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# Geometric 4 manifolds in the sense of Thurston and Seifert 4 manifolds II

Dedicated to Professor Akio Hattori on his 60th birthday

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This paper is a sequel to [22] and characterizes the geometric 4 manifolds of type $S^3\times E$, $S^2\times E^2$, $H^2\times E^2$, and $\widetilde{SL}_2\times E$ in terms of the Seifert 4 manifolds over the 2 orbifolds which are either spherical, bad, or hyperbolic. In [22] we discussed the relations between the Seifert 4 manifolds over the euclidean 2 orbifolds and the geometries of type $E^4$, $Nil^3\times E$, $Nil^4$, and $Sol^3\times E$. Here we call a 4 manifold $S$ a Seifert 4 manifold if $S$ has a structure of a nonsingular fibered orbifold over a 2 orbifold $B$ with general fiber a 2 torus $T^2$ as in [22]. The topology of $S$ can be described by the Seifert invariants defined in [22]. We will recall their descriptions briefly in §1 and §5 when $B$ is either spherical, bad or hyperbolic. If all the monodromies are trivial (including one more case when the base is spherical) we can define the rational euler class $e$ which is a rational number or a pair of rational numbers (§1 and §5). Then the main results (Theorems A and B) which provide the complementary part of Theorems A and B in [22] can be stated as follows.

THEOREM A. *Let $S$ be a closed orientable 4-manifold.* (1) *$S$ is a Seifert 4 manifold over a spherical or bad 2 orbifold whose rational euler class is zero (resp. nonzero) if and only if $S$ is geometric of type $S^2\times E^2$ (resp. $S^3\times E$).* (2) *$S$ is geometric of type $S^3\times E$ if and only if $S$ is diffeomorphic to a bundle over $S^1$ with fiber a spherical 3 manifold. $S$ is geometric of type $S^2\times E^2$ if and only if $S$ is diffeomorphic to a nonsingular fibered orbifold with general fiber $S^2$ over a euclidean orbifold $B'$ where $B'$ is either the torus $T^2$, the Klein bottle $K$, the annulus $A$ or the Möbius band $M$.*

See §3, §4 for the details of the correspondences in Theorem A. We will determine exactly when the Seifert 4 manifolds of the above classes admit complex structures in §3 Corollary 9, §4 Corollary 13. In Corollary 9 we give the explicit correspondence between the Seifert fibrations and the bundle structures over $S^1$ of the Hopf surfaces since some of them were missing in [9] (see [10]) and since not every bundle over $S^1$ with fiber a spherical 3 manifold has a complex structure. On the other hand the situations for the Seifert 4 manifolds over the hyperbolic base orbifolds are somewhat different. Here we describe the result for the cases with orientable base orbifolds as follows. (The statements for the general cases will be given in §5.)

THEOREM B. *A Seifert 4 manifold $S$ over an orientable hyperbolic base orbifold $B$ has a geometric structure of type $X$ if and only if $S$ satisfies one of the following conditions.*

(1) *All the monodromies are represented by powers of a common periodic matrix in $SL_2\boldsymbol{Z}$ or all the monodromies are trivial and the rational euler class is zero. In this case $X=H^2\times E^2$.*

(2) *All the monodromies are trivial and the rational euler class is nonzero. In this case $X=\widetilde{SL}_2\times E$ where $\widetilde{SL}_2$ is the universal covering of $SL_2\boldsymbol{R}$.*

This implies that if $B$ is orientable and hyperbolic then $S$ is geometric if and only if $S$ has a complex structure (Corollary 16). In §5 we will see that every geometric 4 manifold $S$ of type $H^2\times E^2$ or $\widetilde{SL}_2\times E$ has a Seifert fibration over some hyperbolic 2 orbifold (possibly with reflectors) and any such one does not have a geometric structure of type $X$ with $X\neq H^2\times E^2$, $\widetilde{SL}_2\times E$. This implies that some Seifert 4-manifolds (for example those with nonperiodic monodromies) are not geometric in the sense of Thurston since the Seifert fiberings for the cases with hyperbolic base orbifolds are unique ([**25**], also cf. Proposition 14). This is in contrast with the result of [**22**] in which we stated that every Seifert 4 manifold over a euclidean 2 orbifold admits a geometric structure. Thus we have the following list stated in [**22**], §0.

| Type of the bases | The corresponding geometries |
|---|---|
| spherical or bad | $S^2\times E^2$ $S^3\times E$ |
| euclidean | $E^4$ (*) $Nil^3\times E$ $Nil^4$ $Sol^3\times E$ |
| hyperbolic | $H^2\times E^2$ $\widetilde{SL}_2\times E$ non-geometric |

(*) There is one closed orientable euclidean 4-manifold which is not a Seifert 4 manifold in our sense ([**22**]).

The results of this paper together with [**25**] and [**22**] lead us to the following corollaries (cf. Proposition 8).

COROLLARY C. *The diffeomorphism type of a Seifert 4 manifold $S$ over a 2 orbifold $B$ is determined by its fundamental group $\pi_1S$ unless $S$ is diffeomorphic to one of the followings.*

(1) *A bundle over $S^1$ with fiber an arbitrary lens space $L(p, q)$ whose monodromy is either the identity or the involution inducing the multiplication by $-1$ on $H_1(L(p, q), \boldsymbol{Z})$.*

(2) *An $S^2$-bundle over $T^2$ or $K$ with $w_2S=0$ or $w_2S\neq 0$ where $w_2$ is the second Stiefel Whitney class.*

COROLLARY D. *The diffeomorphism types of closed orientable geometric 4 manifolds of 8 types in the above list are determined by their fundamental groups except for the same cases as in Corollary C.*

The Seifert fibrations corresponding to the above exceptions (which are far from unique) will be described explicitly in § 3 List A-1 and § 4 List A-2. The claim of Corollary D for the euclidean cases is deduced from the rigidity theorem for euclidean manifolds and the fact that the closed geometric 4 manifolds of distinct types are not homotopy equivalent ([**23**]). Throughout this paper all the 4 manifolds will be smooth, closed and orientable.

## § 1. The invariants of Seifert 4 manifolds over spherical or bad 2 orbifolds.

In this section all the Seifert 4 manifolds will have the bad or spherical base 2 orbifolds. The types of such 2 orbifolds are listed in Fig. 1 in which the cone point of angle $2\pi/m$ and the corner reflector of angle $\pi/m$ are denoted by $m$ and $\bar{m}$ respectively. The fibering $\pi: S\to B$ of $S$ over $B$ is determined by the following data which we call the Seifert invariants of $S$ ([**22**], § 1). Fix the lattice $l=\boldsymbol{R}/\boldsymbol{Z}\times *$, $h=*\times\boldsymbol{R}/\boldsymbol{Z}$ of the general fiber $T^2=\boldsymbol{R}^2/\boldsymbol{Z}^2$. First suppose that $B$ has no reflectors.

(1) The type $(m, a, b)$ of the multiple torus of multiplicity $m$ over the cone point $p$ of angle $2\pi/m$. In this case $\pi^{-1}(D^2/\boldsymbol{Z}_m)$ for the neighborhood $D^2/\boldsymbol{Z}_m$ of $p$ ($p$ corresponds to 0) is diffeomorphic to $(D^2\times T^2)/\boldsymbol{Z}_m$ where the generator $\rho$ of $\boldsymbol{Z}_m$ acts on $D^2\times T^2$ by $\rho(z, x, y)=(\exp(2\pi i/m)z, x-a/m, y-b/m)$ for $x, y\in\boldsymbol{R}$ mod. 1 and $z\in\boldsymbol{C}$, $|z|=1$. Then the curve on $\partial D^2\times T^2$ represented by $(\exp(2\pi it/m), -ta/m, -tb/m)$ for $0\leqq t\leqq 1$ descends to a cross section $q$ of the meridional curve of $p$ such that $mq+al+bh$ is null-homologous in $\pi^{-1}(D^2/\boldsymbol{Z}_m)$.

(2) The obstruction $(a, b)$ to extending the cross sections $q$'s defined in (1) for all the multiple tori to the lift of $B-\cup$(the disk neighborhood of the cone point). A Seifert 4 manifold with obstruction $(a, b)$ is diffeomorphic to the one with obstruction $(0, 0)$ and with one more (multiple) torus of type $(1, a, b)$.

