

On the Kervaire classes of homotopy real projective spaces

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The purpose of this paper is to study the mutual dependence of Kervaire classes for normal maps of a real projective space P^n .

A smooth free involution T on a homotopy sphere Σ^n defines a homotopy equivalence $\varphi: \Sigma^n/T \rightarrow P^n$ whose normal invariant is denoted by $\nu(\varphi) \in [P^n, F/O]$. By restricting $\nu(\varphi)$ to a subspace $P^m \subset P^n$, we obtain a surgery obstruction that lies in Wall's surgery obstruction group $L_m(\mathbf{Z}/2, (-1)^{m+1})$ which is isomorphic to $\mathbf{Z}/2$ unless $m \equiv 1 \pmod{4}$ ([12], 13A). Suppose that M^m is an even dimensional smooth manifold and let $f \in [M, F/O]$ be a normal map. Then the Kervaire obstruction for f is given by the Sullivan's characteristic variety formula ([10]) as follows:

$$c(f) = \langle V(M)^2 \sum_i f^* H^* K_{2i}, [M] \rangle,$$

where $V(M)$ is the total Wu class of M^m , $[M]$ is the mod 2 fundamental homology class in $H^m(M^m, \mathbf{Z}/2)$, $K_{2i} \in H^{2i}(F/TOP, \mathbf{Z}/2)$ is the Sullivan-Kervaire class and $H: F/O \rightarrow F/TOP$ is the natural map.

When m is even, the formula above enables us to write down the surgery obstruction for $\nu(\varphi)|_{P^m}$ in terms of the Kervaire classes of $\nu(\varphi)$. Giffen, in his works on Brieskorn involutions on homotopy spheres bounding parallelizable manifolds ([4], [5]), showed that in these examples all the Kervaire classes in different degrees (up to the dimension of the manifold) are either all zero or all nonzero. So we may ask if simultaneous vanishing or non-vanishing of the Kervaire classes occurs for arbitrary free involutions on a homotopy n -sphere Σ^n that bound a parallelizable $(n+1)$ -manifold when $n \equiv 1 \pmod{4}$.

Another motivation for the present work comes from the problem of Dovermann, Masuda and Schultz ([3], 4.12). They ask for a reasonable estimate of $M(q)$ such that the restriction map

$$[CP^m, \text{Cok } J_{(2)}] \longrightarrow [CP^q, \text{Cok } J_{(2)}]$$

is trivial for all $m \geq M(q)$. Actually they proved

THEOREM ([3], 4.8). *Let $i+1=2^N$. Then there exists an M such that the*

Kervaire class of degree $4i+2$ vanishes for any $f \in [CP^m, \text{Cok } J_{(2)}]$ if $m \geq M$.

Their result was improved and completed by Stolz ([11]). He proved that M can be taken to be $3 \cdot 2^N - 1$. In [3], there is a question if similar statements hold even when CP^n is replaced by P^n . But unlike the case of CP^n , since the J -map $KO(P^n) \rightarrow JO(P^n)$ is injective, the stable vector bundle of the surgery data for P^n alone, regardless of the bundle map, does not determine the Kervaire classes. So a statement similar to Theorem I of [11] is not possible. However, we can prove an analogue of Theorem II of [11].

THEOREM A. *For any $f \in [P^n, \text{Cok } J_{(2)}]$, we have*

$$\lambda^* H^*(K_{2^{r+1}-2}) = 0,$$

if $n \geq 3 \cdot 2^r - 2$. Here $\lambda: \text{Cok } J_{(2)} \rightarrow F/O_{(2)}$ and $H: F/O_{(2)} \rightarrow F/TOP_{(2)}$ are natural maps.

