J. Math. Soc. Japan
Vol. 43, No. 1, 1991

Strongly nonmultidimensional theories

By Hirotaka KIKYO

(Received Dec. 11, 1989)
(Revised March 7, 1990)

0. Introduction.

It is well-known that a countable theory is $\aleph_1$-categorical if and only if it is ω-stable and unidimensional. Tsuboi [7] has shown that a countable stable theory is almost strongly minimal if and only if it can be extended to a strongly unidimensional theory by adding finitely many constants, and he studied the notion of strongly two-dimensional theories in [8] and obtained some nice structure theorems for the big model of a theory with this property.

In the present paper, we define and study the notion of the strongly κ-dimensional theories. Our results extends many of the results in Tsuboi [8]. A stable theory T is called *strongly κ-dimensional* if all the types of T (with parameters in the big model) can be classified into κ classes such that any two types in the same class are not almost orthogonal. T is called *strongly nonmultidimensional* if it is strongly κ-dimensional for some cardinal κ. We show that a strongly κ-dimensional theory is superstable and a strongly ω-dimensional countable theory is ω-stable. This seems to be significant since there are non-superstable two-dimensional theories and countable non-ω-stable unidimensional theories.

Since a strongly κ-dimensional theory T is superstable and nonmultidimensional, choosing a maximal orthogonal set of regular types on an a-model, every non-algebraic type is not orthogonal to one of the members of this set. The cardinality of this set is called the dimensionality of T and denoted $\mu(T)$. We define the strong dimensionality of T to be the smallest cardinal κ such that T is strongly κ-dimensional. If every non-algebraic type is not almost orthogonal to some member of the given set of regular types, then it is easy to see that the theory is strongly nonmultidimensional. We show that a theory can be extended to a strongly nonmultidimensional theory by adding constants if and only if it can be extended to a theory such that every non-algebraic type is not almost orthogonal to some member of a given maximal orthogonal set of regular types on an a-model by adding constants. Therefore we have that the strong dimensionality coincides with the usual dimensionality. For a countable

Current address: Department of Mathematical Sciences, Tokai University, 1117 Kitakaname, Hiratsuka, Kanagawa 259-12, Japan.

stable theory T, we will show that T can be extended to a strongly ω-dimensional theory by adding countably many constants if and only if it can be extended to a theory such that every non-algebraic type is not almost orthogonal to a strongly regular type on the prime model of T by adding countably many constants. In this case, T will be ω-stable. Now a question arises: If a countable theory can be extended to a strongly ω-dimensional theory by adding constants, can we accomplish this by adding only countably many constants? The answer is no. $\mathrm{Th}(\boldsymbol{Z}_2^\omega, +, H_i)$ where H_i are some subgroups of $\boldsymbol{Z}_2^\omega$ is not ω-stable but it will be strongly unidimensional if we add the elements of $\boldsymbol{Z}_2^\omega$ as constants.

If a theory T is strongly nonmultidimensional, the big model $\mathfrak{C}$ of T has the following uniform structure with respect to the parameters:

There is a set B and a non-orthogonal set $\{p_i\}_{i<\kappa}$ of regular type on B such that for every set A extending B,

$$\mathfrak{C} = \mathrm{acl}(A \cup \bigcup_{i<\kappa} p_i | A^{\mathfrak{C}}).$$

In the case that every p_i has the U-rank 1, then we have the converse of the above claim only assuming that

$$\mathfrak{C} = \mathrm{acl}(B \cup \bigcup_{i<\kappa} p_i^{\mathfrak{C}}).$$

With this theorem, we can see that many of the simple examples of the nonmultidimensional theories fall in this category.

1. Preliminaries.

Our notations are standard. We work in the big model $\mathfrak{C}$ of the given theory. For types p and q, $p \supset_{nf} q$ means that p is a nonforking extension of q and $p \supset_{fk} q$ means that p is a forking extension of q. For a stationary type p, $p|A$ denotes the type parallel to p.

