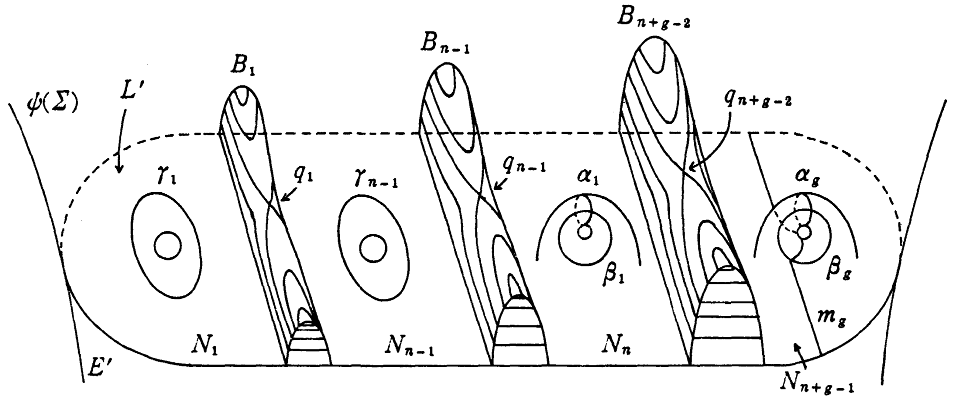

Fig. 7.

Let $\phi' : \Sigma \to M$ be an embedding isotopic to ϕ such that

$$\phi'(\Sigma) = \Big(\phi(\Sigma) - E' - \bigcup_{i=1}^{n+g-2} \partial B_i\Big) \cup \Big(\bigcup_{i=1}^{n+g-1} N_i\Big) \cup \Big(\bigcup_{i=1}^{n+g-2} \overline{\partial B_i - \phi(\Sigma) - L'}\Big).$$

If the annulus N_i $(1 \leqq i \leqq n-1)$ has holonomy, then we can modify a neighborhood of N_i in $\phi'(\Sigma)$ so that $\phi'(\Sigma)$ is transverse to $\mathcal{F}$ there.

If N_{n+j-1} $(1 \leqq j \leqq g)$ has holonomy, then α_j or β_j has holonomy. Now we assume that α_j has holonomy. Let m_j be a properly embedded arc of N_{n+j-1} which intersects β_j at one point (Fig. 7). We make a small block containing

m_j like B_i. Since $N_{n+j-1}-m_j$ is an annulus with holonomy, we can modify a neighborhood of N_{n+j-1} in $\psi'(\Sigma)$ so that $\psi'(\Sigma)$ is transverse to $\mathcal{F}$ there.

Next suppose that N_i $(2\leqq i\leqq n+g-1)$ has no holonomy. By the definition of B_i, the saturated product neighborhood $N_i\times[0, 1]$ is a maximal coordinate of type E. Hence we can modify a neighborhood of $(\partial N_i\times[0, 1])\cup N_i$ in $\psi'(\Sigma)$ so that $\psi'(\Sigma)$ is transverse to $\mathcal{F}$ except at the saddle singularity there.

Since γ_1 has holonomy, the number of singularities of $\mathcal{F}|\psi'(\Sigma)$ is less than that of $\mathcal{F}|\psi(\Sigma)$. However this contradicts the minimality of the number of singularities of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$. Thus it does not happen that γ_1 has holonomy.

In the case where N_i $(2\leqq i\leqq n+g-1)$ has holonomy, we take $l_1, l_2, l_3, \cdots, l_{n+g-2}$ so that N_1 has holonomy, where N_1 may be a one-punctured torus. By the same argument as above, the number of singularities of $\mathcal{F}|\psi(\Sigma)$ decreases.

