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On the mod p cohomology of the spaces of free loops on the Grassman and Stiefel manifolds

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§ 0. Introduction.

Let ΩX be a space of loops on X and ΛX a space of free loops on X. We will call a fibration $\Omega X \hookrightarrow \Lambda X \xrightarrow{\pi} X$ a free loop fibration on X where $\pi(w)=w(1)$ for $w \in \Lambda X$. Let $\boldsymbol{K}_p$ be a field of characteristic p. When is ΩX totally non-homologous to zero in ΛX with respect to a field $\boldsymbol{K}_p$?

We call a commutative algebra $\Lambda(y_1, \cdots, y_l) \otimes \boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m)$ over a field $\boldsymbol{K}_p$ is *GCI* algebra if $\rho_1, \cdots, \rho_m$ is a regular sequence (see [**4**; p. 95]) or $m=0$ where deg y_i is odd and deg x_i is even if $p \neq 2$. (see [**5**; Definition, p. 893].) In [**5**], L. Smith has proved the following.

THEOREM 1 ([**5**; Theorem 4.1]). *Let X be a simply connected space such that $H^*(X; \boldsymbol{K}_0)$ is a GCI algebra. Then ΩX is totally non-homologous to zero in ΛX with respect to $\boldsymbol{K}_0$ if and only if $H^*(X; \boldsymbol{K}_0)$ is a free commutative algebra, in which case $H^*(\Lambda X; \boldsymbol{K}_0) \cong H^*(X; \boldsymbol{K}_0) \otimes H^*(\Omega X; \boldsymbol{K}_0)$ as an algebra.*

In this paper, using methods which L. Smith has given in [**5**], we will examine whether ΩX is totally non-homologous to zero in ΛX with respect to $\boldsymbol{K}_p$ for cases where $X=U(m+n)/U(m)\times U(n)$, $Sp(m+n)/Sp(m)\times Sp(n)$, $Sp(n)/U(n)$, $SO(m+n)/SO(n)$, $SU(m+n)/SU(n)$, $Sp(m+n)/Sp(n)$, $\mathfrak{C}P(2)$ and $p \geqq 2$.

In order to obtain our results, we will consider the Eilenberg-Moore spectral sequence of a fibre square

$$\mathcal{F}(X) := \begin{array}{ccc} \Lambda X & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \Delta \\ X & \underset{\Delta}{\longrightarrow} & X \times X \end{array} \qquad \text{(see [5])},$$

where Δ is a diagonal map. Throughout this paper, $\mathcal{F}(X)$ means the above fibre square.

For a space X, let $T(X)$ denote a set of prime numbers p such that ΩX is totally non-homologous to zero in ΛX with respect to $\boldsymbol{K}_p$.

Our results are stated as follows.

THEOREM 2.
(1) $p \in T(\boldsymbol{C}P(n))$ *iff* $n+1\equiv 0 \bmod p$.
(2) $p \in T(\boldsymbol{H}P(n))$ *iff* $n+1\equiv 0 \bmod p$.
(3) $p \in T(\mathfrak{C}P(2))$ *iff* $p=3$.
(4) *If* $m, n\geqq 2$, *then* $p\notin T(U(m+n)/U(m)\times U(n))$ *for any prime* p.
(5) *If* $m, n\geqq 2$, *then* $p\notin T(Sp(m+n)/Sp(m)\times Sp(n))$ *for any prime* p.
(6) $p\in T(Sp(n)/U(n))$ *iff* $p=2$.
(7) *If* n *is even, then* $p\notin T(SO(m+n)/SO(n))$ *for any odd prime* p.
(8) *If* n *is odd, then* $p\in T(SO(m+n)/SO(n))$ *for any odd prime* p.
(9) $p\in T(SU(m+n)/SU(n))$ *for any prime* p.
(10) $p\in T(Sp(m+n)/Sp(n))$ *for any prime* p.

The problem whether the prime 2 is contained to $T(SU(m+n)/SU(n))$ for any m and n is not expected to be easy. We will consider the problem for some m and n. Before we state the results, we recall the mod 2 cohomology of the real Stiefel manifold and the action of the squaring operations on it.

(0.1) $$H^*(SO(m+n)/SO(n)\,;\,\boldsymbol{Z}/2)\cong \Delta(x_n, x_{n+1}, \cdots, x_{m+n-1}) \cong \bigotimes_{j\in J\cup J'} \boldsymbol{Z}/2[x_j]/(x_j^{2^{r_j}}),$$

where $J=\{j=2t+1 \mid n\leqq j<m+n\}$, $J'=\{j=2t \mid n\leqq j<\min(2n, m+n)\}$, $j\cdot 2^{r_j-1}< m+n\leqq j\cdot 2^{r_j}$ and $Sq^j x_i=\binom{i}{j}x_{i+j}$; $x_{i+j}=0$ if $i+j\geqq m+n$.

L. Smith has proved the following collapse theorem, making use of the k-stage Postnikov system given by D. Kraines [**3**].

THEOREM 3 ([**6**; Theorem]). *Let X be a simply connected space, and suppose that Sq^1 vanishes on $H^*(X\,;\,\boldsymbol{Z}/2)$ and $H^*(X\,;\,\boldsymbol{Z}/2)\cong\boldsymbol{Z}/2[x_1, \cdots, x_t]/(x_1^{2^{r_1}}, \cdots, x_t^{2^{r_t}})$. Then the* mod 2 *Eilenberg-Moore spectral sequence of the fibre square $\mathcal{F}(X)$ collapses at the E_2-term.*

The fact (0.1) implies that Theorem 3 can not be applied in the case where $X=SO(m+n)/SO(n)$. Examining the argument in the proof of Theorem 3, we see that the theorem holds if the action of Sq^1 is trivial on certain important degrees of the cohomology groups of X. Taking notice of this fact, we have the following.

THEOREM 4. *Suppose that the vector space* $V:=\bigoplus_{\substack{s=j\cdot 2^{r_j+k}-2^{k+1}+2 \\ j\in J\cup J',\, k\geqq 1}} (\mathrm{Im}\, Sq^1)^s$ *is zero or, for any non-zero element $\rho\in V$, there exists an integer j such that $(\partial\pi^*\rho/\partial x_j)\neq 0$ in $H^*(\mathrm{Spin}(m+n)\,;\,\boldsymbol{Z}/2)$. Then $2\in T(SO(m+n)/SO(n))$, where*

$\pi : \mathrm{Spin}(m+n) \xrightarrow{\tilde{\pi}} SO(m+n) \xrightarrow{\tilde{p}} SO(m+n)/SO(n)$; *$\tilde{\pi}$ is the universal covering and $\tilde{p}$ is the natural projection.*

In consequence, we get:

COROLLARY 5. (1) *If $m \leqq 4$, then $2 \in T(SO(m+n)/SO(n))$.*

(2) *If $n \geqq m$ and*

$$\{s = j \cdot 2^{k+1} - 2^{k+1} + 2 \mid j \in J \cup J',\ k \geqq 1\} \cap \{j_1 + \cdots + j_t \mid j_1 < \cdots < j_t,\ j_i \in J'\} = \emptyset,$$

then $2 \in T(SO(m+n)/SO(n))$. In particular, when $1 \leqq m \leqq 8$ and $n \geqq 43$, $2 \in T(SO(m+n)/SO(n))$.

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§1. Preliminaries.