Let $B \subset [P^n, F/O]$ be the set of normal cobordism classes given by the Brieskorn involutions. In the exact sequence

$$[P^n, \text{Cok } J] \longrightarrow [P^n, F/O] \longrightarrow [P^n, BSO^{\otimes}] \longrightarrow 0,$$

B is mapped surjectively onto $[P^n, BSO]$ ([5]). Hence the set B and the image of $[P^n, \text{Cok } J]$ generate the whole set $[P^n, F/O]$ using the H -space structure of F/O . For each element of B , the Kervaire classes of different degrees either vanish or do not vanish simultaneously and as to an element coming from $[P^n, \text{Cok } J]$, all the Kervaire classes $H^*(K_{2^{r+1}-2})$ for $3 \cdot 2^r - 2 \leq n$ always vanish. Thus Theorem A implies

THEOREM B. *For a normal map $f \in [P^n, F/O]$, the following conditions are equivalent if $n \geq 3 \cdot 2^r - 2$.*

- a) $f^* H^*(K_2) = 0$
- b) $f^* H^*(K_{2^{r+1}-2}) = 0$.

The assumption for n is best possible. In fact, when $r=2$ and $n=9$, there exists $f \in [P^9, F/O]$ with $f^* H^*(K_2) = 0$ and $f^* H^*(K_6) \neq 0$ ([7]).

We shall give a geometric application of Theorem B. Let n be an integer such that $n \equiv 1 \pmod{4}$ and neither $n-1$ nor $n+3$ is a power of 2. Consider a smooth free involution T on a homotopy n -sphere Σ^n bounding a parallelizable manifold. Then we have a homotopy equivalence $\varphi: \Sigma/T \rightarrow P^n$. The normal map $\nu(\varphi)$ of φ extends to a normal map $f: P^{n+1} \rightarrow F/O$ since Σ bounds a parallelizable manifold. Then Σ^n is diffeomorphic to the standard (resp. Kervaire) sphere if and only if the surgery obstruction for f is zero (resp. nonzero). Let e be the 2-order of $n+3$, i.e. $n+3 = 2^e(2s+1)$ for some integer s . By the characteristic variety formula, the surgery obstruction of f is zero (resp. non-

zero) if and only if the class $f^*H^*K_{2^{e-2}}$ is zero (resp. nonzero). From the assumption that $n+3$ is not a power of 2, we have $n \geq 3 \cdot 2^{e-1} - 2$. It is known that the Browder-Livesay desuspension obstruction for the involution T coincides with the surgery obstruction for $\nu(\varphi)|P^{n-3}$ ([8], III.3). The vanishing of this obstruction is equivalent to the vanishing of the class $f^*H^*K_{2^{q-2}}$, where q is the 2-order of $n-3$. And by the assumption, $n \geq 3 \cdot 2^{q-1} - 2$ also holds. Theorem B implies that the two classes $f^*H^*K_{2^{e-2}}$ and $f^*H^*K_{2^{q-2}}$ vanish or do not vanish simultaneously. Thus we have deduced

COROLLARY C. *Let $n \equiv 1 \pmod 4$ and suppose that neither $n-1$ nor $n+3$ is a power of 2. Then any smooth free involution on the standard sphere S^n desuspends. Equivalently any free involution on the Kervaire n -sphere does not desuspend.*

Before we proceed further, let us fix the notations that will be used in later sections. First we have the fibrations of classifying spaces:

$$SF_{(2)} \xrightarrow{\pi} F/O_{(2)} \xrightarrow{V} BSO_{(2)},$$

$$\text{Cok } J_{(2)} \xrightarrow{\lambda} F/O_{(2)} \xrightarrow{\beta} BSO_{(2)}^{\otimes 2}$$

and

$$\text{Im } J_{(2)} \xrightarrow{\phi^3-1} BSO_{(2)}^{\otimes 2} \xrightarrow{\phi^3-1} BSO_{(2)}.$$

These sequences are combined to give the commutative diagram

$$\begin{array}{ccccc} \text{Cok } J_{(2)} & \xlongequal{\quad} & \text{Cok } J_{(2)} & & \\ \downarrow \lambda' & & \downarrow \lambda & & \\ SF_{(2)} & \xrightarrow{\pi} & F/O_{(2)} & \xrightarrow{V} & BSO_{(2)} \\ \downarrow e & & \downarrow \beta & & \parallel \\ \text{Im } J_{(2)} & \longrightarrow & BSO_{(2)}^{\otimes 2} & \xrightarrow{\phi^3-1} & BSO_{(2)}. \end{array}$$