Thus the maximal coordinate F is not of type A. ∎

LEMMA 6. *If $\psi(\Sigma)$ is isotopic to an interior compact leaf L of $\mathcal{F}$ with holonomy, then there exists an embedding $\psi' : \Sigma\to M\in G(\phi)$ such that all the singularities of $\mathcal{F}|\psi'(\Sigma)$ are saddle singularities.*

PROOF. Since L has holonomy, there is a simple closed curve l on L which has holonomy. Let U be a small closed tubular neighborhood of l in L. Since l has holonomy there is a product neighborhood $U\times[0, 1]$ satisfying the following conditions:

- $U\times\{0\}=U$ and $\mathcal{F}|(U\times\{1\})$ is a product foliation whose leaves are properly embedded arcs.
- $\mathcal{F}|(U\times[0, 1])$ is constructed by turbulizing $\mathcal{F}|(U\times\{1\})$ along $U\times\{0\}$.

Since $\overline{L-U}$ is a surface with boundary and $\overline{L-U}$ has holonomy, there exists a compact surface S with boundary near $\overline{L-U}$ such that $\partial S\subset\partial U\times[0, 1]$ and $\mathcal{F}|S$ is a singular foliation transverse to ∂S with only saddle singularities (by the proof of the case of type A in Lemma 5). Thus there exists an embedding $\psi' : \Sigma\to M\in G(\phi)$ such that each singularity of $\mathcal{F}|\psi'(\Sigma)$ is a saddle singularity. ∎

PROOF OF THEOREM 2. Suppose that $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ contains a center for an element $\psi\in G'(\phi)$. Then there exists a patch of rank 0. If $\psi(\Sigma)$ is not isotopic to an interior compact leaf of $\mathcal{F}$, then there are an embedding $\psi'\in G'(\phi)$ and its patch of any rank by Lemmas 4 and 5. However the number of singularities of $\mathcal{F}|\psi'(\mathrm{int}\,\Sigma)$ is less than that of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$ by the minimality of the number of singularities of $\mathcal{F}|\psi(\mathrm{int}\,\Sigma)$. Thus $\psi(\Sigma)$ is isotopic to an interior compact leaf of $\mathcal{F}$, denoted by L $(\partial\Sigma=\emptyset)$.

If L has holonomy, then there is an embedding $\psi' : \Sigma\to M\in G(\phi)$ such that all the singularities of $\mathcal{F}|\psi'(\Sigma)$ are saddle singularities. Hence the number of singularities of $\mathcal{F}|\psi'(\Sigma)$ is less than that of $\mathcal{F}|\psi(\Sigma)$. Since this contradicts the minimality of the number of singularities of $\mathcal{F}|\psi(\Sigma)$, L has no holonomy.

By Reeb's global stability theorem, $\mathcal{F}$ is a foliation of a Σ-bundle over S^1 without boundary such that $\phi(\Sigma)$ and the leaves of $\mathcal{F}$ are isotopic to the fiber. ■

If $\partial\Sigma\neq\emptyset$, then the element ψ of $G'(\phi)$ is isotopic to ϕ with boundary fixed. Therefore, by the same argument as in the proof of Theorem 2, the next Corollary holds:

COROLLARY. *Let Σ, M, $\mathcal{F}$ be as in the main theorem. Suppose that the pseudo-foliation $\mathcal{F}$ and the immersion $\phi: \Sigma\to M$ satisfy the condition 2) of the main theorem and the following conditions:*

- *$\partial\Sigma\neq\emptyset$ and $\phi(\Sigma)\cap\partial M=\phi(\partial\Sigma)$.*
- *There are finitely many points $r_i\in\partial\Sigma$ $(t=1, 2, 3, \cdots, n)$ such that $\phi\Big|\Big(\Sigma-\bigcup_{i=1}^{n}\{r_i\}\Big)$ is an embedding.*
- *$\phi\Big(\partial\Sigma-\bigcup_{i=1}^{n}\{r_i\}\Big)$ is transverse to $\mathcal{F}|\partial M$.*
- *$\phi(r_i)$ is a saddle singularity of $\mathcal{F}|\partial M$ and $\phi(\partial\Sigma)$ crosses $\phi(r_i)$ (Fig. 8).*
- *$(\phi|(\mathrm{int}\,\Sigma))_*: \pi_1(\mathrm{int}\,\Sigma)\to\pi_1(M)$ is injective.*

Then there exists an immersion ϕ' isotopic to ϕ with boundary fixed such that all the singularities of $\mathcal{F}|\phi'(\mathrm{int}\,\Sigma)$ are saddle singularities and $\phi'\Big|\Big(\Sigma-\bigcup_{i=1}^{n}\{r_i\}\Big)$ is an embedding $(\phi'(\Sigma)\cap\partial M=\phi'(\partial\Sigma))$.