Definition. A stable theory T is said to be *strongly κ-dimensional* if there are κ families of types S_i $(i<\kappa)$ such that

(1) every non-algebraic 1-type of T belongs to some S_i, and

(2) any two types which belong to the same S_i are not almost orthogonal.

A stable theory is said to be *strongly nonmultidimensional* if it is strongly κ-dimensional for some cardinal κ. The *strong dimensionality* of a stable theory T is the smallest cardinal κ such that T is strongly κ-dimensional.

Definition. Let p_i $(i<\kappa)$ be stationary types on a set A. A type q is called a *product of p_i's* if q can be represented as $\mathrm{tp}(\bar{b}/A)$ where $\bar{b}$ is an independent sequence over A such that each element of $\bar{b}$ realizes some p_i.

DEFINITION. Let T be a complete theory. A theory T' is called an extension of T by adding constants if for some model M of T and for some subset A of M, $T'=\mathrm{Th}(M, a)_{a\in A}$.

We state some elementary facts which will be used frequently in this paper.

LEMMA 1.1. (i) *If p and q are not almost orthogonal and $A\supset\mathrm{dom}(q)$, then p is not almost orthogonal to one of the nonforking extensions of q to A.*

(ii) *If p and q are not almost orthogonal and q is stationary, then p is not almost orthogonal to every nonforking extension of q.*

(iii) *If p and q are not almost orthogonal, q stationary and $\mathrm{dom}(p)\supset\mathrm{dom}(q)$, then q is not almost orthogonal to every nonforking extension of p.*

PROOF. (i) and (ii) are easy.

(iii) Let $\mathrm{tp}(a/A)$ be a nonforking extension of p where $A\supset\mathrm{dom}(p)\supset\mathrm{dom}(q)$. Since p and q are not almost orthogonal and q stationary, there is b such that $\mathrm{tp}(b/\mathrm{dom}(p))\supset_{nf} q$ and $a\not\downarrow_{\mathrm{dom}(p)} b$. Let $\mathrm{tp}(b'/Aa)\supset_{nf}\mathrm{tp}(b/\mathrm{dom}(p)\cup\{a\})$. Since $\mathrm{tp}(a/A)\supset_{nf} p=\mathrm{tp}(a/\mathrm{dom}(p))$, we have $\mathrm{tp}(ab'/A)\supset_{nf}\mathrm{tp}(ab'/\mathrm{dom}(p))=\mathrm{tp}(ab/\mathrm{dom}(p))$. Hence, $a\not\downarrow_A b'$ and $\mathrm{tp}(b'/A)\supset_{nf}\mathrm{tp}(b/\mathrm{dom}(p))\supset_{nf} q$. So, $\mathrm{tp}(a/A)$ and q are not almost orthogonal. □

The following lemma is useful in our work.

LEMMA 1.2. *Let $q_i\in S(B)$ $(i<\lambda)$ be stationary types and S a set of types on B. Suppose that every non-algebraic type on a set of cardinality at most $\max(|B|, \kappa(T)\cdot\lambda)$ extending a type in S is not almost orthogonal to some product of q_i's. Then:*

(i) *Let I_i be a countably infinite Morley sequence of q_i for each i such that $\{I_i|i<\lambda\}$ is an independent set over B. Then every type $q\in S$ is realized in* $\mathrm{acl}(B\cup\bigcup_{i<\lambda}I_i)$.

(ii) $\bigcup\{q^{\mathfrak{C}}|q\in S\}\subset\mathrm{acl}(B\cup\bigcup_{i<\lambda}q_i^{\mathfrak{C}})$.

PROOF. Let a realize a type p in S. By stability of T, we can choose a set I of cardinality $\kappa(T)\cdot\lambda$ such that

(a) I is independent over B such that each element of I satisfies some q_i and

(b) if J is an independent set over B such that J extends I and each element of J satisfies some q_i, then $\mathrm{tp}(a/BJ)$ does not fork over BI.