Let $K_{4i+2} \in H^{4i+2}(F/TOP, \mathbf{Z}/2)$ be the Kervaire class. The image of K_{4i+2} by the map $H^*: H^{4i+2}(F/TOP, \mathbf{Z}/2) \rightarrow H^{4i+2}(F/O, \mathbf{Z}/2)$ will be denoted by k_{4i+2} , which is known to vanish unless $i+1$ is a power of 2 ([2]). We shall write $\tilde{k}_{4i+2} = \lambda^*(k_{4i+2}) \in H^{4i+2}(\text{Cok } J_{(2)}, \mathbf{Z}/2)$. In degree 2, $\tilde{k}_2 = 0$ since $\text{Cok } J_{(2)}$ is 5-connected. We shall always work in the 2-local category and all the cohomology coefficients are $\mathbf{Z}/2$ and will be omitted. For a positive integer m , let $\alpha(m)$ be the number of 1's in the dyadic expansion (binary expression) of m , and let $\text{ord}_2(m)$ denote the 2-order of m , i.e. $r = \text{ord}_2(m)$ if $m = 2^r \cdot (2s+1)$ for some integer s . Let \mathcal{A} be the mod 2 Steenrod algebra and its augmentation ideal composed of the set of elements of positive degrees is denoted by $I(\mathcal{A})$.

§ 2. Reduction of the problem.

We shall reduce the proof of Theorem A following the line of Stolz [11]. Let $f \in [P^n, \text{Cok } J_{(2)}]$ and consider the oriented (2-local) stable spherical fibration η_f over the reduced suspension of the real projective space ΣP^n that has $\lambda' \circ f$ as the characteristic map. Then we have

PROPOSITION 1. *The Stiefel-Whitney class $w_i(\eta_f)$ vanishes for all $i > 0$.*

This is equivalent to saying that the Thom class $U(\eta_f)$ of the spherical fibration η_f is annihilated by all elements of $I(\mathcal{A})$. In particular, Proposition 1 implies that the Adams' secondary cohomology operation $\Phi_{i,j}$ is defined for $U(\eta_f)$.

PROPOSITION 2. *For all r satisfying $3 \cdot 2^r - 2 \leq n$, $\Phi_{r,r}(U(\eta_f))$ vanishes with zero total indeterminacy.*

These two propositions together with the following key result of Hegenbarth and Heil give our result.

THEOREM H-H ([6]). *Let $h: \Sigma SF \rightarrow BSF$ be the adjoint of the identity map of SF and let $\tilde{p}: BSF \sim \rightarrow BSF$ be the fibration induced by $\prod_{i>0} w_i: BSF \rightarrow \prod_{i>0} K(\mathbf{Z}/2, i)$. Then there exists a class $\varepsilon_{r+1} \in H^{2^r+1-1}(BSF, \mathbf{Z}/2)$ such that*

- (1) $\tilde{p}^*(\varepsilon_{r+1}) \cup U(\tilde{\gamma}) = \Phi_{r,r}(U(\tilde{\gamma}))$ mod zero indeterminacy and
- (2) $h^*(\varepsilon_{r+1}) = \sigma \pi^*(k_{2^r+1-2})$,

where $\tilde{\gamma}$ is the pull-back of the universal spherical fibration γ over BSF to $BSF \sim$ by \tilde{p} and σ is the usual suspension.