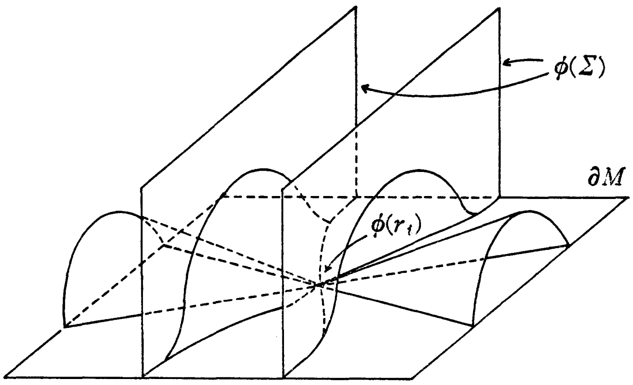

Fig. 8.

3. The existence of cutting surfaces.

In this section, we prove the main theorem. If $\partial\Sigma=\emptyset$, then the main theorem is proved by Theorem 1. In the following, we assume that $\partial\Sigma\neq\emptyset$.

LEMMA 7. *Let Σ, M, $\mathcal{F}$, ϕ and ψ be as in the main theorem. Denote by q_i $(1\leq i\leq m)$ the points of $\psi(\partial\Sigma)$ where $\psi(\partial\Sigma)$ is tangent to $\mathcal{F}|\partial M$. By changing ψ by an isotopy in a neighborhood of $\partial\Sigma$, the continuous maps $F_i: D^1\times[0, 1]\to\partial M$*

($1 \leqq i \leqq m$) *defined in Lemma* 1 *can be taken so that the following properties hold:*

1) $F_i|(D^1 \times (0, 1])$ *is injective and* $q_i \notin F_i(D^1 \times \{1\})$.

2) $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ *is parallel to* $F_i(\partial D^1 \times [0, 1])$ *for each* i (*That is, each connected component of* $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ *except for* $F_i(\partial D^1 \times [0, 1])$ *separates* $h_\phi(q_i)$ *from* $F_i(\partial D^1 \times [0, 1])$).

3) *If* $\mathrm{int}(F_i(D^1 \times [0, 1])) \cap \mathrm{int}(F_j(D^1 \times [0, 1])) \neq \emptyset$, *then* $h_\phi(q_i) = h_\phi(q_j)$, *and* $F_i(D^1 \times [0, 1]) \subset F_j(D^1 \times [0, 1])$ *or* $F_j(D^1 \times [0, 1]) \subset F_i(D^1 \times [0, 1])$.

PROOF. In the proof of Lemma 1 ([**6**]), Roussarie showed that the embedding ϕ could be taken by an isotopy so that $F_i|(D^1 \times (0, 1])$ was injective and $q_i \notin F_i(D^1 \times \{1\})$ for each i.

First we claim that $\phi(\partial \Sigma)$ can be taken by an isotopy so that $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ is parallel to $F_i(\partial D^1 \times [0, 1])$ for each i. Let γ be a connected component of $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ except for $F_i(\partial D^1 \times [0, 1])$. By the minimality of the number of the points of $\phi(\partial \Sigma)$ tangent to $\mathcal{F}|\partial M$, γ has the unique point tangent to $\mathcal{F}|\partial M$. If γ is not parallel to $F_i(\partial D^1 \times [0, 1])$, then γ bounds a disk with $F_i(D^1 \times \{1\})$ which does not contain $h_\phi(q_i)$. As in the proof of the case of type E of Lemma 5, we push γ out of $F_i(D^1 \times [0, 1])$ by changing ϕ by an isotopy, where the points of $\phi(\partial \Sigma)$ tangent to $\mathcal{F}|\partial M$ move but h_ϕ for these tangent points do not change. By the induction on the number of the connected components of $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ not parallel to $F_i(\partial D^1 \times [0, 1])$, $\phi(\partial \Sigma)$ can be taken so that $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ is parallel to $F_i(\partial D^1 \times [0, 1])$ for each i.