In order to study the cohomology of the space of free loops on a simply connected space X, we use the Eilenberg-Moore spectral sequence of the fibre square $\mathcal{F}(X)$. When we use this method, we must computes $\mathrm{Tor}^{**}_{\Lambda \otimes \Lambda}(\Lambda, \Lambda)$, where $\Lambda = H^*(X; \boldsymbol{K}_p)$. In [**5**], L. Smith has given a differential bigraded algebra to compute $\mathrm{Tor}^{**}_{\Lambda \otimes \Lambda}(\Lambda, \Lambda)$ when the field is of characteristic zero. In general, when the field is of characteristic $p \geqq 0$, considering the methods of the proofs of [**5**; Lemma 3.2], [**5**; Lemma 3.3], [**5**; Lemma 3.4] and [**5**; Proposition 3.5], we can obtain the following:

PROPOSITION 1.1 ([**5**; Proposition 3.5]). *Let Λ be a GCI algebra $\Lambda(y_1, \cdots, y_l) \otimes \boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m)$ over $\boldsymbol{K}_p$, where each ρ_i is decomposable, $\deg y_i$ is odd and $\deg x_j$ is even if $p \neq 2$. Then there exists the following proper projective resolution $\mathcal{F} \xrightarrow{\varphi} \Lambda \to 0$ of Λ as a left $\Lambda \otimes \Lambda$-module:*

$$\mathcal{F} := \Lambda \otimes \Lambda \otimes \Gamma[\nu_1, \cdots, \nu_l] \otimes \Lambda(u_1, \cdots, u_n) \otimes \Gamma[w_1, \cdots, w_m],$$

$d(\Lambda \otimes \Lambda) = 0$, $d(\nu_i) = y_i \otimes 1 - 1 \otimes y_i$, $d(u_j) = x_j \otimes 1 - 1 \otimes x_j$, $d(\gamma_r(w_i)) = (\sum_{j=1}^n \zeta_{ij} u_j) \otimes \gamma_{r-1}(w_i)$ and φ is the multiplication of Λ, where $\mathrm{bideg}\,\lambda = (0, \deg \lambda)$; $\lambda \in \Lambda \otimes \Lambda$, $\mathrm{bideg}\,\nu_i = (-1, \deg y_i)$, $\mathrm{bideg}\,u_j = (-1, \deg x_j)$, $\mathrm{bideg}\,w_i = (-2, \deg \rho_i)$ *and $\zeta_{ij} \in \boldsymbol{K}_p[x_1, \cdots, x_n] \otimes \boldsymbol{K}_p[x_1, \cdots, x_n]$ satisfies that $\rho_i \otimes 1 - 1 \otimes \rho_i = \sum_{j=1}^n \zeta_{ij}(x_j \otimes 1 - 1 \otimes x_j)$ and $\mu(\zeta_{ij}) = (\partial \rho_i / \partial x_j)$; $\mu : \boldsymbol{K}_p[x_1, \cdots, x_n] \otimes \boldsymbol{K}_p[x_1, \cdots, x_n] \to \boldsymbol{K}_p[x_1, \cdots, x_n]$ is the multiplication. Hence a differential bigraded algebra*

$$\mathcal{E} := \Lambda \otimes \Gamma[\nu_1, \cdots, \nu_l] \otimes \Lambda(u_1, \cdots, u_n) \otimes \Gamma[w_1, \cdots, w_m],$$

where $d(\lambda) = d(\nu_i) = d(u_j) = 0$; $\lambda \in \Lambda$, $i = 1, \cdots, l$, $j = 1, \cdots, n$, and

$$d(w_i) = \sum_{j=1}^n \frac{\partial \rho_i}{\partial x_j} u_j \qquad \text{for any } i = 1, \cdots, m,$$

computes $\mathrm{Tor}^{*,*}_{A\otimes A}(A, A)$. □

Let A be *GCI* algebra in Proposition 1.1. We can compute $\mathrm{Tor}^{*,*}_{A}(\boldsymbol{K}_p, \boldsymbol{K}_p)$ as follows by making use of [**7**; §3. Theorem 2] in the spirit of [**7**; §2] and [**5**; Lemma 3.3].

PROPOSITION 1.2 ([**5**; Lemma 3.1]). *There exists the following Koszul resolution* $\mathcal{K}\xrightarrow{\varepsilon}\boldsymbol{K}_p\to 0$ *of* $\boldsymbol{K}_p$ *as a left* A*-module*:

$$\mathcal{K} := A\otimes\Gamma[s^{-1}y_1, \cdots, s^{-1}y_l]\otimes\Lambda(s^{-1}x_1, \cdots, s^{-1}x_n)\otimes\Gamma[\tau\rho_1, \cdots, \tau\rho_m],$$

$d(s^{-1}y_i)=y_i$, $d(s^{-1}x_j)=x_j$, $d(\gamma_r(\tau\rho_i))=\xi_i\otimes\gamma_{r-1}(\tau\rho_i)$, *where* bideg $s^{-1}y_i=(-1, \deg y_i)$, bideg $s^{-1}x_j=(-1, \deg x_j)$, bideg $\tau\rho_i=(-2, \deg\rho_i)$ *and* $\xi_i\in A\otimes\Lambda(s^{-1}x_1, \cdots, s^{-1}x_n)$ *satisfies that* $d(\xi_i)=\rho_i$. *Hence*

$$\mathrm{Tor}^{*,*}_{A}(\boldsymbol{K}_p, \boldsymbol{K}_p)\cong\Gamma[s^{-1}y_1, \cdots, s^{-1}y_l]\otimes\Lambda(s^{-1}x_1, \cdots, s^{-1}x_n)\otimes\Gamma[\tau\rho_1, \cdots, \tau\rho_m]$$

as an algebra. □

For the rest of this paper, let X be a simply connected space whose cohomology with coefficients in the field $\boldsymbol{K}_p$ is isomorphic to a *GCI* algebra over $\boldsymbol{K}_p$: $H^*(X;\boldsymbol{K}_p)\cong\Lambda(y_1, \cdots, y_l)\otimes\boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m)$, where $\deg x_j$ is even and $\deg y_i$ is odd if $p\neq 2$, and ρ_i is decomposable. Let $\{E_r, d_r\}$, $\{\hat{E}_r, \hat{d}_r\}$ and $\{\bar{E}_r, \bar{d}_r\}$ (or $\{E_r(X), d_r(X)\}$, $\{\hat{E}_r(X), \hat{d}_r(X)\}$ and $\{\bar{E}_r(X), \bar{d}_r(X)\}$) be the Eilenberg-Moore spectral sequences of the fibre square $\mathcal{F}(X)$, of the path-loop fibration $\Omega X\hookrightarrow PX\to X$, and the Leray-Serre spectral sequence of the free loop fibration $\Omega X\hookrightarrow\Lambda X\to X$ respectively.

We need the following lemma in order to consider relations between the above three spectral sequences.

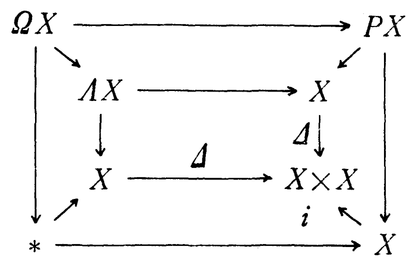
LEMMA 1.3. *There exists a morphism of spectral sequences*

$$\{f_r\}:\{E_r, d_r\}\longrightarrow\{\hat{E}_r, \hat{d}_r\}$$

such that

(1.1) $\qquad f_2(\lambda)=0$ *if* $\lambda\in A$ *and* $\deg\lambda>0$, $f_2(\lambda)=\lambda$ *if* $\lambda\in A$ *and* $\deg\lambda=0$, $f_2(\nu_i)=s^{-1}y_i$ $(1\leqq i\leqq l)$, $f_2(u_j)=s^{-1}x_j$ $(1\leqq j\leqq n)$ *and* $f_2(\gamma_r(w_i))=\gamma_r(\tau\rho_i)$ $(1\leqq i\leqq m)$ *if* $\gamma_r(w_i)$ *is defined in* $\mathrm{Tor}^{*,*}_{A\otimes A}(A, A)$, *where* $A=H^*(X;\boldsymbol{K}_p)$. (*For notations, see Proposition* 1.1 *and* 1.2.)