If $\mathrm{tp}(a/BI)$ is non-algebraic, then it is not almost orthogonal to some q_i $(i<\lambda)$. Then we can choose b realizing $q_i|(BI)$ such that a and b are dependent over BI. But Bb satisfies the condition of (b). This contradicts (b). So, a is in $\mathrm{acl}(B\bar{b})$ for some finite sequence $\bar{b}$ in I. Since q_i's are stationary and $\bar{b}$ is independent over B as a set, we can embed $\bar{b}$ into $\bigcup_{i<\lambda}I_i$ by an automorphism over B. Hence, the lemma follows. □

2. Characterizations of the strongly nonmultidimensional theories.

Our main theorems in this section are Theorem 2.6 and Theorem 2.7. Theorem 2.6 is a characterization of the strongly ω-dimensional countable theories and Theorem 2.7 is a characterization of the strongly nonmultidimensional theories. We give several propositions to prove them. As the proofs almost go parallel to each other, we will concentrate on proving propositions needed for Theorem 2.6 which we have to take care on the cardinality of parameters. We begin with the proposition on the stability class.

PROPOSITION 2.1. *A strongly nonmultidimensional theory is superstable. A strongly ω-dimensional countable theory is ω-stable.*

PROOF. Let T be a strongly λ-dimensional theory and S_i $(i<\lambda)$ be the classes of types witness the strong λ-dimensionality of T. If some S_i does not have a stationary type in it, choose some q in S_i and take the all nonforking extensions to some model. Then every type in S_i is not almost orthogonal to one of the nonforking extensions of q_i. So, we can assume that each S_i has a stationary type in it. Note that the dimensionality might be increased. But if the theory is strongly ω-dimensional, this modification does not affect the dimensionality since we will see below that the theory is ω-stable in this case and the multiplicity of every type will be finite.

Choose a stationary type q_i from each S_i. Let B be a set of cardinality greater than $\kappa(T)\cdot\lambda$ and the cardinality of $\bigcup_{i<\lambda}\mathrm{dom}(q_i)$. Without loss of generality, B contains the domain of each q_i. Then, every type is not almost orthogonal to some $q_i|B$. Choose a countable Morley sequence I_i for each $q_i|B$ so that $\{I_i|i<\lambda\}$ is independent over B. By Lemma 1.2, every type on B is realized in $B^*=\mathrm{acl}(B\cup\bigcup_{i<\lambda}I_i)$ and $|B^*|=|B|$. Hence, T is superstable.

Now suppose that T is strongly ω-dimensional and countable. Choose $\{q_i\}_{i<\omega}$ such that each q_i is a stationary type on a countable set and every stationary type on a countable set is not almost orthogonal to some q_i. Let M be a countable model of T. We want to show that $S_1(M)$ is countable. We can assume that M contains the domain of each q_i. Now suppose contrary that there are uncountably many different types on M. Let a_i be a realization of $q_i|M$ for each $i<\omega$, and Let N be a countable model containing $M\cup\{a_i\}_{i<\omega}$. As every non-algebraic type on M is not almost orthogonal to some q_i, every non-algebraic type on M has a forking extension to N. Since N is countable, uncountably many of them are not algebraic over N. Repeating this process, we get an infinite forking sequence, contradicting the superstability of T. □

THEOREM 2.2. (i) *Suppose that T is a strongly ω-dimensional countable theory and let $\{p_i\}_{i<\kappa}$ be a maximal orthogonal set of strongly regular types on the prime model M_0 of T. Then there is a countable model $M \supset M_0$ such that for every element a and a set $A \supset M$, there is a tuple $\bar{b}$ realizing a product of $p_i|A$'s such that $a \in \mathrm{acl}(A\bar{b})$.*

(ii) *Suppose that T is a strongly nonmultidimensional theory and let $\{p_i\}_{i<\kappa}$ be maximal orthogonal set of regular types on an a-model M_0 of T. Then there is a model $M \supset M_0$ such that for every element a and a set $A \supset M$, there is a tuple $\bar{b}$ realizing a product of $p_i|A$'s such that $a \in \mathrm{acl}(A\bar{b})$.*

PROOF. (i) First of all, we show the theorem in the case A is countable. Let S_i $(i<\omega)$ be families of types such that

(a) every non-algebraic type on a countable set belongs to some S_i and

(b) any two types in the same S_i are not almost orthogonal.