Suppose that $\mathrm{int}(F_i(D^1 \times [0, 1])) \cap \mathrm{int}(F_j(D^1 \times [0, 1])) \neq \emptyset$ for some i, j ($i \neq j$). Then $F_i(D^1 \times [0, 1]) \subset F_j(D^1 \times [0, 1])$ or $F_j(D^1 \times [0, 1]) \subset F_i(D^1 \times [0, 1])$, because $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ is parallel to $F_i(\partial D^1 \times [0, 1])$. ■

In the following, ϕ and F satisfy the conditions 1), 2) and 3) of Lemma 7.

LEMMA 8. *For any saddle singularity* p *of* $\mathcal{F}|\partial M$, *the elements of* $h_\phi^{-1}(p)$ *are either all convex or all concave if* $h_\phi^{-1}(p) \neq \emptyset$.

PROOF. Suppose that a convex tangent point q_i and a concave tangent point q_j satisfy $h_\phi(q_i) = h_\phi(q_j)$. Let γ_j denote the connected component of $F_j(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ nearest to $h_\phi(q_j)$. The connected component of $F_i(D^1 \times [0, 1]) \cap \phi(\partial \Sigma)$ intersecting γ_j on the boundary is denoted by γ_i. By transferring a small neighborhood of $\gamma_i \cup \gamma_j$ in $\phi(\partial \Sigma)$ outside $F_i(D^1 \times [0, 1]) \cup F_j(D^1 \times [0, 1])$ so that $\phi(\partial \Sigma)$ is transverse to $\mathcal{F}|\partial M$ there, we can remove two points of $\phi(\partial \Sigma)$ tangent to $\mathcal{F}|\partial M$. However this contradicts the minimality of tangent points of $\phi(\partial \Sigma)$. Thus Lemma 8 is proved. ■

PROOF OF THE MAIN THEOREM ($\partial \Sigma \neq \emptyset$). Let ϕ be an embedding of $E'(\phi)$ modified by Lemma 7. Denote by $\mathcal{S}_\phi^v$ the image of convex tangent points by h_ϕ and denote by $\mathcal{S}_\phi^c$ that of concave tangent points. By Lemma 8, $\mathcal{S}_\phi^v \cap \mathcal{S}_\phi^c = \emptyset$.

First we will change ϕ so that ϕ satisfies the hypothesis of Corollary in Section 2.

If r is a point of $\mathcal{S}^v_\phi$, then there exists the unique convex tangent point q_i such that $h_\phi(q_i)=r$. Since $F_i(D^1\times[0, 1])\cap\phi(\partial\Sigma)=F_i(\partial D^1\times[0, 1])$, we transfer a neighborhood of $F_i(\partial D^1\times[0, 1])$ in $\phi(\partial\Sigma)$ to a neighborhood of $F_i(D^1\times\{1\})$ in ∂M so that $\phi(\partial\Sigma)$ passes through r and $\phi(\partial\Sigma)-\{r\}$ is transverse to $\mathcal{F}|\partial M$ there.

If r is an element of $\mathcal{S}^c_\phi$, then there exist one or two concave tangent points q_i, q_j such that $F_k(D^1\times[0, 1])\subset F_i(D^1\times[0, 1])$ or $F_k(D^1\times[0, 1])\subset F_j(D^1\times[0, 1])$ for any $q_k\in h_\phi^{-1}(r)$, where $F_i(D^1\times[0, 1])\cap F_j(D^1\times[0, 1])=\{r\}$ if there exist two concave tangent points. Let W denote $(F_i(D^1\times[0, 1])\cup F_j(D^1\times[0, 1]))\cap\phi(\partial\Sigma)$ and let m denote the number of connected components of W. We transfer a neighborhood of W in $\phi(\partial\Sigma)$ to a neighborhood of $F_i(D^1\times\{1\})\cup F_j(D^1\times\{1\})$ in ∂M so that $\phi(\partial\Sigma)$ passes through the saddle singularity m-times, $\phi|\partial\Sigma$ is an embedding except at r and $\phi(\partial\Sigma)-\{r\}$ is transverse to $\mathcal{F}|\partial M$ there (Fig. 9).