Thus if $B$ is orientable $S$ is represented by a series of triples $\{(m_1, a_1, b_1), \cdots (m_k, a_k, b_k)\}$. The rational euler class $e(S)$ of $S$ is defined by $e=(\sum_{i=1}^k a_i/m_i, \sum_{i=1}^k b_i/m_i)\in\boldsymbol{Q}^2$ (mod. the action of $GL_2\boldsymbol{Z}$). When $B=P^2(n)$ we have a monodromy matrix $A\in GL_2\boldsymbol{Z}$ with $\det A=-1$ along the orientation reversing curve $\bar{\gamma}$ in $B$. By an appropriate choice of $l$ and $h$ we can assume that $A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$. Then the Seifert invariants of $S$ are represented by $\{A, (n, a, b)\}$

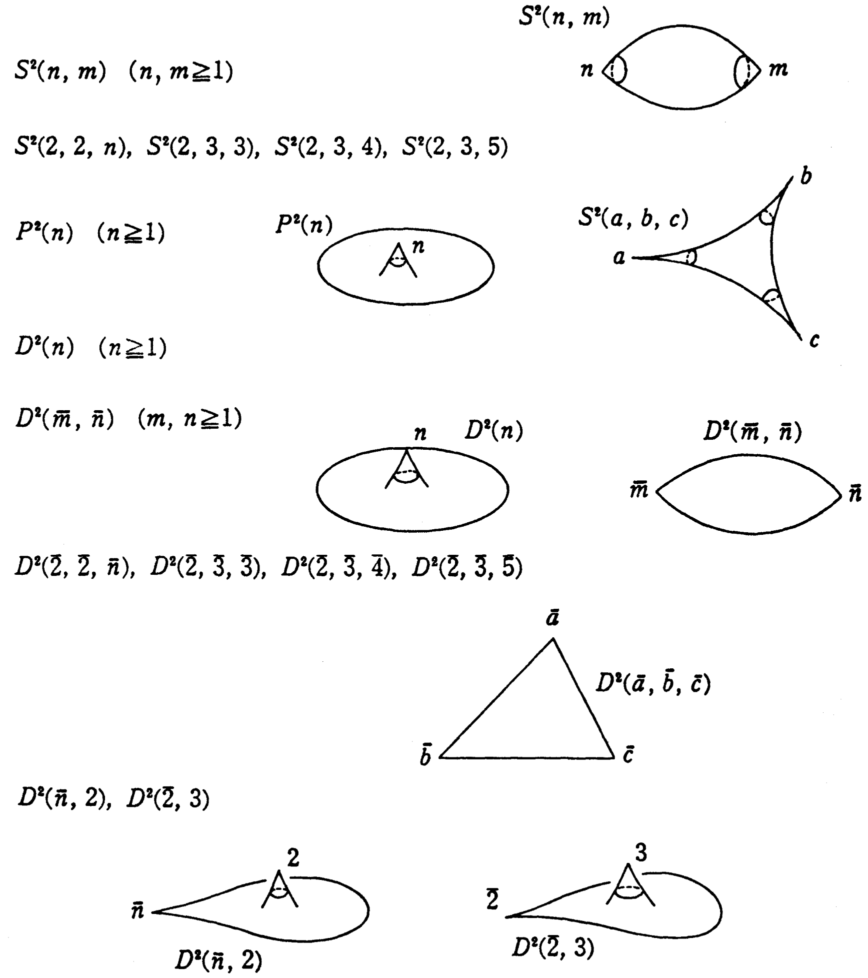


Figure 1. Closed spherical or bad 2-orbifolds.

where the type of the multiple torus is given with respect to the cross section homologous to $\gamma^{-2}$ for the lift $\gamma$ of $\bar{\gamma}$. In this case put $e(S)=b/n$. When $B$ has reflectors take the double covering $\tilde{B}$ of $B$ without reflectors and let $\tilde{S}$ be the unbranched covering of $S$ induced by the natural projection $\tilde{B}\to B$. Let $\iota$ be the covering translation of $\tilde{S}$ satisfying $\iota^2=l$, $\iota h\iota^{-1}=h^{-1}$ in $\pi_1 S$. Take an annular neighborhood $N$ of the reflector circle $C$ and let $\tilde{N}\subset\tilde{B}$ be the inverse image of $N$ (Figure 2). We fix the lift $\alpha$ of the curve $\partial N-C$ which is the boundary of the cross section over $B-N-\cup$(the disk neighborhood of the cone point). Then every corner reflector $p_i$ is covered by a cone point $p_i'$ of the same multiplicity. We further take a point $p_0'$ which projects to a smooth point on $C$. Then we have two kinds of invariants.

(3) The type $(m_i, 0, b_i)$ of the multiple Klein bottle over $p_i$. This is the type of the multiple torus in $\tilde{S}$ over $p_i'$ with respect to the cross section $q_i$

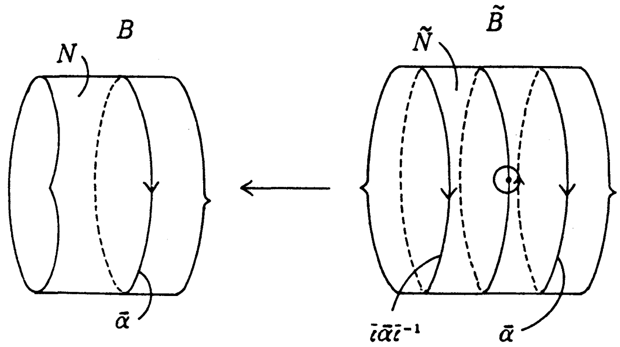


Figure 2.

around $p_i'$ such that $\iota q_i \iota^{-1}$ is conjugate to $q_i^{-1}$.

(4) The euler class $(a, b)$ of the reflector circle $C$. This is the obstruction to extending the cross sections $\cup q_i \cup \alpha \cup \iota\alpha\iota^{-1}$ to the one over $\tilde{N} - \cup$(the disk neighborhood of $p_i'$), i.e., $\alpha \prod q_i \iota \alpha^{-1} \iota^{-1} = l^a h^b$ in $\pi_1 S$. (The convention of the orientation of $\alpha$ is opposite to [22].) In our case here the monodromy along $C$ is trivial and hence $a=0$ ([22]). The Seifert manifold with euler class $(a, b)$ is the same as that with euler class (0, 0) with one extra (multiple) Klein bottle of type $(1, 0, b)$ over $p_0$.

We denote the type $(m, 0, b)$ of the multiple Klein bottle by $\overline{(m, 0, b)}$ (including the cases with $m=1$). If $S$ is represented by $\{\overline{(m_1, 0, a_1)}, \cdots, \overline{(m_s, 0, a_s)}, (n_1, b_1, c_1), \cdots, (n_t, b_t, c_t)\}$, then we define the rational euler class of $S$ by $e = (\sum_{i=1}^s a_i/m_i)/2 + \sum_{j=1}^t c_j/n_j$. We adopt the analogous notation for the Seifert 3 manifolds used in [14]. Note that the invariants $(m, a, b)$ make sense even if $m=1$ or $m<0$. If $m<0$ we assume that $(m, a, b)$ is the same as $(|m|, -a, -b)$.

## § 2. The diffeotopy groups of spherical 3 manifolds.

In this section we summarize the known results about the diffeotopy groups of the spherical 3 manifolds and fix the notations of their generators. Note that in dimension 3 the diffeotopy groups are the same as the homeotopy groups (cf. [8] § 5.8). Let $F$ be a spherical 3 manifold and denote by $D^+(F)$ the group of all orientation preserving self-diffeomorphisms of $F$ modulo diffeomorphisms isotopic to the identity. The structure of $D^+(F)$ was determined by [2], [8], [1], [18], [5] and [4]. In either case every diffeomorphism of $F$ is isotopic to a fiber preserving one with respect to one of the Seifert fibrations of $F$ over some orientable spherical or bad 2-orbifold $\Delta$. Let $f$ be an orientation-preserving and fiber-preserving diffeomorphism of $F$ covering the automorphism $\bar{f}$ of $\Delta$. Let $q_i$'s and $h$ be the cross sections and the general fiber of $F$ such that $n_i q_i + a_i h = 0$, $\sum q_i = 0$ in $H_1 F$ for $n_i \geqq 1$. If $f$ maps the $i$-th multiple fiber to the

$j$-th fiber and preserves (setwise) the $k$-th fiber we can assume that (up to isotopy) $f$ satisfies $f(q_i)=\pm q_j$ and $f(q_k)=\pm q_k$ according as $\bar{f}$ preserves or reverses the orientation of $\Delta$. In this case the type of the $i$-th and the $j$-th fibers are the same. Moreover if $\bar{f}$ fixes every cone point of $\Delta$ and preserves the orientation of $\Delta$ then we can see that $f$ is isotopic to the identity since $\pi_0\mathrm{Diff}^+(S^2, \text{rel. } n \text{ points})=0$ for $n\leqq 3$ (cf. [2] Chapter 4.) Therefore $f$ coincides (up to isotopy) with one of the following diffeomorphisms, their inverses or their compositions.

(1) $\Delta$ is arbitrary.

$id$: the identity.

$\tau$: $\bar{\tau}$ is a reflection along the circle $C$ through all cone points (including the image of the fiber of type $(1, b)$) of $\Delta$ (Figure 3).

(2) $\Delta=S^2(n, n)$, $S^2(2, 3, 3)$ or $S^2(2, 2, n)$.

$\sigma$: $\sigma$ is a fiber map of $F=\{(1, b), (n, a), (n, a)\}$, $\{(1, b), (2, 1), (3, 1), (3, 1)\}$ or $\{(1, b), (2, 1), (2, 1), (n, k)\}$ such that $\bar{\sigma}$ is a reflection along the circle $C'$ in Figure 3 which interchanges the cone points of the same cone angle and which passes through the extra cone points including the image $p_0$ of the fiber of type $(1, b)$. The composition $\sigma\tau=\tau\sigma$ induces the map $\bar{\sigma}\bar{\tau}$ which is a rotation by angle $\pi$ along the axis $A$ (containing $p_0$) in Figure 3.