PROOF OF THEOREM A FROM PROPOSITIONS 1 AND 2. Let $f \in [P^n, \text{Cok } J_{(2)}]$. Then the composition

$$\Sigma P^n \xrightarrow{\Sigma(\lambda' \circ f)} \Sigma SF \xrightarrow{h} BSF$$

classifies η_f . From Proposition 1, we have a lift $\tilde{f}: \Sigma P^n \rightarrow BSF \sim$ of $h \circ \Sigma(\lambda' \circ f)$ and the map $M(\tilde{f}): M(\eta_f) \rightarrow M(\tilde{\gamma})$ of Thom spaces. Then we have

$$\begin{aligned} \Phi_{r,r}(U(\eta_f)) &= M(\tilde{f})^* \Phi_{r,r}(U(\tilde{\gamma})) = M(\tilde{f})^*(\tilde{p}^* \varepsilon_{r+1} \cup U(\tilde{\gamma})) \\ &= \tilde{f}^* \tilde{p}^* \varepsilon_{r+1} \cup U(\eta_f) = (\Sigma(\lambda' \circ f))^* h^* \varepsilon_{r+1} \cup U(\eta_f) \\ &= (\Sigma(\lambda' \circ f))^* \sigma \pi^* k_{2^r+1-2} \cup U(\eta_f) \\ &= \sigma f^* \lambda'^* \pi^* k_{2^r+1-2} \cup U(\eta_f) = \sigma f^*(\tilde{k}_{2^r+1-2}) \cup U(\eta_f). \end{aligned}$$

Thus it follows that $\Phi_{r,r}(U(\eta_f))=0$ if and only if $f^*(\tilde{k}_{2^r+1-2})=0$. Therefore from Proposition 2, the proof of Theorem A is complete.

§ 3. The vanishing of the Stiefel-Whitney classes.

Let $x \in H^1(P^n)$ be the cohomology generator. The following lemma is immediate from the formula

$$Sq^i(x^j) = \binom{j}{i} x^{i+j}.$$

LEMMA 3. Let $i < j$ and $b \in A$ be of degree $j - i$. Then b maps $H^i(P^n)$ to zero if $\alpha(i) < \alpha(j)$ or $\text{ord}_2(i) > \text{ord}_2(j)$ holds.

Let $f \in [P^n, \text{Cok } J_{(2)}]$. Since $\text{Cok } J_{(2)}$ is 5-connected, it is clear that $w_i(\eta_f) = 0$ for $i < 6$.

LEMMA 4. For any $f \in [P^n, \text{Cok } J_{(2)}]$, $w_8(\eta_f)$ vanishes.

PROOF. $H^7(\text{Cok } J_{(2)})$ is generated by $Sq^1 \tilde{k}_6$ (see e. g. [7]). Suppose that $w_8(\eta_f)$ is not zero. Then $(\lambda' \circ f)^* \sigma^*(w_8) = x^7$ where $\sigma^* : H^*(BSF) \rightarrow H^*(SF)$ is the cohomology suspension. Since $(\lambda')^* \sigma^* w_8$ must be nonzero, we have $(\lambda')^* \sigma^* w_8 = Sq^1 \tilde{k}_6$. On the other

$$f^* \lambda'^* \sigma^* w_8 = f^* Sq^1 \tilde{k}_6 = Sq^1 f^* \tilde{k}_6 = 0$$

since $Sq^1 H^6(P^n) = 0$. This is a contradiction.

Recall that for a stable spherical fibration η , the Stiefel-Whitney class $w_i(\eta)$ is characterized by

$$Sq^i(U(\eta)) = w_i(\eta) \cup U(\eta),$$

where $U(\eta) \in H^0(M(\eta))$ is the (stable) Thom class.

LEMMA 5. Let η be a stable spherical fibration over ΣP^n with $w_{2^j}(\eta) = 0$ for $0 \leq j \leq 3$. Then $w_i(\eta) = 0$ for all $i > 0$.