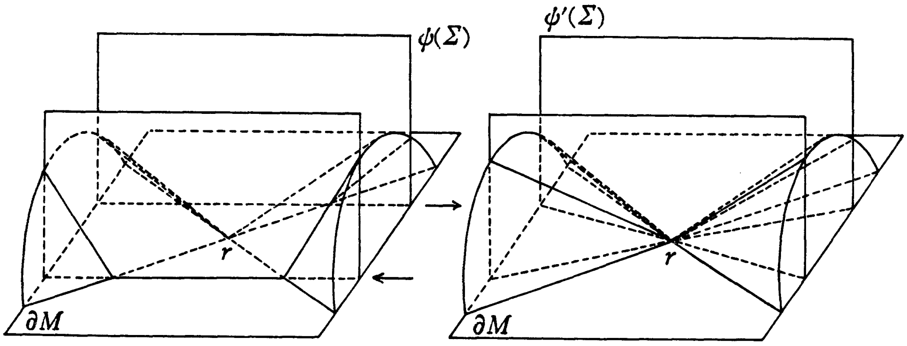

Fig. 9.

Denote by $\phi' : \Sigma\to M$ the immersion obtained by transferring $\phi(\partial\Sigma)$ as above for all the tangent points. Since ϕ' satisfies the hypothesis of Corollary, ϕ' can be taken with boundary fixed so that all the singularities of $\mathcal{F}|\phi'(\mathrm{int}\,\Sigma)$ are saddle singularities.

Next we change the immersion ϕ' in small neighborhoods of all the saddle singularities of $\mathcal{S}^c_\phi$ whose inverse image of h_ϕ contains at least two points so that $\phi'(\Sigma)$ is an embedding isotopic to $\phi(\Sigma)$ and $\mathcal{F}|\phi'(\Sigma)$ has two separatrices at the above singularities on $\phi'(\partial\Sigma)$ (Fig. 9). Then all the singularities of $\mathcal{F}|(\overline{M-\phi'(\Sigma)})$ are saddle singularities. By changing ϕ' slightly, ϕ' can be taken so that $\mathcal{F}|(\overline{M-\phi'(\Sigma)})$ has no leaves connecting distinct saddle singularities. Thus the main theorem is proved. ∎

References

[1] G. Hector and U. Hirsch, Introduction to the geometry of foliations, Part B Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1983.
[2] W. Jaco, Lectures on three-manifold topology, CBMS Regional Conf. Ser. in Math., No. 43 (1979).
[3] H. Nakayama, Transversely affine foliations of some surface bundles over S^1 of pseudo-Anosov type, preprint.
[4] S. P. Novikov, Topology of foliations, Trudy Moskov. Mat. Obshch., **14** (1965), 248-278. Trans. Moscow Math. Soc., **14** (1965), 268-304.
[5] G. Levitt, Pantalons et feuilletages des surfaces, Topology, **21** (1982), 9-33.
[6] R. Roussarie, Sur les feuilletages des variétés de dimension 3, Ann. Inst. Fourier, **21** (1971), 13-81.
[7] R. Roussarie, Plongements dans les variétés feuilletées et classification de feuilletages sans holonomie, Publ. Math. IHES, **43** (1974), 101-141.
[8] W. Thurston, Foliations of 3-manifolds which are circle bundles, Thesis, Univ. of California, Berkeley, 1972.
[9] W. Thurston, A norm for the homology of 3-manifolds, Mem. Amer. Math. Soc., **59** (1986), 99-130.

Hiromichi NAKAYAMA
Tokyo Denki University
College of Science and Engineering
Department of Mathematical Sciences
Hatoyama-machi, Hiki-gun
Saitama-ken 350-03
Japan