PROOF. The morphism of fibre squares



where $i(x)=(x, *)$, yields a morphism of spectral sequences. It remains to examine the stated behavior on E_2^{**}. It suffices to show that there exists a morphism of resolutions

$$(1.2)\qquad \begin{array}{ccccc} \mathcal{F}^{\cdot} & \xrightarrow{\varphi} & \Lambda & \longrightarrow & 0 \\ \Psi=\{\psi_{-n}\}_{n\geq 0}: \downarrow & & \downarrow t & & \\ \mathcal{K}^{\cdot} & \xrightarrow{\mu} & \boldsymbol{K}_p & \longrightarrow & 0 \end{array}$$

which induces the trivial map t such that ψ_{-n} is an i^*-morphism for any n, $\psi_{-1}(\nu_i)=s^{-1}y_i$, $\psi_{-1}(u_j)=s^{-1}x_j$ and $\psi_{-r}(\gamma_r(w_i))=\gamma_r(\tau\rho_i)$. In fact the following diagram

$$\begin{array}{ccccc} E_2^{**} \cong & \mathrm{Tor}_{\Lambda\otimes\Lambda}^{**}(\Lambda, \Lambda) & \cong & H(\Lambda\otimes_{\Lambda\otimes\Lambda}\mathcal{F}^{\cdot}) \\ f_2\downarrow & \downarrow \mathrm{Tor}_{i*} & & \downarrow H(t\otimes_{i*}\Psi) \\ \hat{E}_2 \cong & \mathrm{Tor}_{\Lambda}^{**}(\boldsymbol{K}_p, \boldsymbol{K}_p) & \cong & H(\boldsymbol{K}_p\otimes_{\Lambda}\mathcal{K}^{\cdot}) \end{array}$$

is commutative.

First let us choose the elements ζ_{ij} and ξ_i for any i. In the case where ρ_i is a monomial, suppose that $\rho_i=\lambda x_1^{m_1}\cdots x_n^{m_n}$. Let J mean a set $\{j_1, \cdots, j_t | 1\leqq j_1<\cdots<j_t\leqq n,\ m_{j_i}\geqq 1$ for any $i\}$. Put $\zeta_{ij}=0$ if $j\notin J$,

$$\zeta_{ij_1}=\lambda x_{j_1}^{m_{j_1}-1}x_{j_2}^{m_{j_2}}\cdots x_{j_t}^{m_{j_t}}\otimes 1+\lambda x_{j_1}^{m_{j_1}-2}x_{j_2}^{m_{j_2}}\cdots x_{j_t}^{m_{j_t}}\otimes x_{j_1} + \cdots +\lambda x_{j_2}^{m_{j_2}}\cdots x_{j_t}^{m_{j_t}}\otimes x_{j_1}^{m_{j_1}-1},$$

$$\zeta_{ij_2}=\lambda x_{j_2}^{m_{j_2}-1}\cdots x_{j_t}^{m_{j_t}}\otimes x_{j_1}^{m_{j_1}}+\cdots+\lambda x_{j_3}^{m_{j_3}}\cdots x_{j_t}^{m_{j_t}}\otimes x_{j_1}^{m_{j_1}}x_{j_2}^{m_{j_2}-1},$$

....

and
$$\zeta_{ij_t}=\lambda x_{j_t}^{m_{j_t}-1}\otimes x_{j_1}^{m_{j_1}}\cdots x_{j_{t-1}}^{m_{j_{t-1}}}+\cdots+\lambda 1\otimes x_{j_1}^{m_{j_1}}\cdots x_{j_t}^{m_{j_t}-1}$$

(see [**5**; Lemma 3.4]). Then it follows that $\partial\rho_i/\partial x_j=\mu(\zeta_{ij})$ for any i and j. Put $\xi_i=\lambda x_{j_1}^{m_{j_1}-1}x_{j_2}^{m_{j_2}}\cdots x_{j_t}^{m_{j_t}}\otimes s^{-1}x_{j_1}$. Define a morphism of resolution Ψ as follows: $\psi_{-1}(\nu_i)=s^{-1}y_i$, $\psi_{-1}(u_j)=s^{-1}x_j$ and $\psi_{-r}(\gamma_r(w_i))=\gamma_r(\tau\rho_i)$. We can show that $\psi_{-r+1}d(\gamma_r(w_i))=d\psi_{-r}(\gamma_r(\omega_i))$.

In the general case where $\rho_i=\sum\lambda_k x_1^{m_{k1}}\cdots x_n^{m_{kn}}$, we can choose elements $(\zeta_{ij})_k$ and $(\xi_i)_k$ as the above for any k. Put $\zeta_{ij}=\sum(\zeta_{ij})_k$ and $\xi_i=\sum(\xi_i)_k$, then $\psi_{-r+1}d(\gamma_r(w_i))=d\psi_{-r}(\gamma_r(w_i))$. We obtain the required morphism of resolutions.

q. e. d.

Applying the same argument as above, we have:

LEMMA 1.4. *Suppose* $H^*(X; \boldsymbol{K}_p)\cong\Lambda(y)\otimes\boldsymbol{K}_p[x]/(x^{p^s})\otimes(GCI\text{-}alg)$, $H^*(Y; \boldsymbol{K}_p)\cong\Lambda(y')\otimes\boldsymbol{K}_p[x']/(x'^{p^s})\otimes(GCI\text{-}alg)$ *and there exists a map* $f: Y\to X$ *such that* $f^*(x)=x'$ $(f^*(y)=y')$. *Then the map* f *induces a morphism of spectral sequences* $\{g_r\}: \{E_r(X), d_r(X)\}\to\{E_r(Y), d_r(Y)\}$ *satisfying that* $g_2(u)=u'$, $g_2(\gamma_r(w))=\gamma_r(w')$ $(g_2(\gamma_r(\nu))=\gamma_r(\nu'))$, *where* u *and* w *are elements associated with* x, ν *is an* *element associated with y. u', w' and ν' are elements associated with x' and y' respectively.* □

Next we examine the relation between the torsion $\mathrm{Tor}^{**}_{\Lambda\otimes\Lambda}(\Lambda, \Lambda)$ obtained from Proposition 1.1 and the torsion $\mathrm{Tor}^{**}_{\Lambda\otimes\Lambda}(\Lambda, \Lambda)$ obtained from the bar resolution of Λ as a $\Lambda\otimes\Lambda$-module. Let $\mathrm{Tor}^{**}_{\Lambda\otimes\Lambda}(\Lambda, \Lambda)_B$ denote the latter.

LEMMA 1.5. *Suppose Λ is a GCI algebra $\Lambda(y_1, \cdots, y_l)\otimes \boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m)$, where each ρ_i is decomposable and $l=0$ if $p=2$. Then there exists an isomorphism of algebras*

$$\phi: \mathrm{Tor}^{**}_{\Lambda\otimes\Lambda}(\Lambda, \Lambda)_B \longrightarrow \mathrm{Tor}^{**}_{\Lambda\otimes\Lambda}(\Lambda, \Lambda)$$

such that $\phi(1[y_i\otimes 1-1\otimes y_i]1)=\nu_i$ and $\phi(1[x_j\otimes 1-1\otimes x_j]1)=u_j$, moreover $\phi(1[z\otimes 1-1\otimes z]1)=\sum_{j=1}^n(\partial z/\partial x_j)u_j$ if $p=2$.

PROOF. Let us define a morphism of resolutions

$$\begin{array}{ccccc} B^{\cdot}(\Lambda\otimes\Lambda, \Lambda) & \xrightarrow{\varepsilon} & \Lambda & \longrightarrow & 0 \\ \Psi^{\#}=\{\phi^{\#}_{-n}\}_{n\geqq 0}: \downarrow & & \| & & \\ \mathcal{F} & \xrightarrow{\mu} & \Lambda & \longrightarrow & 0\,. \end{array}$$

First define $\phi^{\#}_0: (\Lambda\otimes\Lambda)\otimes\Lambda\to\Lambda\otimes\Lambda$ by $\phi^{\#}_0(a\otimes b\otimes c)=a\otimes bc$. Then $\phi^{\#}_0$ is a morphism of $\Lambda\otimes\Lambda$-modules. From the definition of the external differential δ in the bar resolution, it follows that

$$(1.3)\qquad \begin{aligned} \delta(1\otimes 1[y_i\otimes 1-1\otimes y_i]1) &= y_i\otimes 1\otimes 1-1\otimes y_i\otimes 1 \quad \text{and} \\ \delta(1\otimes 1[x_j\otimes 1-1\otimes x_j]1) &= x_j\otimes 1\otimes 1-1\otimes x_j\otimes 1\,. \end{aligned}$$

We define $\Psi^{\#}_{-1}$ as follows.

$$(1.4)\qquad \phi^{\#}_{-1}(1\otimes 1[y_i\otimes 1-1\otimes y_i]1)=\nu_i \quad\text{and}\quad \phi^{\#}_{-1}(1\otimes 1[x_j\otimes 1-1\otimes x_j]1)=u_j.$$