For each $i<\omega$, choose q_i from S_i such that $RM(q_i)$ is the smallest in S_i. Choose countable model $M \supset M_0$ containing the domain of q_i for all $i<\omega$. Since T is ω-stable, there are only finitely many nonforking extensions of q_i to M for each $i<\omega$. Since every type in S_i is not almost orthogonal to some nonforking extension of q_i to M, by dividing each S_i into finitely many families, we can assume that each q_i has already been a type on M and thus stationary. Note that each q_i is not almost orthogonal to some strongly regular type p_j on M_0. For, since T is nonmultidimensional, each q_i is not orthogonal to M_0, and thus to some p_j. Since $M \supset M_0$, $q_i \not\perp p_j|M$. Since M is a model and T ω-stable, we have $q_i \not\perp^a p_j|M$.

To prove the theorem for the case A is countable, it is sufficient to prove the following claim:

CLAIM. For all countable set $A \supset M$ and an element a, there is a tuple $\bar{b}$ realizing a product of $p_i|A$'s such that $a \in \mathrm{acl}(A\bar{b})$.

We prove the claim by induction on $\alpha = RM(a/A)$. So, let $A \supset M$ be countable and $RM(a/A) = \alpha$. Then, every non-algebraic extension of $\mathrm{tp}(a/A)$ to a countable set is not in S_i such that $RM(q_i) > \alpha$ by the minimality of $RM(q_i)$ in S_i. Hence, every non-algebraic extension of $\mathrm{tp}(a/A)$ to a countable set belongs to some S_i such that $RM(q_i) \leqq \alpha$ and thus is not almost orthogonal to some q_i such that $RM(q_i) \leqq \alpha$. By Lemma 1.2, there is a tuple $\bar{c}$ such that $a \in \mathrm{acl}(A\bar{c})$, $\bar{c}$ realizes a product of $q_i|A$'s such that $RM(q_i) \leqq \alpha$ ($\lambda = \omega$ and $\kappa(T) = \omega$). We show that there is a tuple $\bar{b}$ realizing a product of $p_i|A$'s such that $\bar{c} \in \mathrm{acl}(A\bar{b})$. Then the claim follows. Suppose that for some $\bar{c}' \subset \bar{c}$, there is a tuple $\bar{b}$ realizing a product of $p_i|A$'s such that $\bar{c}' \in \mathrm{acl}(A\bar{b})$. It is sufficient to show that if $\bar{c}' \neq \bar{c}$ then we can expand $\bar{c}'$. Suppose that there is $c'' \in \bar{c} - \bar{c}'$. If $c'' \not\downarrow_A \bar{b}$, then

$RM(c''/A\bar{b})<RM(c''/A)\leqq\alpha$ and thus there is a tuple $\bar{b}'$ realizing a product of $(p_i|A\bar{b})$'s such that $c''\in\mathrm{acl}(A\bar{b}\bar{b}')$ by induction hypothesis. $\bar{b}\bar{b}'$ realizes a product of p_iA's. If $c''\downarrow_A\bar{b}$, then c'' realizes $q_j|A\bar{b}$ for some j. Since q_j is not almost orthogonal to some p_k, there is b_0 realizing $p_k|A\bar{b}$ such that $c''\not\downarrow_{A\bar{b}}b_0$. Then $RM(c''/A\bar{b}b_0)<RM(q_j)\leqq\alpha$ and thus there is a tuple $\bar{b}'$ realizing a product of $(p_i|A\bar{b}b_0)$'s such that $c''\in\mathrm{acl}(A\bar{b}b_0\bar{b}')$ by induction hypothesis. $\bar{b}b_0\bar{b}'$ realizes a product of $p_i|A$'s and $\bar{c}c''\in\mathrm{acl}(A\bar{b}b_0\bar{b}')$. The claim follows.