PROOF. Let $u = U(\eta) \in H^0(M(\eta))$ be the Thom class. Since the elements of the form Sq^{2^j} generate A as an algebra, we have only to show that $Sq^{2^j} u = 0$ for all $j > 0$. Assume that $Sq^{2^j} u = 0$ for all $j \leq r$ ($r \geq 3$). We use a famous decomposition of $Sq^{2^{r+1}}$ by secondary cohomology operations due to Adams [1]:

$$Sq^{2^{r+1}} u = \sum a_{i,j,r} \Phi_{i,j}(u), \quad a_{i,j,r} \in A,$$

where $\text{deg}(a_{i,j,r}) = 2^{r+1} - 2^i - 2^j + 1$ and the summation runs over $0 \leq i \leq j \leq r$, $i+1 \neq j$. We have $\Phi_{i,j}(u) \in H^{2^i+2^j-1}(M(\eta))$ which has generator $\sigma x^{2^i+2^j-2} \cup u$ and $a_{i,j,r}$ maps $\Phi_{i,j}(u)$ into $H^{2^{r+1}}(M(\eta))$ whose generator is $\sigma x^{2^{r+1}-1} \cup u$. By the inductive assumption, $a_{i,j,r}$ annihilates u and so

$$a_{i,j,r}(\sigma x^{2^i+2^j-2} \cup u) = \sigma(a_{i,j,r} x^{2^i+2^j-2}) \cup u.$$

This is zero by Lemma 3, since $\alpha(2^{r+1}-1) = r+1 > \alpha(2^i+2^j-2)$. Therefore each

term $a_{i,j,r}\Phi_{i,j}(u)$ must be zero. Thus the proof is complete.

§ 4. Proof of Proposition 2.

Let $\{h_i = [\xi_1^{2^i}]\}_{i \geq 0} \in \text{Ext}_A^1(\mathbf{Z}/2, \mathbf{Z}/2)$ be the set of generators and $(C(r), d(r))$ be the partial minimal resolution of $\mathbf{Z}/2$ by a free A -module:

$$\longrightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} \mathbf{Z}/2 \longrightarrow 0.$$

C_0 is a free A -module on one generator c of degree 0. C_1 is free on generators $\{c_i\}_{0 \leq i \leq r}$ and $d_1 c_i = Sq^{2^i} c$. Actually C_1 can be constructed as follows. By the minimality assumption, we have the identification

$$\mathbf{Z}/2 \otimes_A C_1 \cong \text{Tor}_1^A(\mathbf{Z}/2, \mathbf{Z}/2) = \text{Hom}(\text{Ext}_A^1(\mathbf{Z}/2, \mathbf{Z}/2), \mathbf{Z}/2).$$

Take $c_i \in C_1$ so that $\{1 \otimes_A c_i\}_{0 \leq i \leq r}$ form a basis dual to $\{h_i\}_{0 \leq i \leq r}$. Similarly, let C_2 be a free A -module on $\{c_{i,j}\}_{0 \leq i \leq j \leq r, i+1 \neq j}$ so that $\{1 \otimes_A c_{i,j}\}$ becomes a basis of $\text{Tor}_2^A(\mathbf{Z}/2, \mathbf{Z}/2)$ dual to $\{h_i h_j\}$. From the Adem relation, we pick up two relations:

$$R = \sum_{0 \leq k \leq r} Sq^{2^{r+1}-2^k} Sq^{2^k} = 0,$$

and

$$R' = Sq^{2^r} Sq^1 + (Sq^2 Sq^{2^{r-1}-1}) Sq^{2^{r-1}} + Sq^1 Sq^{2^r} = 0.$$

These relations induce $\Phi_{r,r}$ and $\Phi_{0,r}$. To be precise, put

$$z = \sum_{0 \leq k \leq r} Sq^{2^{r+1}-2^k} c_k,$$

and

$$z' = Sq^{2^r} c_0 + Sq^2 Sq^{2^{r-1}-1} c_{r-1} + Sq^1 c_r.$$

LEMMA 6. (z, d_1) (resp. (z', d_1)) defines the secondary operation $\Phi_{r,r}$ (resp. $\Phi_{0,r}$).