From [5: Lemma 3.4], for any element $z\in\boldsymbol{K}_p[x_1, \cdots, x_n]$, there exist elements $\zeta_j\in\boldsymbol{K}_p[x_1, \cdots, x_n]\otimes\boldsymbol{K}_p[x_1, \cdots, x_n]$ $(1\leqq j\leqq n)$ such that $z\otimes 1-1\otimes z=\sum_{j=1}^n\zeta_j(x_j\otimes 1-1\otimes x_j)$ and $\mu(\zeta_j)=\partial z/\partial x_j$. We define that

$$(1.5)\qquad \phi^{\#}_{-1}(1\otimes 1[z\otimes 1-1\otimes z]1)=\sum_{j=1}^n\zeta_j u_j\,.$$

From (1.3), (1.4) and (1.5), $\phi^{\#}_{-1}$ can be defined in $(\overline{\Lambda\otimes\Lambda})\otimes\Lambda$ $(\subset B^{-1}(\Lambda\otimes\Lambda, \Lambda))$ extending the above $\phi^{\#}_{-1}$. Furthermore, we require that $\phi^{\#}_{-1}((a\otimes b)\otimes v)=(a\otimes b)\cdot\phi^{\#}_{-1}(v)$ for any $(a\otimes b)\otimes v\in\Lambda\otimes\Lambda\otimes(\overline{\Lambda\otimes\Lambda})\otimes\Lambda=B^{-1}(\Lambda\otimes\Lambda, \Lambda)$, where $v\in(\overline{\Lambda\otimes\Lambda})\otimes\Lambda$. Since δ, d and $\Psi^{\#}_0$ are morphisms of $\Lambda\otimes\Lambda$-modules, it follows that $d\phi^{\#}_{-1}=\phi^{\#}_0\delta$ and $\phi^{\#}_{-1}$ is a morphism of $\Lambda\otimes\Lambda$-modules. Finally, we obtain a morphism of resolutions $\Psi^{\#}$ extending $\phi^{\#}_0$ and $\phi^{\#}_{-1}$. Since $\Psi^{\#}$ induces the identity map of Λ, $\Psi^{\#}$ is a chain equivalence map, and so $1\otimes_{\Lambda\otimes\Lambda}\Psi^{\#}$ induces the required isomorphism. q. e. d.

From Proposition 1.1 and 1.2, it follows that a condition which $\partial\rho_i/\partial x_j=0$ in $H^*(X;\boldsymbol{K}_p)$ for any i and j does not depend on the choice of algebra generators x_j and generators of the ideal ρ_i.

The condition whether $\partial\rho_i/\partial x_j$ is zero in $H^*(X;\boldsymbol{K}_p)$ for each i and j is important for the collapse problem of the spectral sequence $\{\bar{E}_r, \bar{d}_r\}$, because we have the following two propositions.

PROPOSITION 1.6. *Suppose $\partial\rho_i/\partial x_j=0$ in $H^*(X):=H^*(X;\boldsymbol{K}_p)$ for any i and j, moreover $d_r(\langle\oplus_{n\geq 1}(QE_r^{**})^{-n,*}\rangle)\subset\langle\oplus_{n\geq 1}(QE_r^{**})^{-n,*}\rangle$ for any r in the spectral sequence $\{E_r, d_r\}$, where $\langle M\rangle$ denotes the subalgeba generated by M. Then $\{\bar{E}_r, \bar{d}_r\}$ collapses at the E_2-term.*

PROOF. From assumptions, we can define an isomorphism of spectral sequences $\{g_r\}:\{E_r, d_r\}\to\{\hat{E}_r\otimes H^*(X), \hat{d}_r\otimes 1\}$ by $g_r(a\otimes\lambda)=f_r(a)\otimes\lambda$ where $\{f_r\}$: $\{E_r, d_r\}\to\{\hat{E}_r, \hat{d}_r\}$ is the morphism of spectral sequences given by Lemma 1.3, $\lambda\in E_r^{0,*}\cong H^*(X)$, $a\in\langle\oplus_{n\geq 1}(QE_r^{**})^{-n,*}\rangle$ and $[\hat{E}_r\otimes H^*(X)]^{s,t}=\oplus_{u+v=t}E_r^{s,u}\otimes H^v(X)$. Therefore, we conclude that $E_\infty^{**}\cong\hat{E}_\infty^{**}\otimes H^*(X)$. Since the spectral sequence $\{E_r, d_r\}$ converges to $H^*(\Lambda X;\boldsymbol{K}_p)$ and the spectral sequence $\{\hat{E}_r, \hat{d}_r\}$ converges to $H^*(\Omega X;\boldsymbol{K}_p)$, it follows that $H^*(\Lambda X;\boldsymbol{K}_p)$ is isomorphic to $H^*(\Omega X;\boldsymbol{K}_p)\otimes H^*(X;\boldsymbol{K}_p)$ as a vector space. Hence $\{\bar{E}_r, \bar{d}_r\}$ collapses at the E_2-term. q. e. d.

PROPOSITION 1.7. (1) *Suppose that there exist integers i $(1\leqq i\leqq m)$ and j $(1\leqq j\leqq n)$ such that $\partial\rho_i/\partial x_j\neq 0$ in $\boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m)$ and $\hat{d}_r{}^{s,t}=0$ for any $r\geqq 2$, s and t; $s+t\leqq\deg\rho_i-2$. Then there exist integers $r(\geqq 2)$, s and t such that $s+t\leqq\deg\rho_i-2$ and $\bar{d}_r{}^{s,t}\neq 0$.*

(2) *Suppose that $\partial\rho_i/\partial x_j=0$ in $\boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m)$ for any i and j. Then the spectral sequence $\{E_r, d_r\}$ collapses at the E_2-term if and only if two spectral sequences $\{\hat{E}_r, \hat{d}_r\}$ and $\{\bar{E}_r, \bar{d}_r\}$ collapse at the E_2-term.*

Before we prove Proposition 1.7, let us define the Poincaré series and notations which will be used in its proof.

DEFINITION 1.8. If V is a graded vector space, we define the Poincaré series of V to be the formal power series

$$P(V, t)=\sum_{n=0}^{\infty}(\dim V^n)\cdot t^n\,.$$

DEFINITION 1.9. If V is a bigraded vector space, we define the Poincaré series of V to be the formal power series

$$P(V, t)=\sum_{n=0}^{\infty}(\dim \bigoplus_{i+j=n} V^{i,j})\cdot t^n .$$

DEFINITION 1.10. For power series $A=\sum_{n=0}^{\infty}a_n\cdot t^n$ and $B=\sum_{n=0}^{\infty}b_n\cdot t^n$ $(a_n, b_n\in \boldsymbol{Z})$, we call that A is less than B and denote it by $A<B$, if $a_i-b_i\leqq 0$ for any $i\geqq 0$, and there exists some integer n such that $a_n-b_n<0$.

NOTATION 1.11. For a power series $A=\sum_{n=0}^{\infty}a_n\cdot t^n$, put $A^{\leq N}=\sum_{n=0}^{N}a_n\cdot t^n$.

PROOF OF PROPOSITION 1.7. (1) Since $E_2^{**}\cong \mathrm{Tor}_{\Lambda\otimes\Lambda}^{**}(\Lambda, \Lambda)$ as an algebra, where $\Lambda=H^*(X;\boldsymbol{K}_p)$, by Proposition 1.1, we see that

$$E_2^{**}\cong H(\mathcal{E}, d) \quad \text{as an algebra.} \tag{1.6}$$

From the assumption that there exist integers i and j such that

$$\frac{\partial \rho_i}{\partial x_j}\neq 0 \qquad \text{in } \boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m),$$

it follows that $d(w_i)=\sum_{j=1}^{n}(\partial\rho_i/\partial x_j)u_j\neq 0$ in $\mathcal{E}$. Put $k(i)=\deg \rho_i-2$. From (1.6) and the above fact, we have an inequality:

$$P(E_2^{**}, t)^{\leq k(i)}<P(\mathcal{E}, t)^{\leq k(i)} .$$

Since the spectral sequence $\{E_r, d_r\}$ converges to $H^*(\Lambda X;\boldsymbol{K}_p)$,

$$P(H^*(\Lambda X;\boldsymbol{K}_p), t)\leqq P(E_\infty^{**}, t)\leqq \cdots \leqq P(E_2^{**}, t).$$

Therefore,

$$P(H^*(\Lambda X;\boldsymbol{K}_p), t)^{\leq k(i)}<P(\mathcal{E}, t)^{\leq k(i)} . \tag{1.7}$$

Next let us consider the spectral sequence $\{\hat{E}_r, \hat{d}_r\}$. Then $\hat{E}_2^{**}\cong \mathrm{Tor}_{\Lambda}^{**}(\boldsymbol{K}_p, \boldsymbol{K}_p)$. By Proposition 1.2, it follows that

$$\hat{E}_2^{**}\cong \Gamma[s^{-1}y_1, \cdots, s^{-1}y_l]\otimes\Lambda(s^{-1}x_1, \cdots, s^{-1}x_n)\otimes\Gamma[\tau\rho_1, \cdots, \tau\rho_m]=:\mathcal{K}$$

as an algebra.