Now we have the theorem for the case the set A is countable. In general case, choose countable set $A_0\subset A$ such that $\mathrm{tp}(a/A)$ does not fork over A_0. Then there is a tuple $\bar{b}$ realizing a product of $p_i|A_0$'s such that $a\in\mathrm{acl}(A_0\bar{b})$. Choose $\bar{b}'$ such that $\mathrm{tp}(\bar{b}'/aA)\supset_{nf}\mathrm{tp}(\bar{b}/aA_0)$. Then, $a\in\mathrm{acl}(A\bar{b}')$ and $\mathrm{tp}(\bar{b}'/A)\supset_{nf}\mathrm{tp}(\bar{b}/M_0)$.

(ii) Proof goes almost parallel to that of (i) by making use of D-rank or U-rank instead of the Morley rank. Even, we can make use of these ranks in (i). □

LEMMA 2.3. *Suppose that T is superstable, a not algebraic over A, $a\in$* $\mathrm{acl}(A\bar{b})$, *and $\bar{b}$ realizes a product of orthogonal regular types p_i $(i<k)$ on A. Then* $\mathrm{tp}(a/A)$ *is not almost orthogonal to a power of some p_i.*

PROOF. Choose an a-model $M\supset A$ such that $a\bar{b}\downarrow_A M$. Then a belongs to $M[\bar{b}]$. So, we can make $M[a]\subset M[\bar{b}]$. Since a is not algebraic over A, $M[a]\neq M$. Then, we can choose $c\in M[a]-M$ such that $\mathrm{tp}(c/M)$ is regular. Also, we have $c\not\downarrow_M a$. On the other hand, since $c\in M[\bar{b}]$, $c\not\downarrow_M\bar{b}'$ where $\bar{b}'$ is the maximal subsequence of $\bar{b}$ realizing the unique p_i not orthogonal to $\mathrm{tp}(c/M)$. Since $\mathrm{wt}(c/M)=1$, $a\not\downarrow_M\bar{b}'$. As $a\bar{b}'\downarrow_A M$, $a\not\downarrow_A\bar{b}'$. Hence, $\mathrm{tp}(a/A)$ is not almost orthogonal to a power of p_i. □

PROPOSITION 2.4. (i) *Suppose that T is countable. Let B be a countable set and $\{p_i\}_{i<\kappa}(\kappa\leqq\omega)$ be maximal orthogonal set of (strongly) regular types on B. Suppose that every type on any set $A\supset B$ is not almost orthogonal to a power of some p_i. Then, there is a countable set $B'\supset B$ such that every non-algebraic type on any set $A\supset B'$ is not almost orthogonal to some p_i.*

(ii) *Let B be a set and $\{p_i\}_{i<\lambda}$ be maximal orthogonal set of regular types on B. Suppose that every type on any set $A\supset B$ is not almost orthogonal to a power of some p_i. Then, there is a set $B'\supset B$ such that every non-algebraic type on any set $A\supset B'$ is not almost orthogonal to some p_i.*

PROOF. (i) Note that T is ω-stable by Lemma 1.2. Let I_i be the countable Morley sequence of p_i for each $i<\kappa$ such that the set $\{I_i|i<\kappa\}$ is independent over B, $I=\bigcup_{i<\kappa}I_i$, and $B'=BI$. Suppose $A\supset B'$ and $\mathrm{tp}(a/A)$ is not algebraic. Choose $\bar{c}\in A$ such that $\mathrm{tp}(a/A)$ does not fork over $B\bar{c}$ and choose finite subset

$I_0 \subset I$ such that $\mathrm{tp}(a\bar{c}/IB)$ does not fork over I_0B. Let $I' = I - I_0$. Since $I_0 \downarrow_B I'$, we have $a\bar{c}I_0 \downarrow_B I'$. Without loss of generality, $\bar{c} \supset I_0$. So, we have,

$$a \downarrow_{B\bar{c}} A \tag{1}$$

and

$$a\bar{c} \downarrow_B I' . \tag{2}$$

Now, choose p_i such that $\mathrm{tp}(a/B\bar{c})$ is not almost orthogonal to a power of p_i. Choose $\bar{b}b_0$ realizing a power of $p_i | B\bar{c}$ such that

$$a \downarrow_{B\bar{c}} \bar{b} \quad \text{and} \quad a \not\downarrow_{B\bar{c}\bar{b}} b_0 .$$