(3) $\Delta=S^2(2, 2, 2)$.

$\rho$: $\rho$ acts on $\{(1, k), (1, b-k), (2, 1), (2, 1), (2, 1)\}$ for arbitrary $b, k\in \boldsymbol{Z}$ so that $\bar{\rho}$ is a rotation along the axis $A'$ (containing the images of the first 2 fibers) in Figure 3 which induces the cyclic permutation of the 3 cone points. The isotopy class of $\rho$ does not depend on $k$ (cf. [**1**], [**18**]).

Then $D^+(F)$ is generated by $\tau, \sigma, \rho$ (if they exist) which satisfy the following obvious relations: $\tau^2=\sigma^2=\rho^3=1$, $\tau\sigma=\sigma\tau$, $\tau\rho=\rho\tau$, $\rho$ and $\sigma\tau$ form the symmetric group of degree 3.

In the case when $F$ is a lens space $L(p, q)$ which corresponds to $\{(1, b), (\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$ with

(*) $p=b\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1$, $q=m\alpha_2+n\beta_2$ for $m\alpha_1-n(b\alpha_1+\beta_1)=1$, $m, n\in \boldsymbol{Z}$ ([**16**], [**5**]. Here if $p<0$ we assume that $L(p, q)=L(|p|, -q)$.)

The group $D^+(F)=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ generated by $\tau$ and $\sigma$ if $q^2\equiv 1$ and $q\neq\pm 1 \pmod p$; $D^+(F)=1$ if $|p|=1$ or 2, and $D^+(F)=\boldsymbol{Z}_2$ generated by $\tau$ otherwise ([**2**], [**8**]). The isotopy class of the diffeomorphism of $F=L(p, q)$ is determined by its action of $H_1(F, \boldsymbol{Z})$ ([**2**], [**5**]). The involution $\tau$ on $F$ induces the multiplication by $-1$ on $H_1F=\boldsymbol{Z}_p$ and so its isotopy class does not depend on the fibrations. On the other hand it is easy to see that $L(p, q)$ has the fibration of the form $\{(1, b), (\alpha, \beta), (\alpha, \beta)\}$ satisfying (*) if $q^2\equiv 1 \pmod p$ and $\sigma$ with respect to this fibration induces the multiplication by $-q$ on $H_1F$. Hence $\sigma=1$ if $q\equiv -1 \pmod p$ and $\sigma=\tau$ if $q\equiv 1 \pmod p$. Note that $L(p, -q)$ has a fibration $\{(1, b'),$

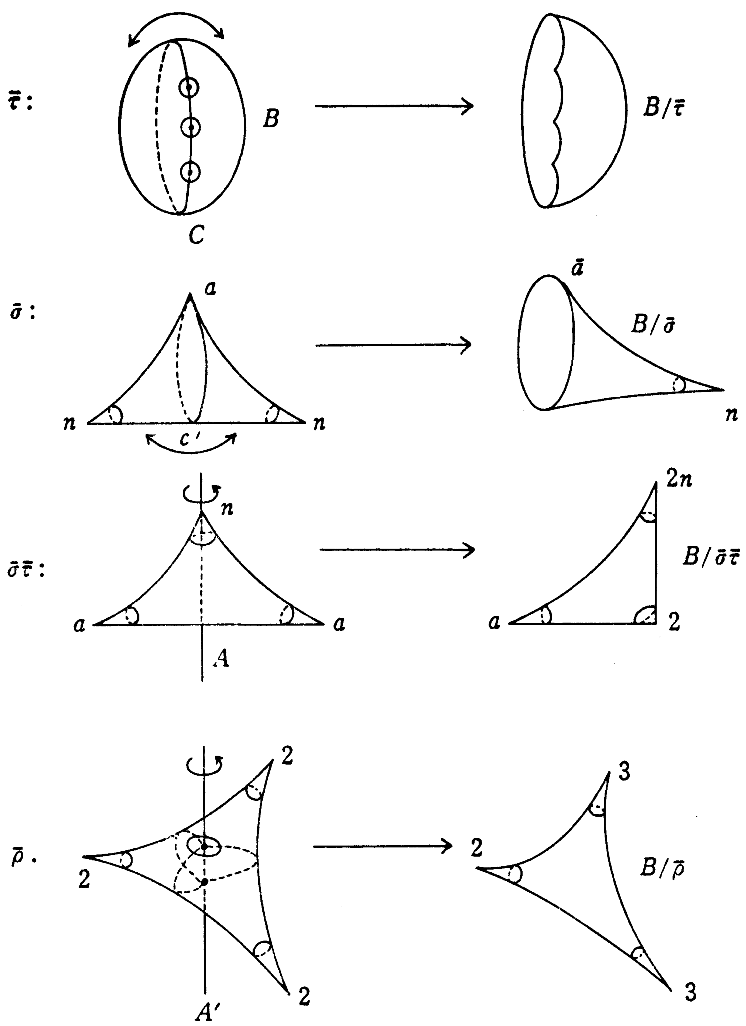


Figure 3.

$(\alpha', \beta'), (\alpha', \beta')\}$ satisfying (*) for $(p, -q)$ in place of $(p, q)$ and $\tau\lambda$ on $L(p, -q)$ with respect to this fibration corresponds to $\lambda$ on $L(p, q)$ with respect to the original fibration for $\lambda=\sigma$ or $\tau\sigma$.

If $F$ is not a lens space then the Seifert fibration of $F$ over an orientable base is unique and we can see that all the relations in $D^+(F)$ except for the obvious relations above come from Proposition 1 below (see [**1**], [**18**], [**4**], [**5**] for the details). Here we note that every element of $D^+(F)$ given above can be realized by an isometry of $F$. It is sufficient for our purpose in § 3 to check this for $\tau, \sigma, \tau\sigma, \rho, \tau\rho$. Each of them induces the cyclic action on $F$ such that the quotient is a spherical 3 orbifold (see the classification by Dunbar [**6**] and [**8**] for the lens spaces). Hence they can be represented by isometries as the covering translations of $F$. The following proposition is involved in the classification of $D^+(F)$ but we give its proof for completeness.

PROPOSITION 1. (i) *If* $F=\{(1, b), (2, 1), (2, 1), (n, k)\}$ *with* $n(b+1)+k=\pm 1$, *then* $\tau=id$ *for* $n$ *even and* $\sigma=id$ *for* $n$ *odd.*

(ii) *If* $F=\{(1, -1), (2, 1), (3, 1), (3, 1)\}$, *then* $\sigma=id$.

(iii) *If* $F=\{(1, -1), (2, 1), (3, 1), (4, 1)\}$ *or* $\{(1, -1), (2, 1), (3, 1), (5, 1)\}$ *then* $\tau=id$.

PROOF. In every case listed above $\Gamma=\pi_1 F$ is a subgroup of $SU_2$ up to conjugation in $O_4$. If we identify $S^3$ or $SU_2$ with the set of unit quoternions then $\Gamma$ acts on $S^3$ as the right multiplication by the elements in $S^3$. Consider the map $j: z_1+z_2 j \to j(z_1+z_2 j)$ which preserves the Hopf fibration, and induces the antipodal map $\bar{j}: \lambda \to -\bar{\lambda}^{-1}$ on $S^2=\boldsymbol{C}\cup\infty$. Since $j$ commutes with the right multiplication $j$ descends to the map $j'$ which preserves the induced fibration $S^3/\Gamma \to S^2/\bar{\Gamma}$ and covers the isomorphism $\bar{j}'$ of $\Delta=S^2/\bar{\Gamma}$. Consider the polyhedron $P$ such that $\bar{\Gamma}$ is the group of the symmetric transformations of $P$. Then every cone point of each base $\Delta$ comes from either the vertices, the midpoints of the edges, or the centers of the faces of $P$. Since $\bar{j}'$ is induced by the antipodal map of $S^2$ we can see that $j'=\tau$ (up to isotopy) in the cases (i) for $n$ even, (iii), and $j'=\sigma$ in the cases (i) for $n$ odd, (ii). On the other hand left multiplication on $S^3$ by a path $\gamma_t$ from $j$ to 1 in $S^3$ also descends to the map $\gamma_t'$: $F \to F$ since $\gamma_t$ commutes with the action of $\Gamma$. The map $\gamma_t'$ gives the desired isotopy between $j'$ and $id$.