PROOF. Let $p_k: C_k \rightarrow \mathbf{Z}/2 \otimes_A C_k \cong \text{Tor}_k^A(\mathbf{Z}/2, \mathbf{Z}/2)$ be the natural projection and put

$$\theta = p_k \circ d_k^{-1}: \text{Ker } d_{k-1} \longrightarrow \text{Tor}_k^A(\mathbf{Z}/2, \mathbf{Z}/2).$$

From [1], Lemma 2.2.2, we have

$$h_i h_j(\theta z) = \sum_{0 \leq k \leq r} \xi_1^{2^i}(Sq^{2^{r+1}-2^k}) h_j(1 \otimes_A c_k) = \xi_1^{2^i}(Sq^{2^{r+1}-2^j}),$$

which is nonzero if and only if $i=j=r$. This shows that θz is dual to h_r^2 proving that (z, d_1) defines $\Phi_{r,r}$. Similarly, we have

$$\begin{aligned} h_i h_j(\theta z') &= \xi_1^{2^i}(Sq^{2^r}) h_j(1 \otimes_A c_0) + \xi_1^{2^i}(Sq^2 Sq^{2^{r-1}-1}) h_j(1 \otimes_A c_{r-1}) \\ &\quad + \xi_1^{2^i}(Sq^1) h_j(1 \otimes_A c_r). \end{aligned}$$

Since $\xi_1^{2^i}$ is primitive, the second term in the right hand side of the above expression is zero. The term $\xi_1^{2^i}(Sq^{2^r})h_j(1 \otimes_A c_0)$ is nonzero if and only if $i=r$ and $j=0$ and $\xi_1^{2^i}(Sq^1)h_j(1 \otimes_A c_r)$ is nonzero if and only if $i=0$ and $j=r$. This shows that (z', d_1) defines $\Phi_{0,r}$.

LEMMA 7. *Let η be a stable spherical fibration over ΣP^n with vanishing Stiefel-Whitney classes. Then both $\Phi_{r,r}U(\eta)$ and $\Phi_{0,r}U(\eta)$ have total indeterminacy zero.*

PROOF. For $\Phi_{r,r}U(\eta)$, the indeterminacy is contained in

$$\sum Sq^{2^{r+1}-2^k} H^{2^k-1}(M(\eta)).$$

This is zero since

$$Sq^{2^{r+1}-2^k}(\sigma x^{2^k-2} \cup U(\eta)) = 0 \quad \text{for } k \leq r.$$

As to $\Phi_{0,r}(U(\eta))$, the total indeterminacy is

$$Sq^{2^r} H^0(M(\eta)) + Sq^2 Sq^{2^r-1} H^{2^r-1}(M(\eta)) + Sq^1 H^{2^r-1}(M(\eta)),$$

which is also zero.

LEMMA 8. *Let η be as in Lemma 7 and suppose that $\Phi_{j,j}(u)=0$ for all $j < r$ and $\Phi_{0,j}(u)=0$ for all $j \leq r$ for some integer $r > 0$, where u is the Thom class of η . Then we have $\Phi_{r,r}(u)=0$.*

PROOF. Let

$$z_{r,r,r} = \sum b_{i,j} c_{i,j} \quad (b_{i,j} \in A)$$

be an element of C_2 such that $\theta z_{r,r,r}$ is dual to h_r^3 where the summation runs over $0 \leq i \leq j \leq r, i+1 \neq j$ as usual. $z_{r,r,r}$ induces a relation

$$b_{r,r} \Phi_{r,r}(u) + \sum_{0 < i < j \leq r, i+1 \neq j} b_{i,j} \Phi_{i,j}(u) = 0,$$

where $b_{i,j}$ has degree $3 \cdot 2^r - 2^i - 2^j (> 0)$ and maps $H^{2^i+2^j-1}(M(\eta))$ to $H^{3 \cdot 2^r-1}(M(\eta))$. Since $\alpha(2^i+2^j-2) = i < \alpha(3 \cdot 2^r - 2) = r$, it follows that $b_{i,j} H^{2^i+2^j-1}(M(\eta)) = 0$ by Lemma 3. Hence $b_{r,r} \Phi_{r,r}(u) = 0$. On the other, we have

$$1 = h_r^3(\theta z_{r,r,r}) = \xi_1^{2^r}(b_{r,r})$$

and this implies that $b_{r,r} = Sq^{2^r} + \text{decomposables}$. This shows that $b_{r,r}$ maps $H^{2^{r+1}-1}(M(\eta))$ isomorphically onto $H^{3 \cdot 2^r-1}(M(\eta))$ proving that $\Phi_{r,r}(u) = 0$.