Then,

$$\mathrm{tp}(a/B\bar{c}\bar{b}) \not\perp^a p_i \tag{3}$$

and

$$a\bar{c} \downarrow_B \bar{b} . \tag{4}$$

By (2) and (4), we can embed $\bar{b}$ into I' over $a\bar{c}B$. So, we can assume that $\bar{b} \subset I' \subset A$. Then $\mathrm{tp}(a/A) \supset_{nf} \mathrm{tp}(a/B\bar{c}\bar{b})$ by (1). Hence by (3) and Lemma 1.1 (iii), we have $\mathrm{tp}(a/A) \not\perp^a p_i$.

(ii) The proof goes parallel to that for (i). In this case, choose a Morley sequence I_i of cardinality $\kappa_r(T)$ for each p_i. Then choose $C \subset A$ of cardinality less than $\kappa(T)$ in place of $\bar{c}$, and then choose I_0 of cardinality less than $\kappa_r(T)$ as in (i). The rest of the argument goes exactly in the same way. □

LEMMA 2.5. *Let p be a stationary regular type and let q_1 and q_2 be types such that for both $i=1, 2$, $\mathrm{dom}(q_i) \supset \mathrm{dom}(p)$ and q_i is not almost orthogonal to p. Then, q_1 and q_2 are not almost orthogonal.*

PROOF. Let $A = \mathrm{dom}(q_1) \cup \mathrm{dom}(q_2)$ and $\mathrm{tp}(a_i/A) \supset_{nf} q_i$ for $i=1, 2$. Then $\mathrm{tp}(a_i/A) \not\perp^a p$. Let b_i be such that $a_i \not\downarrow_A b_i$ and $\mathrm{tp}(b_i/A) \supset_{nf} p$ for $i=1, 2$. Since p is stationary, $\mathrm{tp}(b_1/A) = \mathrm{tp}(b_2/A)$. Hence, there is a_2' such that $\mathrm{tp}(a_2'/A) = \mathrm{tp}(a_2/A)$ and $a_2' \not\downarrow_A b_1$. Since $\mathrm{wt}(b_1/A) = \mathrm{wt}(p) = 1$, we have $a_1 \not\downarrow_A a_2'$. □

Now, we have the following two theorems.

THEOREM 2.6. *The following are equivalent for a countable stable theory T.*

(i) *T can be extended to a strongly ω-dimensional theory by adding at most countably many constants.*

(ii) *There are a countable set B and an at most countable orthogonal set $\{p_i\}_{i<\kappa}$ $(\kappa \leq \omega)$ of strongly regular types on B such that for every element a and a set $A \supset B$, there is a tuple $\bar{b}$ realizing a product of $p_i | A$'s such that $a \in \mathrm{acl}(A\bar{b})$.*

(iii) *There are a countable set B and an at most countable orthogonal set $\{p_i\}_{i<\kappa}$ $(\kappa \leq \omega)$ of strongly regular types on B such that every type over a set $A \supset B$ is not almost orthogonal to some power of p_i.*

(iv) *There are a countable set B and an at most countable orthogonal set*

$\{p_i\}_{i<\kappa}$ ($\kappa \leqq \omega$) of strongly regular types on B such that every type over a set $A \supset B$ is not almost orthogonal to some p_i.