## §3. Geometric 4 manlifolds of type $S^3\times E$.

In this section we restrict our attention to the geometric 4 manifolds of type $S^3\times E$ and their fibrations. First suppose that $S=\Gamma\backslash S^3\times E$ where $\Gamma$ is a discrete subgroup of $Isom^+(S^3\times E)=(O(4)\times Isom\, E)^+$ acting freely on $S^3\times E$. Put $\Gamma'=\Gamma\cap$(The kernel of the projection $p: Isom(S^3\times E)\to Isom\, E$) and $\bar{\Gamma}=p(\Gamma)$. Then we have a fibering $\Gamma'\backslash S^3 \to \Gamma\backslash S^3\times E \to \bar{\Gamma}\backslash E$ induced from the natural projection where $\bar{\Gamma}\backslash E=S^1$ and $\Gamma'\backslash S^3$ is a spherical 3 manifold ([**23**] §2) since there is no orientation reversing diffeomorphism on $\Gamma'\backslash S^3$ without fixed points ([**19**]) and hence $\bar{\Gamma}$ contains no reflections. Conversely let $S$ be an orientable bundle over $S^1$ with fiber a spherical 3 manifold $F$. Since the diffeomorphism type of $S$ is determined by the isotopy class of the monodromy $\lambda: F\to F$, we can assume by the classification of $D^+(F)$ that $\lambda$ is conjugate to one of $1, \tau, \sigma, \tau\sigma, \rho, \tau\rho$ which can be realized by an isometry (§2). Hence $S$ is the orbit space $F\underset{\lambda}{\times}S^1$ of the $\boldsymbol{Z}_k$ action on $F\times S^1$ (where $k$ is the period of $\lambda$) generated by $\lambda(x, \theta)=(\lambda x, \theta-1/k)$ for $x\in F, \theta\in\boldsymbol{R}$ mod 1 ($\lambda=1, \tau, \sigma, \tau\sigma, \rho, \tau\rho$). Therefore every $F$-bundle $S=F\underset{\lambda}{\times}S^1$ over $S^1$ has a geometric structure of type $S^3\times E$.

Since $\lambda$ induces an automorphism $\bar{\lambda}$ of $\Delta$ the composition of the natural projection $F\times S^1\to F$ and $\pi: F\to\Delta$ induces a Seifert fibration $S\to B=\Delta/\bar{\lambda}$. The local structure of the fibration is given by the following lemma.

LEMMA 2. *Let $\pi: S_0\to D^2$ be the fibration over the 2 disk $D^2$ with general fiber $S^1$ such that $\pi^{-1}(0)$ is a multiple fiber of type $(m, a)$ with respect to a cross section $q$. (1) Let $\iota$ be the fiber preserving involution inducing the reflection $\bar{\iota}$ on $D^2$ (the rotation by angle $\pi$ of the solid torus). Consider $S_0\times_\iota S^1$ where $\iota(x, \theta)=(\iota x, \theta-1/2)$ for $x\in S_0$, $\theta\in\boldsymbol{R}$ mod 1. Then the fibration $\pi': S_0\times_\iota S^1\to D^2/\bar{\iota}$ induced by the composition of the canonical projection and $\pi$ gives a neighborhood of a multiple Klein bottle of type $\overline{(m, 0, a)}$. (2) Let $\lambda$ be an orientation preserving fiber map of $S_0$ mapping $q$ to itself with $\lambda^k=1$ which induces the rotation $\bar{\lambda}$ of $D^2$ of angle $2\pi/k$. Consider $S_0\times_\lambda S^1$ where $\lambda(x, \theta)=(\lambda x, \theta-b/k)$ for $x\in S_0$, $\theta\in\boldsymbol{R}$ mod 1, g.c.d.$(k, b)=1$. Then the fibration $\pi': S_0\times_\lambda S^1\to D^2/\bar{\lambda}=D^2$ induced by the composition of the natural projection and $\pi$ gives us a neighborhood of a multiple torus $\pi'^{-1}(0)$ of type $(mk, mb, a)$ with respect to $(q', l, h)$ where $q'$ is the image of $q$ by $\lambda$, $l$ is the $S^1$ factor and $h$ is the general fiber of $S_0$.*

PROOF. (1) We have an isomorphism $S_0=D^2\times_{\lambda_0}S^1$ where $\lambda_0(z, \varphi)=(\exp(2\pi i/m)z, \varphi-a/m)$ for $\varphi\in\boldsymbol{R}$ mod 1, $z\in\boldsymbol{C}$, $|z|\leqq 1$ such that $\pi: S_0\to D^2$ is induced by the natural projection $D^2\times S^1\to D^2$. Then $S_0\times_\iota S^1=(D^2\times S^1\times S^1)/D_{2m}$ where $D_{2m}$ is the dihedral group generated by $\lambda$ and $\iota$ which act on $D^2\times S^1\times S^1$ by $\lambda(z, \theta, \varphi)=(\exp(2\pi i/m)z, \theta, \varphi-a/m)$, $\iota(z, \theta, \varphi)=(\bar{z}, \theta+1/2, \varphi)$ for $z\in\boldsymbol{C}$, $|z|\leqq 1$, $\theta, \varphi\in\boldsymbol{R}/\boldsymbol{Z}$. This gives us a neighborhood of the desired form. (2) The curve in $D^2\times S^1$ represented by $(\exp(2\pi it/m), -at/m)$, $0\leqq t\leqq 1$ descends to a cross section $q$ such that $mq+ah$ is null-homologous in $S_0$. Then the action of $\lambda$ on $S_0$ is induced by that on $D^2\times S^1$ defined by $\lambda'(z, \varphi)=(\exp(2\pi i/mk)z, \varphi-a/km)$. Hence we have a diffeomorphism between $S_0\times_\lambda S^1$ and $(D^2\times S^1\times S^1)/\lambda'$ where $\lambda'(z, \theta, \varphi)=(\exp(2\pi i/km)z, \theta-b/k, \varphi-a/km)$ for $(z, \theta, \varphi)\in D^2\times S^1\times S^1$ such that $\pi'$ is induced by the projection $D^2\times S^1\times S^1\to D^2$. This gives the desired representation of the multiple torus $\pi'^{-1}(0)$.

Now using Lemma 2 we will describe the Seifert fibration for each geometric 4 manifold $S=F\times_\lambda S^1$ of type $S^3\times E$.

*Case* 0. For any $F=\{(n_1, a_1), \cdots, (n_k, a_k)\}$ we have

(0) $F\times S^1=\{(n_1, 0, a_1), \cdots, (n_k, 0, a_k)\}$ with $B=S^2(n_1, \cdots, n_k)$.

(1) $F\times_\tau S^1=\{\overline{(n_1, 0, a_1)}, \cdots, \overline{(n_k, 0, a_k)}\}$ with $B=D^2(\bar{n}_1, \cdots, \bar{n}_k)$.

*Case* 1. $F=L(p, q)$ with $q^2\equiv 1$ (mod. $p$) $\lambda=\sigma$, $\sigma\tau$ with respect to the fibration of the form $\{(1, b), (\alpha, \beta), (\alpha, \beta)\}$ satisfying

(*) $p=\alpha(2\beta+\alpha b)$, $q=m\alpha+n\beta$ for $m\alpha-n(\beta+\alpha b)=1$, $m, n\in\boldsymbol{Z}$.

(1-2) $L(p, q)\underset{\sigma}{\times}S^1=\{\overline{(1, 0, b)}, (\alpha, 0, \beta)\}$ with $B=D^2(\alpha)$ satisfying (*).

The action of $\sigma\tau$ on $\{(1, b'), (1, b-b'), (\alpha, \beta), (\alpha, \beta)\}$ yields

(1-3) $L(p, q)\underset{\sigma\tau}{\times}S^1=\{(2, 1, b'), (2, -1, b-b'), (\alpha, 0, \beta)\}$ with $B=S^2(2, 2, \alpha)$

with arbitrary $b'$ which is equivalent to the case with $b'=0$ (change the cross sections or replace $(l, h)$ by $(lh, h)$).

REMARK 1-4. We note that $L(p, q)\underset{\lambda}{\times}S^1=L(-p, q)\underset{\lambda}{\times}S^1$ for $\lambda=\sigma$ or $\tau\sigma$ where $\lambda$ on $L(-p, q)$ is defined with respect to the fibration $\{(1, -b), (\alpha, -\beta), (\alpha, -\beta)\}$. On the other hand $L(p, -q)$ has a fibration $\{(1, b'), (\alpha', \beta'), (\alpha', \beta')\}$ satisfying (*) for $(p, -q)$ in place of $(p, q)$. Then we have $L(p, q)\underset{\lambda}{\times}S^1=L(p, -q)\underset{\tau\lambda}{\times}S^1$ for $\lambda=\sigma$ or $\tau\sigma$ where $\tau\lambda$ on $L(p, -q)$ is defined with respect to the above fibration.

*Case* 2. $F=\{(1, b), (2, 1), (2, 1), (n, k)\}$.

(2-2) $F\underset{\sigma}{\times}S^1=\{\overline{(1, 0, b)}, \overline{(n, 0, k)}, (2, 0, 1)\}$ with $B=D^2(2, \bar{n})$.

(2-3) $F\underset{\sigma\tau}{\times}S^1=\{(2n, n, k), (2, -1, b), (2, 0, 1)\}$ with $B=S^2(2, 2, 2n)$

which is equivalent to $\{(2n, n, k+nb), (2, -1, 0), (2, 0, 1)\}$ (change the cross sections or replace $(l, h)$ by $(lh, h)$ as in case (1-3)).