We see that

$$P(H^*(\Omega X;\boldsymbol{K}_p), t)^{\leq k(i)}=P(\mathcal{K}, t)^{\leq k(i)} \tag{1.8}$$

because $\hat{d}_r^{s,t}=0$ for any $r(\geqq 2)$, s and t $(s+t\leqq \deg \rho_i-2)$, and the spectral sequence $\{\hat{E}_r, \hat{d}_r\}$ converges to $H^*(\Omega X;\boldsymbol{K}_p)$. Suppose $\bar{d}_r^{s,t}=0$ for any $r(\geqq 2)$, s and t $(s+t\leqq \deg \rho_i-2)$. Then

$$P(H^*(\Lambda X;\boldsymbol{K}_p), t)^{\leq k(i)}=P(H^*(X;\boldsymbol{K}_p)\otimes H^*(\Omega X;\boldsymbol{K}_p), t)^{\leq k(i)}$$

because $\bar{E}_2^{**}\cong H^*(X;\boldsymbol{K}_p)\otimes H^*(\Omega X;\boldsymbol{K}_p)$ and the spectral sequence $\{\bar{E}_r, \bar{d}_r\}$ converges to $H^*(\Lambda X;\boldsymbol{K}_p)$. Moreover

$$P(H^*(\Lambda X; \boldsymbol{K}_p), t)^{\leq k(i)} = [P(H^*(X; \boldsymbol{K}_p), t) \cdot P(H^*(\Omega X; \boldsymbol{K}_p), t)]^{\leq k(i)}$$

$$= [P(H^*(X; \boldsymbol{K}_p), t) \cdot P(\mathcal{K}, t)]^{\leq k(i)} \quad \text{(from (1.8))}$$

$$= P(H^*(X; \boldsymbol{K}_p) \otimes \mathcal{K}, t)^{\leq k(i)}$$

(Regard $H^*(X; \boldsymbol{K}_p)$ as a bigraded algebra by a bigrading such that bideg $\lambda = (0, \deg \lambda)$ for $\lambda \in H^*(X; \boldsymbol{K}_p)$.)

$$= P(\mathcal{E}, t)^{\leq k(i)} \quad \text{(see Proposition 1.1).}$$

This consequence contradicts (1.7). Hence we have Proposition 1.7 (1).

(2) If $\{E_r, d_r\}$ collapses at the E_2-term, from Proposition 1.6 and Lemma 1.3, it follows that $\{\hat{E}_r, \hat{d}_r\}$ and $\{\bar{E}_r, \bar{d}_r\}$ collapse at the E_2-term. The converse is obtained from Proposition 1.1 and 1.2. q.e.d.

§2. Proof of Theorem 2.

The proof of Theorem 2 is based on Proposition 1.7. We must examine whether the cohomology ring of the given space is a *GCI* algebra.

Let G be a compact, connected Lie group and H a maximal rank subgroup of G.

LEMMA 2.1 ([**1**; 6.3 Theorem]). *If $p \neq 0$ and $H^*(G)$, $H^*(H)$ have no p-torsion, or $p=0$, then $H^*(G/H; \boldsymbol{K}_p)$ is isomorphic to a GCI algebra $\boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_n)$ such that $\rho_i \neq 0$ for $1 \leqq i \leqq n$, where $n = \operatorname{rank} G$.* □

REMARK 2.2. For a *GCI* algebra $\Gamma = \boldsymbol{K}_p[x_1, \cdots, x_n]/(\rho_1, \cdots, \rho_m)$, suppose that there exists an indecomposable element ρ_i, and that $\rho_i = \sum \lambda_s x_s + W$, where W is decomposable and some λ_j is non-zero. Then Γ is isomorphic to a *GCI* algebra

$$\boldsymbol{K}_p[x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n]/(\tau_1, \cdots, \tau_{j-1}, \tau_{j+1}, \cdots, \tau_n),$$

where τ_s $(s \neq i)$ is an element that x_j was replaced with

$$-\sum_{s \neq j} \frac{\lambda_s}{\lambda_j} x_s - \frac{1}{\lambda_j} W \quad \text{in } \rho_S.$$

Using such replacements till all elements generating the ideal become decomposable, the algebra Γ may be regarded as a *GCI* algebra constructed from a polynomial algebra and an ideal consisting of decomposable elements.

As is known

(2.1) $$H^*(U(m+n)/U(m) \times U(n); \boldsymbol{K}_p)$$

$$\cong \boldsymbol{K}_p[c_1', \cdots, c_m', c_1, \cdots, c_n]/(\sum_{i+j=k} c_i' \cdot c_j; k \geqq 1),$$

$$\deg c_i = \deg c_i' = 2 \cdot i,$$

(2.2) $$H^*(Sp(m+n)/Sp(m)\times Sp(n);\boldsymbol{K}_p)$$
$$\cong \boldsymbol{K}_p[q_1', \cdots, q_m', q_1, \cdots, q_n]/(\sum_{i+j=k} q_i'\cdot q_j;\ k\geqq 1),$$
$$\deg q_i = \deg q_i' = 4\cdot i$$

and

(2.3) $$H^*(Sp(n)/U(n);\boldsymbol{K}_p)$$
$$\cong \boldsymbol{K}_p[c_1, c_2, \cdots, c_n]/(\sum_{i+j=2k}(-1)^i c_i\cdot c_j;\ k\geqq 1),$$
$$\deg c_i = 2\cdot i.$$

From Lemma 2.1, we see that the above algebras are *GCI* algebras. By Remark 2.2, the each algebra can be expressed as *GCI* algebra constructed from a polynomial algebra and an ideal which consists of decomposable elements. We consider this problem in the concrete.

LEMMA 2.3.

(1) *Suppose* $m\geqq n\geqq 2$. *Then* $H^*(U(m+n)/U(m)\times U(n);\boldsymbol{K}_p)\cong\boldsymbol{K}_p[c_1, \cdots, c_n]/(\rho_1, \cdots, \rho_n)$, *where* $\rho_1, \cdots, \rho_n\in\boldsymbol{K}_p[c_1, \cdots, c_n]$ *is a regular sequence,* ρ_i $(1\leqq i\leqq n)$ *is decomposable,* $\deg\rho_i=2m+2i$, *and*

$$\rho_j=\sum_{p_1+2p_2=m+j}(-1)^{p_1+p_2}\binom{p_1+p_2}{p_2}c_1^{p_1}\cdot c_2^{p_2}+W_j \qquad (j=1 \text{ or } 2);$$

W_j *consists of terms which include the factors* c_i $(3\leqq i\leqq n)$ *if* $n\geqq 3$, *and* $W_j=0$ *if* $n=2$.