PROOF OF PROPOSITION 2. As for the total indeterminacy, the result is already shown in Lemma 7. So we shall only prove that $\Phi_{r,r}(U(\eta_f)) = 0$. Let $u = U(\eta_f) \in H^0(M(\eta_f))$ be the Thom class of η_f . We shall prove that for all r such that $3 \cdot 2^r - 2 \leq n, \Phi_{0,r}(u) = 0$ and $\Phi_{r,r}(u) = 0$ hold. The assertion is true for $r = 0$. For $r = 1$, we have only to show that $\Phi_{1,1}(u) = 0$. But this follows from the previous lemma. So we proceed inductively and assume that for $r \geq 2$ we

have shown $\Phi_{0,j}(u)=0$ and $\Phi_{j,j}(u)=0$ for all $j < r$.

Case $r=2$: Let

$$z_{0,0,2} = b_{0,0}c_{0,0} + b_{0,2}c_{0,2} + b_{1,1}c_{1,1}$$

be an element of degree 6 such that $\theta_{z_{0,0,2}}$ is dual to $h_0^2 h_2$. Then by the inductive assumption we have a relation

$$b_{0,2}\Phi_{0,2}(u) = 0.$$

On the other, since $\theta_{z_{0,0,2}}$ is dual to $h_0^2 h_2$ we have

$$1 = h_0^2 h_2 (\theta_{z_{0,0,2}}) = \xi_1(b_{0,2}).$$

Therefore $b_{0,2} = Sq^1$ and $b_{0,2}$ maps $H^4(M(\eta_f))$ isomorphically onto $H^5(M(\eta_f))$. Thus we must have $\Phi_{0,2}(u)=0$. Hence by Lemma 8, we proved that $\Phi_{2,2}(u)=0$.

The case when $r=3$ is similar. Just consider $z_{0,0,3} = \sum b_{i,j}c_{i,j}$ such that $\theta_{z_{0,0,3}}$ is dual to $h_0^2 h_3$.

Case $r \geq 4$: Consider the element

$$z_{0,2,r} = \sum_{0 \leq i \leq j \leq r, i+1 \neq j} b_{i,j}c_{i,j} \in C_2$$

of degree $2^r + 5$ such that $\theta_{z_{0,2,r}}$ is dual to $h_0 h_2 h_r$. The element $b_{i,j}$ is of degree $2^r - 2^i - 2^j + 5$, so the term $i=j=r$ does not appear. This element and the assumption of the present lemma imply the relation

$$b_{0,r}\Phi_{0,r}(u) + \sum_{0 < i < j \leq r, i+1 \neq j} b_{i,j}\Phi_{i,j}(u) = 0.$$

We have $b_{i,j}H^{2^i+2^j-1}(M(\eta_f))=0$ from Lemma 3, since

$$\text{ord}_2(2^i + 2^j - 2) > \text{ord}_2(2^r + 3) = 0.$$

Hence $b_{0,r}\Phi_{0,r}(u)=0$. On the other hand, since

$$1 = h_0 h_2 h_r (\theta_{z_{0,2,r}}) = \xi_1^i(b_{0,r}),$$

we have

$$b_{0,r} = Sq^4 + \mu Sq^3 Sq^1 \quad (\mu \in \mathbf{Z}/2).$$

This shows that $b_{0,r}$ maps $H^{2^r}(M(\eta_f))$ isomorphically onto $H^{2^r+4}(M(\eta_f))$. Therefore $\Phi_{0,r}(u)=0$ and Lemma 8 implies that $\Phi_{r,r}(u)=0$.

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