(v) *T can be extended to a strongly $\mu(T)$-dimensional theory by adding countably many constants.*

PROOF. (i) implies (ii) by Theorem 2.2 (i), (ii) implies (iii) by Lemma 2.3, (iii) implies (iv) by Proposition 2.4 (i), and (v) implies (i) trivially. Now we show the implication from (iv) to (v). Assume (iv). Consider the classes of types $S_i = \{q \mid \mathrm{dom}(q) \supset B,\ q \not\perp^a p_i\}$ for $i < \kappa$. By Lemma 2.5, any two types in the same S_i are not almost orthogonal. By adding every element of B as a constant, the theory will be strongly κ-dimensional. κ must be equal to $\mu(T)$ by the definition of the dimensionality (in the usual sense). Hence, the theory will be strongly $\mu(T)$-dimensional. Note that for the implications from (i) to (ii), (iii) to (iv) and (iv) to (v), we may have to add countably many constants to the set B. □

THEOREM 2.7. *The following are equivalent for a stable complete theory T.*

(i) *T can be extended to a strongly nonmultidimensional theory by adding constants.*

(ii) *There are a set B and an orthogonal set $\{p_i\}_{i<\kappa}$ of regular types on B such that for every element a and a set $A \supset B$, there is a tuple $\bar{b}$ realizing a product of $p_i|A$'s such that $a \in \mathrm{acl}(A\bar{b})$.*

(iii) *There are a set B and an orthogonal set $\{p_i\}_{i<\kappa}$ of regular types on B such that every type over a set $A \supset B$ is not almost orthogonal to some power of p_i.*

(iv) *There are a set B and an orthogonal set $\{p_i\}_{i<\kappa}$ of regular types on B such that every type over a set $A \supset B$ is not almost orthogonal to some p_i.*

(v) *T can be extended to a strongly $\mu(T)$-dimensional theory by adding constants.*

PROOF. The proof goes parallel to that of Theorem 2.6. We give some comments on the parameters. For the implication from (i) to (ii), we may have to add $\lambda \cdot \mathrm{mult}(T)$ constants to the set B where λ is the strong dimensionality in (i) and $\mathrm{mult}(T)$ is the multiplicity of T. For the implication from (ii) to (iii), we do not have to add any parameters, and for (iii) to (iv), we may have to add $\kappa \cdot \kappa_r(T)$ constants to the set B. □

3. Structure theorem.

Now, we have the following structure theorem for the strongly nonmultidimensional theories by Lemma 1.2.

THEOREM 3.1. (i) *If a countable theory T has a strongly ω-dimensional extension by adding at most countably many constants, then there is a countable set*

B and an orthogonal set $\{p_i\}_{i<\kappa}$ $(\kappa \leqq \omega)$ *of strongly regular types on the prime model of* T *such that for every set* A *extending* B,

$$\mathfrak{C} = \mathrm{acl}(A \cup \bigcup_{i<\kappa} p_i | A^{\mathfrak{C}}) .$$

(ii) *If a theory* T *has a strongly nonmultidimensional extension by adding constants, then there is a set* B *and an orthogonal set* $\{p_i\}_{i<\kappa}$ *of regular types on* B *such that for every set* A *extending* B,

$$\mathfrak{C} = \mathrm{acl}(A \cup \bigcup_{i<\kappa} p_i | A^{\mathfrak{C}}) .$$

We have a weak converse of Theorem 3.1.

THEOREM 3.2. (i) *Let* T *be a countable stable theory. If the big model of* T *has the structure*

$$\mathfrak{C} = \mathrm{acl}(B \cup \bigcup_{i<\kappa} p_i^{\mathfrak{C}})$$

for some countable set B *and an orthogonal set* $\{p_i\}_{i<\kappa}$ $(\kappa \leqq \omega)$ *of (strongly) regular types on* B *of U-rank* 1, *then* T *can be extended to a strongly* κ*-dimensional theory by adding countably many constants.*

(ii) *Let* T *be a stable theory. If the big model of* T *has the structure*

$$\mathfrak{C} = \mathrm{acl}(B \cup \bigcup_{i<\kappa} p_i^{\mathfrak{C}})$$

for some set B *and an orthogonal set* $\{p_i\}_{i<\kappa}$ *of regular types on* B *of U-rank* 1, *then* T *can be extended to a strongly* κ*-dimensional theory by adding constants.*

PROOF. To prove (i), we show that the condition (ii) in Theorem 2.6 holds. The same proof shows that the condition (ii) in Theorem 2.7 holds in the case of (ii).