(2) *Suppose* $m\geqq n\geqq 2$. *Then* $H^*(Sp(m+n)/Sp(m)\times Sp(n);\boldsymbol{K}_p)\cong\boldsymbol{K}_p[q_1, \cdots, q_n]/(\tau_1, \cdots, \tau_n)$, *where* $\tau_1, \cdots, \tau_n\in\boldsymbol{K}_p[q_1, \cdots, q_n]$ *is a regular sequence and* $\deg\tau_i=4m+4i$. *Moreover, if we replace* c_i *and* ρ_i *with* q_i *and* τ_i, *a similar equality to* (1) *holds.*

(3) *Suppose* $n\geqq 1$ *and* $p\neq 2$. *Then*

$$H^*(Sp(n)/U(n);\boldsymbol{K}_p)\cong\boldsymbol{K}_p[c_1, c_3, \cdots, c_{2[(n+1)/2]-1}]/(\mu_{[n/2]+1}, \cdots, \mu_n),$$

where $\deg\mu_i=4i$ $([n/2]+1\leqq i\leqq n)$,

$$\mu_{[n/2]+1}=\begin{cases} c_1^3c_{n-1}-2c_3c_{n-1}+W & \text{if } n \text{ is even and } n\geqq 4,\\ c_1^4 & \text{if } n=2,\end{cases}$$

and $\mu_{[n/2]+1}=-2c_1c_n+W'$ *if* n *is odd;* W *does not have terms which include the factor* c_{n-1} *and* W' *does not have terms which include the factor* c_n.

Suppose $n\geqq 1$ *and* $p=2$. *Then*

$$H^*(Sp(n)/U(n);\boldsymbol{K}_p)\cong\boldsymbol{K}_p[c_1, c_2, \cdots, c_n]/(c_1^2, \cdots, c_n^2).$$

PROOF. Put $R_k=\sum_{i+j=k}c_i'\cdot c_j$ $(k\geqq 1)$. From (2.1), the algebra $H^*(U(m+n)/U(m)\times U(n);\boldsymbol{K}_p)$ has exactly $m+n$ relations such that $R_k=0$. Since $R_k=0$ $(1\leqq k\leqq m)$, $c_1', \cdots, c_m'$ are inductively expressed with $c_1, \cdots, c_n$ in

$\boldsymbol{K}_p[c'_1, \cdots, c'_m, c_1, \cdots, c_n]/(R_k; k\geqq 1)$, that is, let $c'_j(c_1, \cdots, c_n)$ denote c'_j expressed with $c_1, \cdots, c_n$, then $c'_1(c_1, \cdots, c_n)=-c_1$ and

$$(2.4)\quad c'_t(c_1, \cdots, c_n) = \begin{cases} -c'_{t-1}(c_1, \cdots, c_n)c_1-c'_{t-2}(c_1, \cdots, c_n)c_2-\cdots-c'_{t-n}(c_1, \cdots, c_n)c_n & (t>n) \\ -c'_{t-1}(c_1, \cdots, c_n)c_1-c'_{t-2}(c_1, \cdots, c_n)c_2-\cdots-c'_1(c_1, \cdots, c_n)c_{t-1}-c_t & (t\leqq n). \end{cases}$$

Therefore, R_{m+i} $(1\leqq i\leqq n)$ are expressed with $c_1, \cdots, c_n$. Let ρ'_i mean R_{m+i} expressed with $c_1, \cdots, c_n$. $H^*(U(m+n)/U(m)\times U(n); \boldsymbol{K}_p)$ is isomorphic to a *GCI* algebra $\boldsymbol{K}_p[c_1, \cdots, c_n]/(\rho'_1, \cdots, \rho'_n)$ by Remark 2.2.

In order to prove (1), it suffices to prove (1) in the case where $n=2$. From (2.4), we see that $c'_t(c_1, c_2)=\sum_{p_1+2p_2=t}(-1)^{p_1+p_2}\binom{p_1+p_2}{p_2}c_1^{p_1}\cdot c_2^{p_2}$ by induction on integers t. Since $\rho'_1=c'_m(c_1, c_2)c_1+c'_{m-1}(c_1, c_2)c_2$ and $\rho'_2=c'_m(c_1, c_2)c_2$, it follows that

$$\rho'_1=\sum_{p_1+2p_2=m+1}(-1)^{p_1+p_2+1}\binom{p_1+p_2}{p_2}c_1^{p_1}\cdot c_2^{p_2}$$

and

$$\rho'_2=\sum_{p_1+2p_2=m}(-1)^{p_1+p_2}\binom{p_1+p_2}{p_2}c_1^{p_1}\cdot c_2^{p_2+1}.$$

Put $\rho_2=c_1\rho'_1-\rho'_2$ and $\rho_1=-\rho'_1$. We have (1). Similarly, we get (2).

(3) Put $R_k=\sum_{i+j=2k}(-1)^i c_i c_j$. It follows that

$$(2.5)\qquad R_k=\begin{cases} 2c_{2k}-2c_1c_{2k-1}+2c_2c_{2k-2}+\cdots+(-1)^k c_k^2 & (2k\leqq n) \\ (-1)^{2k-n}2c_{2k-n}c_n+(-1)^{2k-n+1}2c_{2k-n+1}c_{n-1}+\cdots \\ \qquad\qquad +(-1)^k c_k^2 & (2n\geqq 2k>n) \end{cases}$$

Using (2.5), we can inductively replace elements c_{2k} $(2k\leqq n)$ with $c_1, c_3, \cdots, c_{2[(n+1/2)]-1}$. Let μ_i $([n/2]+1\leqq i\leqq n)$ mean R_i expressed with $c_1, c_3, \cdots, c_{2[(n+1/2)]-1}$. By the replacement, we have (3). q.e.d.

PROOF OF THEOREM 2. In order to examine whether ΩX is totally non-homologous to zero in ΛX with respect to $\boldsymbol{K}_p$, it suffices to examine it with respect to $\boldsymbol{Z}/p$.

(1), (2) and (3). When $n+1\equiv 0 \bmod p$, applying Proposition 1.1, it follows that $E_2^{**}(\boldsymbol{CP}(n))\cong \mathrm{Tor}^{**}_{\Lambda\otimes\Lambda}(\Lambda, \Lambda)\cong \boldsymbol{K}_p[x_2]/(x_2^{n+1})\otimes\Lambda(u)\otimes\Gamma[w]$ as an algebra, where $\Lambda=H^*(\boldsymbol{CP}(n); \boldsymbol{K}_p)$, bideg $x_2=(0, 2)$, bideg $u=(-1, 2)$ and bideg $w=(-2, 2(n+1))$.

If there exist an element $\gamma_{p^s}(w)$ and an integer r such that $d_r(\gamma_{p^s}(w))\neq 0$, then $d_r(\gamma_{p^s}(w))$ is decomposable and the factors have an element u. We can consider as follows:

$$d_r(\gamma_{p^s}(w))=\lambda u\cdot\gamma_{p^{s-1}}(w)^{k(1)}\cdot\gamma_{p^{s-2}}(w)^{k(2)}\cdots w^{k(s)}\cdot x_2^t+\cdots,$$

where $0\leqq t\leqq n$ and $\lambda\neq 0$. Comparing the total degrees, we have

(2.6) $$2n\cdot p^s+1=1+2n\cdot k(1)\cdot p^{s-1}+2n\cdot k(2)\cdot p^{s-2}+\cdots+2n\cdot k(s)+2t.$$

Comparing the filtration degrees, we get an inequality: $2\cdot p^s-1>1+2\cdot k(1)\cdot p^{s-1}+\cdots+2\cdot k(s)$ and so $2n\cdot p^s>2n+2n\cdot k(1)\cdot p^{s-1}+\cdots+2n\cdot k(s)$. Since $n\geqq 1$ and $0\leqq t\leqq n$, the above inequality contradicts the equality (2.6). Hence this spectral sequence $\{E_r(\boldsymbol{CP}(n)), d_r(\boldsymbol{CP}(n))\}$ collapses at the E_2-term. From Proposition 1.7 (2), we can conclude that $p\in T(\boldsymbol{CP}(n))$ if $n+1\equiv 0 \bmod p$. Suppose that $n+1\not\equiv 0 \bmod p$. Using the differential Hopf algebra structure in $\{\hat{E}_r(\boldsymbol{CP}(n)), \hat{d}_r(\boldsymbol{CP}(n))\}$, it follows that $\{\hat{E}_r(\boldsymbol{CP}(n)), \hat{d}_r(\boldsymbol{CP}(n))\}$ collapses at the E_2-term. From Proposition 1.7 (1), $p\notin T(\boldsymbol{CP}(n))$ if $n+1\not\equiv 0 \bmod p$. Hence the proof of Theorem 2 (1) is complete. Similarly, we have (2) and (3).