The action of $\rho$ on $\{(1, b'), (1, b-b'), (2, 1), (2, 1), (2, 1)\}$ yields

(2-4) $F\underset{\rho}{\times}S^1=\{(2, 0, 1), (3, 1, b'), (3, -1, b-b')\}$ with $B=S^2(2, 3, 3)$

which is equivalent to $\{(2, 0, 1), (3, 1, 0), (3, -1, b)\}$ by the same reason as before.

On the other hand $\rho$ and $\tau$ act on $\{(1, s), (1, s), (1, t), (1, t), (1, t), (2, 1), (2, 1), (2, 1)\}$ so that the images of the first two fibers on $\Delta$ are fixed by $\bar{\rho}$, interchanged by $\bar{\tau}$ and the image of three fibers of type $(1, t)$ are fixed by $\bar{\tau}$ and are permuted cyclically by $\bar{\rho}$. Then we have

(2-5) $F\underset{\tau\rho}{\times}S^1=\{\overline{(1, 0, t)}, \overline{(2, 0, 1)}, (3, 1, s)\}$ with $B=D^2(\bar{2}, 3)$, $2s+3t=b$.

*Case* 3. $F=\{(1, b), (2, 1), (3, 1), (3, k)\}$ with $k=\pm1$.

(3-2) $F\underset{\sigma}{\times}S^1=\{\overline{(1, 0, b)}, \overline{(2, 0, 1)}, (3, 0, 1)\}$ with $B=D^2(\bar{2}, 3)$, $k=1$.

(3-3) $F\underset{\sigma\tau}{\times}S^1=\{(2, 1, b), (4, -2, 1), (3, 0, 1)\}$ with $B=S^2(2, 3, 4)$, $k=1$.

Note that $\{(2, 1, b'), (4, -2, a), (3, 0, a')\}$ is equivalent to the one of the above forms by the same reason as before.

*Case* 4. $F=\{(1, b), (2, 1), (3, 1), (4, k)\}$ with $k=\pm1$.

*Case* 5. $F=\{(1, b), (2, 1), (3, 1), (5, k)\}$ with $k=\pm1, \pm2$.

In Cases 4 and 5 there are at most two fibrations, $F\times S^1$ and $F\underset{\tau}{\times}S^1$.

Every example listed above has nonzero rational euler class. Next we will show that any Seifert 4 manifold $S$ over a spherical or bad orbifold $B$ with $e\neq 0$ is diffeomorphic to the one already appeared in Cases $0\sim 5$. First suppose that $B=S^2(n_1, n_2, n_3)$ with $n_i\geqq 2$. Then by an appropriate choice of the lattice of the fiber $S$ has the representation of the form $\{(n_1, a_1, b_1), (n_2, a_2, b_2), (n_3, a_3, b_3)\}$ with $\sum_{i=1}^{3} a_i/n_i=0$. Then by further change of the cross sectional curves we can see that $S$ coincides with one of the classes in Cases $0\sim 5$. For $B\neq S^2(n_1, n_2, n_3)$ we can see that all the possible cases have already appeared in Cases $0\sim 5$ except for the followings which we will treat in the next stage.

(1) The cases with $B=S^2(m, n)$, $P^2(n)$,

(2) $\{\overline{(1, 0, b')}, (n, a, b)\}$ with $b=D^2(n)$, $a\neq 0$,

(3)$=\{\overline{(n, 0, b')}, (2, a, b)\}$ with $B=D^2(2, \bar{n})$ with $n\geqq 2$, $a\neq 0$ (we may assume that $a=1$).

PROPOSITION 3. *$S=\{(n_1, a_1, b_1), (n_2, a_2, b_2)\}$ with $(a_1/n_1+a_2/n_2, b_1/n_1+b_2/n_2)\neq(0, 0)$ is diffeomorphic to $L(p, q)\times S^1$ where $p$ and $q$ are determined as follows. Let $k_i=\text{g.c.d.}(a_i, b_i)$, $s_i, t_i, u_i, v_i$ be the integers satisfying $a_is_i+b_it_i=k_i$, $n_iu_i+k_iv_i=1$. Then $p=\text{g.c.d.}(n_2k_1+n_1(s_1a_2+t_1b_2), (-a_2b_1+a_1b_2)/k_1)$, $q=-u_1n_2+v_1(s_1a_2+t_1b_2)$.*

*Note.* The signs of $p$ and $q$ are not important since there is an oriented diffeomorphism betweeh $L(p, q)\times S^1$ and $L(p, -q)\times S^1$.

PROOF. $S$ is diffeomorphic to a union of two copies $(D^2\times T^2)_1$, $(D^2\times T^2)_2$ of $D^2\times T^2$ each of which is a tubular neighborhood of a multiple torus. Put $m_i=(\partial D^2\times *\times *)_i$, $l_i=(*\times S^1\times *)_i$, $h_i=(*\times *\times S^1)_i$. Then $(m_i, l_i, h_i)$ forms a base of $H_1(\partial(D^2\times T^2)_i)=H_1(T^3)$ such that $(m_2, l_2, h_2)=(m_1, l_1, h_1)A$ for some $A\in GL_3\boldsymbol{Z}\setminus SL_3\boldsymbol{Z}$. On the other hand fix a base of the general fiber $l$, $h$ and a cross sectional curve $q_i$ for the $i$-th multiple fiber such that $q_1=-q_2$ on $H_1(T^3)$ and $(m_i, l_i, h_i)=(q_i, l, h)B_i$ where $B_i$ is of the form $\begin{bmatrix} n_i & * & * \\ a_i & * & * \\ b_i & * & * \end{bmatrix}\in SL_3\boldsymbol{Z}$. Then $A=B_1^{-1}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}B_2$. Let $\mathcal{M}$ be the set of all the matrices in $SL_3\boldsymbol{Z}$ of the form $\begin{bmatrix} 1 & * & * \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$P_0$}} \\ 0 & & \end{bmatrix}$ for $P_0\in SL_2\boldsymbol{Z}$. Then we can replace $(m_i, l_i, h_i)$ by the new base $(m_i, l_i, h_i)P$ for $P\in\mathcal{M}$ since the self-diffeomorphism $P$ of $\partial(D^2\times T^2)$ can be extended to the one of $D^2\times T^2$. Hence $B_i$ can be replaced by $B_iP$, and $A$ can be replaced by $P'AQ'$ for any $P, P', Q'\in\mathcal{M}$. We use this fact to reduce the matrices $A$, $B_1$, $B_2$ to the simpler ones. First note that $B_i=C_iD_i$ with $C_i=$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_i' & -t_i \\ 0 & b_i' & s_i \end{bmatrix}$ and $D_i = \begin{bmatrix} n_i & * & * \\ k_i & * & * \\ 0 & * & * \end{bmatrix}$ where $k_i, s_i, t_i$ are defined in the statement of Proposition 1 and $a_i' = a_i/k_i$, $b_i' = b_i/k_i$. Then $D_i$ can be replaced by $\begin{bmatrix} P_i & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for $P_i = \begin{pmatrix} n_i & -v_i \\ k_i & u_i \end{pmatrix} \in SL_2\boldsymbol{Z}$ as follows. $D_i = \begin{bmatrix} n_i & * & * \\ k_i & * & * \\ 0 & x & y \end{bmatrix} \to \begin{bmatrix} n_i & * & * \\ k_i & * & * \\ 0 & x & y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & y & z \\ 0 & -x & w \end{bmatrix} =$ $\begin{bmatrix} n_i & -v_i & e \\ k_i & u_i & f \\ 0 & 0 & 1 \end{bmatrix}$ (for $z, w \in \boldsymbol{Z}$ with $xz + yw = 1$) $\to \begin{bmatrix} n_i & -v_i & e \\ k_i & u_i & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} P_i & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for $\begin{pmatrix} x' \\ y' \end{pmatrix} = -P_i^{-1} \begin{pmatrix} e \\ f \end{pmatrix}$. Then by simple calculation we can replace $A$ by the matrix of the form $\begin{bmatrix} c & * & * \\ d & * & * \\ e & * & * \end{bmatrix}$ with $c = -u_1 n_2 + v_1(s_1 a_2 + t_1 b_2)$, $d = n_2 k_1 + n_1(s_1 a_2 + t_1 b_2)$, $e = -b_1' a_2 + a_1' b_2$. Furthermore $A$ is transformed as follows. $A \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & \end{bmatrix}_{P} A = \begin{bmatrix} c & * & * \\ d' & * & * \\ 0 & f & g \end{bmatrix}$ (for some $P \in SL_2\boldsymbol{Z}$, $d' = g.c.d.(d, e)$) $\to \begin{bmatrix} c & * & * \\ d' & * & * \\ 0 & f & g \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & \end{bmatrix}_{Q} = \begin{bmatrix} c & x' & z' \\ d' & y' & w' \\ 0 & 0 & 1 \end{bmatrix}$ (for some $Q \in SL_2\boldsymbol{Z}$) $\to \begin{bmatrix} 1 & 0 & -z' \\ 0 & 1 & -w' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & x' & z' \\ d' & y' & w' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c & x' & 0 \\ d' & y' & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This implies that $D^2 \times T^2 \cup_A D^2 \times T^2 = L(d', c) \times S^1$. This completes the proof.