Let $A \supset B$ and $\mathrm{tp}(a/A)$ a non-algebraic type. Choose $\bar{b} = b_0 \cdots b_n$ such that each b_j realizes some p_i and $a \in \mathrm{acl}(B\bar{b})$. If $\mathrm{tp}(b_j/Ab_0 \cdots b_{j-1})$ forks over B for some j, then $U(b_j/Ab_0 \cdots b_{j-1}) = 0$. Therefore, b_j is algebraic over $Ab_0 \cdots b_{j-1}$. So, we can assume that $\mathrm{tp}(b_j/Ab_0 \cdots b_{j-1})$ does not fork over B for every j. Hence, $\mathrm{tp}(\bar{b}/A)$ is a product of $p_i | A$'s. □

Theorem 3.2 is not true in general for regular types p_i. For example, consider the structure with only one equivalence relation with infinitely many infinite classes. Then the big model has the structure described in Theorem 3.2, but it is multidimensional. If we consider the countable structure with one equivalence relation such that it has countably many finite classes but has arbitrarily large class, then the big model has the structure described in Theorem 3.2 with the single p_i, but also in this case, the theory is multidimensional. Finally, we give some examples of strongly nonmultidimensional theories.

EXAMPLE 3.3. Consider the structure $(M, P_i)_{i<\kappa}$ where P_i are unary predicates representing infinite sets which are pairwise disjoint and the union of them covers M. Then its theory is strongly κ-dimensional.

EXAMPLE 3.4. Consider the structure $(M, P_i)_{i<\omega}$ where P_i are unary predicates such that for every two finite disjoint sets $\omega, \omega' \subset \omega$, $M \models \exists x(\bigwedge_{i\in\omega} P_i \wedge \bigwedge_{i\in\omega'} \neg P_i)$. Then its theory is countable and strongly 2^ω-dimensional.

EXAMPLE 3.5. Consider the abelian group $(\boldsymbol{Z}_2^\omega, +, H_i)_{i<\omega}$ with some distinguished subgroups H_i where $\boldsymbol{Z}_2^\omega$ is the set of all functions $f: \omega \to \{0, 1\}$ and $H_i = \{f \in \boldsymbol{Z}_2^\omega \mid f(j)=0 \text{ for } j<i\}$ for $i<\omega$. Its theory is countable, non-ω-stable and unidimensional. By adding all the elements of $\boldsymbol{Z}_2^\omega$ as constants, the theory becomes strongly unidimensional.

Example 3.5 shows that for a countable theory, even if we could extend it to a strongly ω-dimensional theory by adding constants, it is not necessarily true that it can be extended to a strongly ω-dimensional theory by adding only countably many constants.

References

[1] E. Bouscaren and D. Lascar, Countable models of nonmultidimensional $\aleph_0$-stable theories, J. Symbolic Logic, **48** (1983), 197-205.

[2] S. Buechler, The classification of small weakly minimal sets II., J. Symbolic Logic, **53** (1988), 625-635.

[3] M. Makkai, A survey of basic stability theory, with particular emphasis on orthogonality and regular types, Israel J. of Math., **49** (1984), 181-238.

[4] A. Pillay, An Introduction to Stability Theory, Oxford University Press, London, 1983.

[5] A. Pillay, Regular types in nonmultidimensional ω-stable theories, J. Symbolic Logic, **49** (1984), 880-891.

[6] S. Shelah, Classification Theory and the Number of Non-Isomorphic Models, North-Holland, Amsterdam, 1978.

[7] A. Tsuboi, Categoricity and non-orthogonality of types, Z. Math. Logik Grund-lag. Math., **33** (1987), 335-338.

[8] A. Tsuboi, Strongly 2-dimensional theories, J. Symbolic Logic, **53** (1988), 931-936.

Hirotaka KIKYO
Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1 Okubo, Shinjuku-ku
Tokyo 169
Japan