(4) and (5). By Lemma 2.3, we have

$$\rho_1=-c_1^{2t+1}+(-1)^{t+1}(t+1)c_1c_2^t+2tc_1^{2t-1}c_2+\cdots \quad \text{if } m=2t,$$

and

$$\rho_1=c_1^{2t+2}-(2t+1)c_1^{2t}c_2+(-1)^{t+1}c_2^{t+1}+\cdots \quad \text{if } m=2t+1.$$

Hence, in $H^*(U(m+n)/U(m)\times U(n); \boldsymbol{K}_p)$, $\partial\rho_1/\partial c_1=-(2t+1)c_1^{2t}+(-1)^{t+1}(t+1)c_2^t+2tc_1^{2t-2}c_2+\cdots\neq 0$ for any p if $m=2t$, $\partial\rho_1/\partial c_1=(2t+2)c_1^{2t+1}-2t(2t+1)c_1^{2t-1}c_2+\cdots\neq 0$ for any odd prime p, and $\partial\rho_1/\partial c_2=-(2t+1)c_1^{2t}+(-1)^{t+1}(t+1)c_2^t+\cdots\neq 0$ for $p=2$, if $m=2t+1$. Moreover the spectral sequence $\{\hat{E}_r(U(m+n)/U(m)\times U(n)), \hat{d}_r(U(m+n)/U(m)\times U(n))\}$ satisfies the condition of Proposition 1.7 (1). Therefore we have Theorem 2 (4). Similarly, we get (5).

(6) Suppose $p=2$. From Lemma 2.3 (3), Theorem 3 can be applied in this case. We see that $2\in T(Sp(n)/U(n))$. Suppose $p\neq 2$. By Lemma 2.3 (3), we obtain that

$$\frac{\partial(\mu_{[n/2]+1})}{\partial c_{n-1}}=\begin{cases} c_1^3-2c_3\neq 0 & \text{if } n \text{ is even, and } n\geqq 4,\\ 4c_1^3\neq 0 & \text{if } n=2,\end{cases}$$

and

$$\frac{\partial(\mu_{[n/2]+1})}{\partial c_n}=-2c_1\neq 0 \quad \text{if } n \text{ is odd,}$$

in $H^*(Sp(n)/U(n); \boldsymbol{K}_p)$. Since the spectral sequence $\{E_r(Sp(n)/U(n)), d_r(Sp(n)/U(n))\}$ collapses at the E_2-term, applying Proposition 1.7 (1), it follows that $p\notin T(Sp(n)/U(n))$.

(7) As is known

$$H^*(SO(m+n)/SO(n);\boldsymbol{K}_p)$$
$$\cong\begin{cases}\Lambda(e_{2n+3}, e_{2n+7}, \cdots, e_{2n+2m-3})\otimes\boldsymbol{K}_p[x_n]/(x_n^2) \\ \qquad\text{if } n \text{ is even and } m \text{ is odd} \\ \Lambda(e_{2n+3}, e_{2n+7}, \cdots, e_{2n+2m-5}, e'_{n+m-1})\otimes\boldsymbol{K}_p[x_n]/(x_n^2) \\ \qquad\text{if } n \text{ and } m \text{ are even,}\end{cases}$$

where $p\neq 2$. From Proposition 1.7 (1), it follows that $p\notin T(SO(m+n)/SO(n))$ for any odd prime p.

(8), (9) and (10). Suppose n is odd and $p\neq 2$. The morphism $(\Omega\pi)^*: QH^*(\Omega(SO(m+n)/SO(n));\boldsymbol{K}_p)\to QH^*(\Omega(\mathrm{Spin}(m+n));\boldsymbol{K}_p)$ is monomorphic where π is composite of the universal covering $\tilde{\pi}:\mathrm{Spin}(m+n)\to SO(m+n)$ and the natural projection $\tilde{p}: SO(m+n)\to SO(m+n)/SO(n)$. Since $\Omega(\mathrm{Spin}(m+n))$ is totally non-homologous to zero in $\Lambda(\mathrm{Spin}(m+n))$ with respect to $\boldsymbol{K}_p$ (see [**6**; Lemma 3]), comparing $\{\bar{E}_r(SO(m+n)/SO(n)), \bar{d}_r(SO(m+n)/SO(n))\}$ with $\{\bar{E}_r(\mathrm{Spin}(m+n)), \bar{d}_r(\mathrm{Spin}(m+n))\}$, it follows that the spectral sequence $\{\bar{E}_r(SO(m+n)/SO(n)), \bar{d}_r(SO(m+n)/SO(n))\}$ collapses at the E_2-term. Therefore $p\in T(SO(m+n)/SO(n))$ for any odd prime p if n is odd. Similarly, we have (9) and (10). q.e.d.

§ 3. Proof of Theorem 4.

L. Smith [**6**] has prove Theorem 3 by comparing $\{E_r(X), d_r(X)\}$ with $\{E_r(P_k), d_r(P_k)\}$, where the space P_k is in the $(k+1)$-stage Postnikov system which D. Kraines has given in [**3**]. We prove Theorem 4 with the same method.

REMARK 3.1. Consider the $(k+1)$-stage Postnikov system [**3**; Theorem 4.2] which D. Kraines has given:

(3.1)
$$\begin{array}{ccccc}
K(\boldsymbol{Z}/2, 2^k m+1) & \hookrightarrow & P_k & & \\
 & & \downarrow \pi_k & & \\
 & & P_{k-1} & \xrightarrow{\lambda_{k-1}} & K(\boldsymbol{Z}/2, 2^k m+2) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
K(\boldsymbol{Z}/2, 2m+1) & \hookrightarrow & P_1 & \xrightarrow{\lambda_1} & K(\boldsymbol{Z}/2, 4m+2) \\
 & & \downarrow \pi_1 & & \\
K(\boldsymbol{Z}/2, t) = & & P_0 & \xrightarrow{\lambda_0} & K(\boldsymbol{Z}/2, 2^u t),
\end{array}$$

where $m=2^{u-1}t-1$. Using the fact P_s is a H-space and considering the mod 2 Eilenberg-Moore spectral sequence of the fiber square

$$\begin{array}{ccc} P_s & \longrightarrow & PK(\boldsymbol{Z}/2, 2^{s+1}m+2) \\ \pi_s \downarrow & & \downarrow \\ P_{s-1} & \xrightarrow{\lambda_{s-1}} & K(\boldsymbol{Z}/2, 2^s m+2) \end{array} \quad \text{and} \quad \{\hat{E}_r(P_s), \hat{d}_r(P_s)\},$$

we see that each P_s ($s\geqq 1$) satisfies the following ([2], [3]):

(3.2) $$H^*(P_s; \boldsymbol{Z}/2) \cong \boldsymbol{Z}_2[\iota_t]/(\iota_t^{2^u})\otimes \boldsymbol{Z}/2[s^{-1}Sq^{2^s m}\iota_{2^s m+2}, s^{-1}Sq^1Sq^{2^s m}\iota_{2^s m+2}]\otimes \text{Poly},$$
$$Sq^1(s^{-1}Sq^{2^s m}\iota_{2^s m+2}) = s^{-1}Sq^1Sq^{2^s m}\iota_{2^s m+2},$$

where Poly is an appropriate polynomial algebra.

(3.3) $$\pi_s^*(\iota_t) = \iota_t \quad \text{and} \quad \lambda_s^*(\iota_{2^{s+1}m+2}) = s^{-1}Sq^1Sq^{2^s m}\iota_{2^s m+2}.$$

(3.4) In the spectral sequence $\{\hat{E}_r(P_s), \hat{d}_r(P_s)\}$, $\hat{d}_r(\gamma_{2^i}(\tau(\iota_t^{2^u}))$ is zero for any $0\leqq i\leqq s-1$ and r. Moreover, the elements $\gamma_{2^s}(\tau(\iota_t^{2^u}))$ and $s^{-1}(s^{-1}Sq^1Sq^{2^s m}\iota_{2^s m+2})$ survive to the $E_{2^{s+1}-1}$-term and $\hat{d}_{2^{s+1}-1}(P_s)(\gamma_{2^s}(\tau(\iota_t^{2^u}))=s^{-1}(s^{-1}Sq^1Sq^{2^s m}\iota_{2^s m+2})$.