PROPOSITION 4. *$S = \{A, (n, a, b)\}$ over $B = P^2(n)$ with $b \neq 0$ ($e(S) \neq 0$) is diffeomorphic to the Seifert 4 manifold over $S^2(2, 2, |b|)$ of the form*

$$\{(2, 0, 1), (2, 0, -1), (b, -a, n)\} \qquad \textit{if} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{and}$$

$$\{(2, -1, 1), (2, 0, -1), (b, -a, n)\} \qquad \textit{if} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

PROOF. First suppose that $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since $P^2(n)$ is a union of a Moebius band $M_0$ and a disk with a cone point $D^2(n)$, $S$ is a union of $\pi^{-1}(M_0) = N \times S^1$ and $\pi^{-1}(D^2(n)) = D^2 \times T^2$ where $N$ is a twisted $S^1$ bundle over $M_0$ whose fiber corresponds to $h$. Let $\alpha$ be the orientation reversing curve of $M_0$ and $l$ be the other $S^1$ factor. Then $N$ is also a twisted $I$ bundle over the Klein bottle $K$ on which $\alpha$ and $\alpha h^{-1}$ are the orientation reversing loops. On the other hand $N$ is also a Seifert fibration over $D^2(2, 2)$. Let $h'$ be its general fiber and $q_1'$ and $q_2'$ be the cross sectional curves for the exceptional fibers such that $2q_1' + h' = 2q_2' - h' = 0$ in $H_1(N)$. Then we have $q_1' = h\alpha^{-1}$, $q_2' = \alpha$, $h' = \alpha^2$ where $\alpha^2$ is the boundary curve of $M_0$. Moreover $\alpha$ and $\alpha h^{-1}$ are isotopic to the exceptional fibers of $N$ as the fibering over $D^2(2, 2)$. Then $\pi^{-1}(D^2(n))$ is attached to $N \times S^1$ in $S$ so that $-n\alpha^2 + al + bh$ is null homologous in $\pi^{-1}(D^2(n))$. This implies that

$b(q'_1q'_2)+al'-nh'$ is null homologous in $D^2\times T^2$ for $(l', h')=(l, \alpha^2)$ and hence $S=\{(2, 0, 1), (2, 0, -1), (b, -a, n)\}$. Next suppose that $A=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$. Then $S$ is a union of $\pi^{-1}(M_0)$ and $\pi^{-1}(D^2(n))$ as before. To describe $\pi^{-1}(M_0)$ we consider the tubular neighborhood $U$ in $N$ of the fiber of $N$ as an $I$ bundle over $K$ (Figure 4). Remove $U\times S^1$ from $N\times S^1$ and reglue $D^2\times I\times S^1$ so that $m-l$ for the meridian $m$ of $U$ corresponds to the meridian of the attached $D^2\times I\times S^1$. Then we get a new $T^2$ bundle over $M_0$ diffeomorphic to $\pi^{-1}(M_0)$. On the other hand there are two loops $\alpha_1$ and $\alpha_2$ near $\alpha$ and $\alpha h^{-1}$ on $K$ isotopic to $\alpha^2$ in $N$. They correspond to the general fibers near the exceptional fibers in $N$ (as the fibering over $D^2(2, 2)$). Then after the Dehn surgery given above $\alpha_2$ is identified with $\alpha_1+l$ (Figure 4). Therefore the Seifert invariant $(2, 0, 1)$ is replaced by $(2, -1, 1)$ with respect to $(\alpha_2, l)$. Then the correspondence $q'_1=h\alpha^{-1}$, $q'_2=\alpha$, $l'=l$, $h'=\alpha^2$ gives a diffeomorphism between $\pi^{-1}(M_0)$ and a Seifert fibration over $D^2(2, 2)$ with the multiple tori of type $(2, -1, 1)$ and $(2, 0, -1)$. Thus we get a desired diffeomorphism.

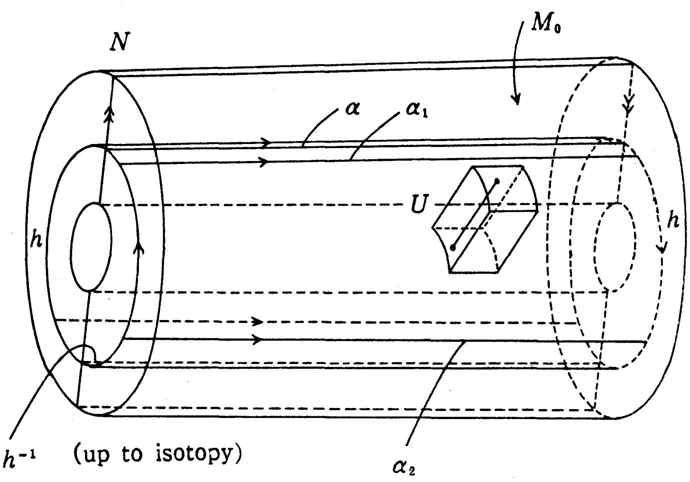


Figure 4.

By simple calculation and Proposition 4 we have

COROLLARY 5. (1) *Put* $S=\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (n, a, b)\right\}$, $k=$g. c. d.$(a, n)$, $a'=a/k$, *and* $n'=n/k$. *Then* $S=F\times S^1$ *for* $F=\{(2, 1), (2, -1), (b, k)\}$ *if* $a'$ *is even and* $S=F\underset{\sigma\tau}{\times}S^1$ *for* $F=\{(b, k), (b, k)\}$ *if* $a'$ *is odd.* (2) *Let* $S=\left\{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, (n, a, b)\right\}$. (i) *If* $b$ *is odd,* $S=F\underset{\sigma\tau}{\times}S^1$ *for* $F=\{(1, 1), (b, (k-b)/2), (b, (k-b)/2)\}$ *where* $k=$g. c. d.$(2a+b, n)$. (ii) *If* $b=2b'$, *then putting* $k=$g. c. d.$(b'+a, n)$, $n'=n/k$, *we have* $S=F\underset{\sigma\tau}{\times}S^1$ *for* $F=\{(1, -1), (2, 1), (2, 1), (b', k)\}$ (*if* $n'$ *is odd*) *or* $F=$

$\{(1, -1), (2b', k+b'), (2b', k+b')\}$ (*if* $n'$ *is even*).

PROPOSITION 6. *Let* $S=\{\overline{(1, 0, b')}, (n, a, b)\}$ *with* $a\neq 0$, $nb'+2b\neq 0$. *Put* $k=\text{g.c.d.}(n, a)$, $n'=n/k$, $a'=a/k$. *Then* $S$ *is diffeomorphic to*

(1) $L\underset{\sigma}{\times}S^1$ *for* $L=\{(1, n'b'), (k, b), (k, b)\}$ *if* $n'$ *is odd*,

(2) $F\times S^1$ (*if* $b'$ *is even*) *or* $F\underset{\sigma\tau}{\times}S^1$ (*if* $b'$ *is odd*) *if* $n'=2m'$ *where* $F=\{(2, 1), (2, -1), (b+b'km', k)\}$.

PROOF. Put $L=\{\overbrace{(1, b'), \cdots, (1, b')}^{n'}, (k, b), (k, b)\}$. Then there is a fiber-preserving self-diffeomorphism $\mu$ of $L$ which preserves the orientation of the general fiber, fixes the two exceptional fibers and induces the cyclic permutation of the first $n'$ fibers of type $(1, b')$. There is also an involution $\sigma$ on $L$ which reverses the orientation of the general fiber and induces the reflection on $S^2(k, k)$ along the circle through the images of the $n'$ fibers of type $(1, b')$. Extend the actions of $\mu$ and $\sigma$ to those on $L\times S^1$ by $\mu(x, \theta)=(\mu x, \theta+a'/n')$, $\sigma(x, \theta)=(\sigma x, \theta+1/2)$ for $x\in L$, $\theta\in \boldsymbol{R}$ (mod 1). Then $S$ is the orbit space of the group $\langle\mu, \sigma\rangle$ generated by $\mu$ and $\sigma$ acting on $L\times S^1$. Note that $\mu$ commutes with $\sigma$.

*Case* 1. $n'$ is odd. Then for some $s\in\boldsymbol{Z}$ satisfying $2sa'+n'=1$ (mod $2n'$), $\langle\mu, \sigma\rangle=\boldsymbol{Z}_{2n}$, generated by $\mu^s\sigma$ which acts on the $S^1$ factor as $\theta\to\theta+1/2n'$. On the other hand $\mu$ is isotopic to the identity on $L$ since $\mu$ induces the identity on $H_1(L)$ ([2], [8]). Hence $S=L\underset{\mu^s\sigma}{\times}S^1$ is diffeomorphic to $L\underset{\sigma}{\times}S^1$.

*Case* 2. $n'=2m'$ for some $m'\in\boldsymbol{Z}$. For some $s\in\boldsymbol{Z}$ satisfying $sa'=1$ (mod. $2m'$), $\langle\mu, \sigma\rangle=\boldsymbol{Z}_2+\boldsymbol{Z}_{2m'}$ generated by $\mu^{m'}\sigma$ and $\mu^s$. Here $\mu^{m'}\sigma$ acts trivially on the $S^1$-factor and $L/\mu^{m'}\sigma=\{(1, b'm'), (k, b)\}$ over $P^2(k)$. To check the action of $\mu^s$ on $L/\mu^{m'}\sigma$ consider the original representation of $\pi_1 S$: $\{\iota, h, q\mid [q, h]=[q, \iota^2]=1,\ \iota h\iota^{-1}=h^{-1},\ q^{2m'k}\iota^{2a'k}h^b=1,\ q\iota q^{-1}\iota^{-1}=h^{b'}\}$. Choose $u, v\in\boldsymbol{Z}$ such that $2m'v-a'u=1$ and put $\phi^{-1}=q^{m'}\iota^{a'}$, $\lambda=q^u\iota^{2v}$, $h'=h$. Then $\pi_1 S=\{\phi, \lambda, h'\mid \phi^{2k}=h'^{b+b'm'k},\ \phi h'\phi^{-1}=h'^{-1},\ \lambda h'\lambda^{-1}=h',\ \lambda\phi\lambda^{-1}=\phi h'^{-ub'}\}$. Here $\phi$ and $h'$ generate the subgroup isomorphic to $\pi_1(L/\mu^{m'}\sigma)$ such that $\phi$ and $\phi h'^{-1}$ correspond to the exceptional fibers of multiplicity 2 and $\phi^2$ corresponds to the general fiber for the Seifert fibration of $L/\mu^{m'}\sigma$ over $S^2(2, 2, |b+b'm'k|)$. On the other hand $u$ must be odd and hence by the above relation we can see that $\pi_1 S=\pi_1(L/\mu^{m'}\sigma\times S^1)$ if $b'$ is even and $\pi_1 S=\pi_1(L/\mu^{m'}\sigma\underset{\sigma\tau}{\times}S^1)$ if $b'$ is odd. Since the bundle structure of $S$ over $S^1$ (with fiber fixed) is determined by $\pi_1 S$ (Proposition 8) we obtain the desired result.

PROPOSITION 7. *Let* $S=\{\overline{(n, 0, b')}, (2, 1, b)\}$ *on* $D^2(2, \bar{n})$ *with* $n\geqq 2$. *Then* $S$ *is diffeomorphic to* $F\underset{\tau}{\times}S^1$ *if* $b$ *is even and* $F\underset{\sigma}{\times}S^1$ *if* $b$ *is odd where* $F=\{(2, 1), (2, -1), (b'+nb, n)\}$.

PROOF. Let $L=\{(1, b), (1, b), (n, b'), (n, b')\}$ and let $\rho$ be the fiber-preserving involution of $L$ which preserves the orientation of the fiber and induces the rotation of angle $\pi$ on $S^2(n, n)$ along the axis through the image of the fiber of type $(1, b)$. Also consider the involution $\lambda$ of $L$ which reverses the orientation of the fiber and induces the reflection of $S^2(n, n)$ along the circle through the images of the exceptional fibers. Extend the actions of $\rho$ and $\lambda$ to those on $L\times S^1$ by $\rho(x, \theta)=(\rho x, \theta+1/2)$, $\lambda(x, \theta)=(\lambda x, \theta+1/2)$ for $x\in L$, $\theta\in\boldsymbol{R} \bmod 1$. Then $S=L\times S^1/\langle\rho, \lambda\rangle$ where $\langle\rho, \lambda\rangle$ is the group generated by $\rho$ and $\lambda$. We note that $\rho\lambda$ $(=\lambda\rho)$ induces the identity on the $S^1$ factor and the fiber-preserving map on $L$ such that $L'=L/\rho\lambda=\{(1, b), (n, b')\}$ on $P^2(n)$ which is diffeomorphic to $\{(2, 1), (2, -1), (b'+nb, n)\}$. Then $\lambda$ induces the involution $\lambda'$ on $L'$ and $S=L'\underset{\lambda'}{\times}S^1$ where $\lambda'(x, \theta)=(\lambda' x, \theta+1/2)$ for $x\in L'$, $\theta\in S^1$. On the other hand consider the original representation of $\pi_1 S$: $\{q_0, q_1, \iota, h, l \mid l=\iota^2, \iota h\iota^{-1}=h^{-1}, q_1q_0\iota q_1^{-1}\iota^{-1}=1, \iota q_0\iota^{-1}=q_0^{-1}, q_0^n h^{b'}=1, q_1^2 l h^b=1, [q_i, l]=[q_i, h]=1\ (i=0, 1)\}$. Putting $\gamma=(q_1\iota)^{-1}$, $\lambda=\iota$, $h'=h$ we have the equivalent representation $\pi_1 S=\{\gamma, h', \lambda \mid \gamma^{2n}=h'^{b'+bn}, \lambda h'\lambda^{-1}=h'^{-1}, \lambda\gamma\lambda^{-1}=h'^{b}\gamma^{-1}, \gamma h'\gamma^{-1}=h'^{-1}\}$. Then $\gamma$ and $h'$ generate the subgroup isomorphic to $\pi_1 L'$ such that $\gamma$ and $\gamma h'$ corresponds to the exceptional fibers of multiplicity 2 for the fibration of $L'$ on $S^2(2, 2, |b'+nb|)$. Note that $\gamma h'^b$ is conjugate to $\gamma$ (for $b$ even) or $\gamma h'$ (for $b$ odd). Thus we can see that $\pi_1 S=\pi_1(L'\underset{\sigma}{\times}S^1)$ if $b$ is odd and $=\pi_1(L'\underset{\tau}{\times}S^1)$ if $b$ is even. Since the bundle structure over $S^1$ with fiber $L'$ of $L'\underset{\lambda'}{\times}S^1$ is unique (Proposition 8) we obtain the desired results.

Summarizing the above results together with Proposition 8 below we obtain Theorem A for the geometric manifolds of type $S^3\times E$. The correspondence between such manifolds and the Seifert fibrations can be seen by the above Propositions and the observations in § 2. We only indicate such correspondences (up to fiber preserving diffeomorphisms) in the exceptional cases appeared in Corollaries C and D in the following list.

LIST A-I.

(I) $L(p, q)\times S^1$

(1) $\{(n_1, a_1, b_1), (n_2, a_2, b_2)\}$ on $S^2(n_1, n_2)$ with $(\Sigma a_i/n_i, \Sigma b_i/n_i)\neq(0, 0)$ where the relations between $p$, $q$ and the other integers are indicated in Proposition 3.

(2) The cases when $L(p, q)=L(4k, 2k+1)$.

$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (m'k, 2a'k, 1)\right\}$ over $P^2(m'k)$ with g.c.d. $(m', 2a')=1$.

$\{\overline{(1, 0, 0)}, (2n'k, a'k, 1)\}$ over $D^2(2n'k)$ with g.c.d. $(2n', a')=1$ and $a'\neq 0$.

(3) The cases when $L(p, q)=L(2k, 1)$.

$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (m'k, a'k, 1)\right\}$ over $P^2(m'k)$ with g. c. d. $(m', a')=1$, $a'$ odd.

$\{\overline{(1, 0, 0)}, (n'k, a'k, 1)\}$ over $D^2(n'k)$ with g. c. d. $(n', a')=1$, $n'$ odd (if $a'=0$ then $n'=1$).

(4) The cases when $L(p, q)=L(k, 1)$ with $k$ odd.

$\left\{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, (n, a, 1)\right\}$ over $P^2(n)$ with $k=$g. c. d. $(2a+1, n)$.

$\{\overline{(1, 0, 1)}, (n'k, a'k, -(n'k+1)/2)\}$ over $D^2(n'k)$ with g. c. d. $(n', a')=1$ and $n'$ odd (if $a'=0$ then $n'=1$).

(II) $L(p, q)\underset{\tau}{\times}S^1$

(1) $\{\overline{(1, 0, b)}, \overline{(a_1, 0, b_1)}, \overline{(a_2, 0, b_2)}\}$ on $D^2(\bar{a}_1, \bar{a}_2)$ where $p=ba_1a_2+a_1b_2+a_2b_1$, $q=ma_2+nb_2$ for $m, n\in \boldsymbol{Z}$, $ma_1-n(ba_1+b_1)=1$.

(2) The cases when $L(p, q)=L(4k, 2k+1)$.
$\{\overline{(k, 0, 1)}, (2, 1, 0)\}$ on $D^2(2, \bar{k})$ with $k>1$.

(3) The cases when $L(p, q)=L(2k, 1)$.
$\{(2, 1, 0), (2, -1, 0), (k, 0, 1)\}$ over $S^2(2, 2, k)$.

$\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (n, a, k)\right\}$ over $P^2(n)$ with g. c. d. $(n, a)=1$, $a$ odd.

$\{\overline{(1, 0, 0)}, (n, a, k)\}$ over $D^2(n)$ with g. c. d. $(n, a)=1$, $a\neq 0$, and $n$ odd.

(4) The cases when $L(p, q)=L(4, 1)$.

$\left\{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, (n, a, 2)\right\}$ over $P^2(n)$ with g. c. d. $(a+1, n)=1$, $n$ odd.

$\{\overline{(1, 0, 1)}, (2m, a, 1-m)\}$ over $D^2(2m)$ with g. c. d. $(2m, a)=1$.