In order to prove Theorem 4, it suffices to show that $\Omega(SO(m+n)/SO(n))$ is totally non-homologous to zero in $\Lambda(SO(m+n)/SO(n))$ with respect to $\boldsymbol{Z}/2$. Let all coefficients of cohomologies be the field $\boldsymbol{Z}/2$ and spectral sequences in the field $\boldsymbol{Z}/2$.

PROOF OF THEOREM 4. For any $j\in J\cup J'$, choose a map $g_j: SO(k+n)/SO(n) \to K(\boldsymbol{Z}/2, j)=P_0$ such that $g_i^*(\iota_j)=x_j$. In (3.1), put $u=r_j$. Then it follows that g_j has a lift $g_{j,1}$ in P_1. Suppose g_j has a lift $g_{j,s}$ up to P_s and does not have a lift in P_{s+1}. Put $V_{k,n}=SO(k+n)/SO(n)$. Let $\{\bar{g}_r\}: \{E_r(P_s), d_r(P_s)\}\to \{E_r(V_{k,n}), d_r(V_{k,n})\}$ be the morphism of spectral sequences induced by $g_{j,s}$ and $\{f_r\}: \{E_r(V_{k,n}), d_r(V_{k,n})\}\to\{E_r(\text{Spin}(k+n)), d_r(\text{Spin}(k+n))\}$ the morphism of spectral sequences induced by π. From Lemmas 1.4, 1.5 and Remark 3.1, we have

$$\begin{aligned} d_{2^{s+1}-1}(V_{k,n})(\gamma_{2^s}(w_j)) &= d_{2^{s+1}-1}(V_{k,n})\bar{g}_{2^{s+1}-1}(\gamma_{2^s}(\tilde{w}_j)) \\ &= \bar{g}_{2^{s+1}-1}d_{2^{s+1}-1}(P_s)(\gamma_{2^s}(\tilde{w}_j)) = \bar{g}_{2^{s+1}-1}([\alpha\otimes 1-1\otimes\alpha]) \\ &= [g_{j,s}{}^*(\alpha)\otimes 1-1\otimes g_{j,s}{}^*(\alpha)], \end{aligned}$$

where w_j and $\tilde{w}_j$ are the algebra generators in $E_2(V_{k,n})$ and $E_2(P_s)$ associated with x_j and ι_j respectively, and $\alpha=s^{-1}Sq^1Sq^{2^s m}\iota_{2^s m+2}$; $m=j\cdot 2^{r_j-1}-1$. Moreover,

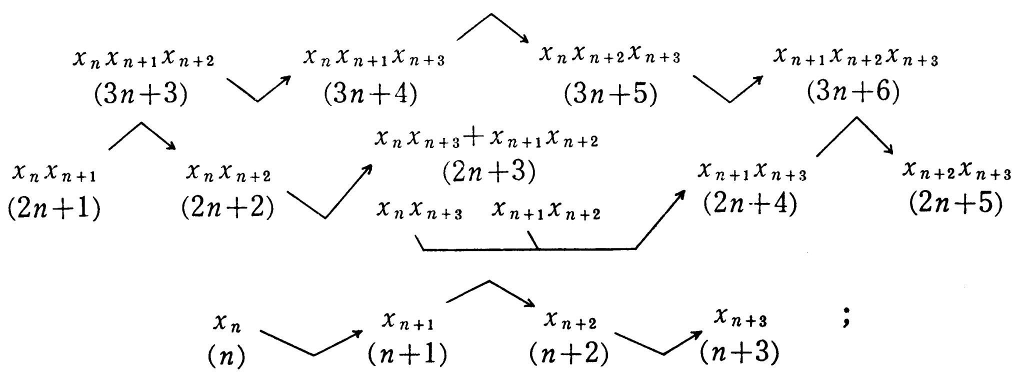
$$\begin{aligned} d_{2^{s+1}-1}(\text{Spin}(k+n))f_{2^{s+1}-1}(\gamma_{2^s}(w_j)) &= f_{2^{s+1}-1}([g_{j,s}{}^*(\alpha)\otimes 1-1\otimes g_{j,s}{}^*(\alpha)]) \\ &= [\pi^*g_{j,s}{}^*(\alpha)\otimes 1-1\otimes\pi^*g_{j,s}{}^*(\alpha)]. \end{aligned}$$

Since g_j does not have a lift in P_{s+1}, it follows that $g_{j,s}^*(\alpha)\neq 0$. From the assumption, there exists an integer i such that $\partial(\pi^*g_{j,s}{}^*(\alpha))/\partial x_i\neq 0$ in

$H^*(\mathrm{Spin}(k+n); \boldsymbol{Z}/2)$. Applying Lemma 1.5, we have that $d_{2^{s+1}-1}(\mathrm{Spin}(k+n)) \neq 0$, which contradicts Proposition 1.7 (2). Therefore, g_j has a lift $g_{j,s}$ in P_s for any j and s. Using the same argument as the proof of [**6**; Theorem], we see that $\{E_r(V_{k,n}), d_r(V_{k,n})\}$ collapses at the E_2-term. From Proposition 1.7 (2), it follows that $2 \in T(SO(k+n)/SO(n))$. q.e.d.

Proof of Corollary 5. (1) It suffices to show that $2 \in T(SO(4+n)/SO(n))$. In fact, since $\tilde{i}^*: QH^*(SO(4+n)/SO(n); \boldsymbol{Z}/2) \to QH^*(SO(m+n)/SO(n); \boldsymbol{Z}/2)$ is epimorphic, where $m \leqq 4$ and $\tilde{i}$ is the map induced by the inclusion map $i: SO(m+n) \hookrightarrow SO(4+n)$, it follows that $2 \in T(SO(m+n)/SO(n))$ for any $1 \leqq m \leqq 4$, by applying the Leray-Serre spectral sequence (or Proposition 1.7 (2)).

Now let us prove that $2 \in T(SO(4+n)/SO(n))$. We have the following table of the behavior of Sq^1 in $H^*(SO(4+n)/SO(n); \boldsymbol{Z}/2) \cong \Delta(x_n, x_{n+1}, x_{n+2}, x_{n+3})$:



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the upper arrows mean the behavior of Sq^1 when n is even and the lower arrows mean those when n is odd.

Suppose $n \geqq m = 4$. Let s be the dimensions of V such that V^s may be non-zero. Since $3n+6$ is the largest of the dimensions i such that $(\mathrm{Im}\, Sq^1)^i$ is not trivial, s is equal to $n \cdot 2^2 - 2^2 + 2$ or $(n+1) \cdot 2^2 - 2^2 + 2$. From the above table, we see that $V^{4n-2} = V^{4n+2} = \{0\}$. When $n = 3$ or 2, we can verify that $V = \{0\}$ with concrete computation. Applying Theorem 4, the proof is completed.

(2) Suppose $n \geqq m$. Then

$$\{s = j \cdot 2^{1+k} - 2^{k+1} + 2 \mid j \in J \cup J',\ k \geqq 1\}$$
$$\cap \{j_1 + \cdots + j_t \mid j_1 < \cdots < j_t,\ j_i \in J'\} = \emptyset$$

if and only if there exists an integer j such that $\partial \pi^* \rho / \partial x_j \neq 0$ in $H^*(\mathrm{Spin}(m+n); \boldsymbol{Z}/2)$ for any non-zero element $\rho \in V$. Therefore, we have the former of this corollary by applying Theorem 4. The latter is obtained by solving an inequality $\sum_{j_i \in J'} j_i < n \cdot 2^2 - 2^2 + 2$. q.e.d.

Remark 3.2. When $m = 5$ and $n = 8$, we see that $Sq^1(x_8 x_{10} x_{11}) = x_8 x_{10} x_{12}$, $x_8 x_{10} x_{12} \in (\mathrm{Im}\, Sq^1)^{8 \cdot 2^2 - 2^2 + 2} \subset V$ and that $\pi^*(x_8 x_{10} x_{12}) = x_4^2 x_5^2 x_6^2$ in $H^*(\mathrm{Spin}(13); \boldsymbol{Z}/2)$.

Hence we have that $\partial(x_4^2x_5^2x_6^2)/\partial x_j=0$ for any free algebra generator x_j in $H^*(\mathrm{Spin}(13); \boldsymbol{Z}/2)$, so we can not apply Theorem 4 in this case.

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