(5) The cases when $L(p, q)=L(k, 1)$ with $k$ odd.
$\{(2, 1, 0), (2, -1, 0), (k, 0, -(k+1)/2)\}$ over $S^2(2, 2, k)$.

$\left\{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, (n, a, k)\right\}$ over $P^2(n)$ with g. c. d. $(2a+k, n)=1$, $b$ odd.

$\{\overline{(1, 0, 1)}, (n, a, (k-n)/2)\}$ over $D^2(n)$ with g. c. d. $(n, a)=1$, $n$ odd and $a\neq 0$.

(*) $L(p, q)\underset{\lambda\sigma\tau}{\times}S^1=L(p, q)\underset{\lambda}{\times}S^1$ if $q\equiv 1 \pmod{p}$, $L(p, q)\underset{\lambda\sigma}{\times}S^1=L(p, q)\underset{\lambda}{\times}S^1$ if $q\equiv -1 \pmod{p}$, $L(2, 1)\underset{\lambda}{\times}S^1=L(2, 1)\times S^1$, $S^3\underset{\lambda}{\times}S^1=S^3\times S^1$ for any $\lambda$. Furthermore we have $L(p, q)\underset{\lambda}{\times}S^1=L(p, -q)\underset{\tau\lambda}{\times}S^1$ for $\lambda=\sigma$ or $\tau\sigma$ (cf. Remark 1-4).

The following proposition is proved by a direct calculation (cf. [**15**], [**9**]) and so we only give the sketch of the proof.

PROPOSITION 8. *Let* $S_i=F_i\underset{\lambda_i}{\times}S^1$ *be an* $F_i$*-bundle over* $S^1$ *with monodromy* $\lambda_i$ *where* $F_i$ *is a spherical* 3*-manifold* $(i=1, 2)$. *If* $S_1$ *is diffeomorphic to* $S_2$ *or* $\pi_1S_1=\pi_1S_2$ *with* $F_1=F_2$ *then there is a (weak) bundle isomorphism betweem them.*

*Moreover* $S_1$ *is diffeomorphic to* $S_2$ *if and only if* $\pi_1 S_1 = \pi_1 S_2$ *unless* $S_i = L_i \times S^1$ *or* $L_i \underset{\tau}{\times} S^1$ *for some lens space* $L_i$ $(i=1, 2)$.

SKETCH OF PROOF. Suppose that there is a diffeomorphism $\varphi$ between $S_1$ and $S_2$. Let $p_i : S_i \to S^1$ be the bundle projection. We may assume that $\lambda_i$ is one of the diffeomorphisms appeared in § 2. Since the map $p_* : \pi_1 S_i \to \pi_1 S^1$ induced by $p$ is a canonical map $\pi_1 S_i \to H_1 S_i / Torsions$ there is a following commutative diagram.

$$\begin{array}{ccccc} \pi_1 F_1 & \longrightarrow & \pi_1 S_1 & \longrightarrow & \boldsymbol{Z} \\ \downarrow \phi & & \downarrow \varphi_* & & \downarrow \bar{\varphi}_* \\ \pi_1 F_2 & \longrightarrow & \pi_1 S_2 & \longrightarrow & \boldsymbol{Z} \end{array}$$

where $\phi$ is the isomorphism induced by $\varphi_*$, $\bar{\varphi}_* = \pm id$. Then we have a diffeomorphism between the infinite cyclic coverings of $S_1$ ond $S_2$ induced by $p_1$ which yields an $h$-cobordism between $F_1$ and $F_2$. Hence $F_1$ is diffeomorphic to $F_2$ (cf. [**15**], [**9**]). Put $F = F_1 = F_2$. The above diagram shows that there is an element $\gamma \in \pi_1 F$ such that $\lambda_{2*}^{\pm 1}(x) = \gamma(\phi_* \lambda_{1*} \phi_*^{-1}(x))\gamma^{-1}$ for any $x \in \pi_1 F$ where $\lambda_{i*}$ is an automorphism of $\pi_1 F$ induced by $\lambda_i$. If $F$ is a lens space then $\lambda_{1*} = \lambda_{2*}$ on $\pi_1 F$ ($\lambda_* = \lambda_*^{-1}$ in either case) and hence $\lambda_1$ must be isotopic to $\lambda_2$ ([**2**], [**8**]). If $F$ is not a lens space then the general fiber $h$ of the Seifert fibering of $F$ over $\Delta = S^2(n_1, n_2, n_3)$ generates the center of $\pi_1 F$. Thus we have $\lambda_{1*}(h) = \lambda_{2*}(h) = h^{\varepsilon}$ with $\varepsilon = \pm 1$. Conosider $F \underset{\lambda}{\times} S^1$ with $\lambda = id, \tau, \sigma, \tau\sigma, \rho, \tau\rho$ (if they exist). Note that $h$ has order 2 in $\pi_1 F$ if and only if $F$ belongs to the cases in Proposition 1. In the other cases we can distinguish $F \underset{\lambda}{\times} S^1$ with $\lambda = id, \tau\sigma, \rho$ from those with $\lambda = \tau, \sigma, \tau\rho$. If $\pi_1(F \underset{\lambda}{\times} S^1) = \pi_1(F \underset{\mu}{\times} S^1)$ then $\lambda_*^{\pm 1} = \phi_* \mu_* \phi_*^{-1}$ on $H_1 F$ for some isomorphism $\phi_*$ of $H_1 F$. Thus considering the actions of $\lambda$ on $H_1 F$ explicitly we can see that $F \underset{\lambda}{\times} S^1$'s for different $\lambda$'s listed above have mutually different fundamental groups. The above proof is still valid under the assumption $\pi_1(F_1 \underset{\lambda_1}{\times} S^1) = \pi_1(F_2 \underset{\lambda_2}{\times} S^1)$ if $F_1 = F_2$ or under the same assumption unless $F_1$ and $F_2$ are lens spaces with the same fundamental groups. If $F_i = L(p, q_i)$ and $\lambda_{i*}(x_i) = x_i^{t_i}$ for the generator $x_i$ of $H_1(F_i) = \boldsymbol{Z}_p$ we can see by the above argument that $t_2^{\pm 1} \equiv t_1$ (mod $p$). If $t_i \equiv \pm 1$ (mod $p$) then $\lambda_i = id$ or $\tau$ (up to isotopy). If $t_i \not\equiv \pm 1$ (mod $p$) then $q_1^2 \equiv q_2^2 \equiv 1$ (mod $p$), $q_1, q_2 \neq \pm 1$, and $t_i \equiv \pm q_i$ (mod $p$) (and hence $\lambda_i = \sigma$ or $\tau\sigma$). Therefore $q_1 \equiv \pm q_2$ (mod $p$) and $F_1 \underset{\lambda_1}{\times} S^1$ is bundle isomorphic to $F_2 \underset{\lambda_2}{\times} S^1$ (cf. [**9**] Lemma 13). This proves Proposition 8.

COROLLARY 9. $S = F \underset{\lambda}{\times} S^1$ *has a complex structure* (*and is diffeomorphic to a Hopf surface*) *if and only if* $\lambda$ *preserves the orientation of the fiber of some Sei-*

*fert fibration of $F$ over an orientable orbifold up to isotopy. In this case $S$ is one of the Followings. The corresponding Seifert fibrations are given by* (2-3), (2-4), (3-3) *and the previous propositions.*

(1) *$L(p, q)\times S^1$ and extra cases $L(p, q)\underset{\lambda}{\times} S^1$ for $\lambda=\tau\sigma$, $\sigma$ (which induces the multiplication by $\pm q$ on $\pi_1 L(p, q)$ if $q^2\equiv 1 \pmod p$) (see Remark* 1-4).

(2) *$F\times S^1$, $F\underset{\tau\sigma}{\times} S^1$ for $F=\{(1, b), (2, 1), (2, 1), (n, k)\}$ and one extra case $F\underset{\rho}{\times} S^1$ if $(n, k)=(2, 1)$ for every $b\in\boldsymbol{Z}$.*

(3) *$F\times S^1$ if $F=\{(1, b), (2, 1), (3, 1), (3, \varepsilon)\}$ with $\varepsilon=\pm 1$ and one extra case $F\underset{\sigma\tau}{\times} S^1$ if $\varepsilon=1$ for every $b\in\boldsymbol{Z}$.*

(4) *$F\times S^1$ for $F=\{(1, b), (2, 1), (2, 1), (4, \pm 1)\}$ for every $b\in\boldsymbol{Z}$.*

(5) *$F\times S^1$ for $F=\{(1, b), (2, 1), (3, 1), (5, k)\}$ with $k=\pm 1, \pm 2, b\in\boldsymbol{Z}$.*

REMARK. (1) The Hopf surface of the form $F\underset{\rho}{\times} S^1$ in (2) is missing in [**9**] Theorem 9 (see [**10**]). (2) Every manifold $F\underset{\lambda}{\times} S^1$ of type $S^3\times E$ satisfies Wu's criterion and so has an almost complex structure.

PROOF. Any manifold listed above is a Seifert 4-manifold over an orientable spherical or bad 2 orbifold $S^2(m_1, \cdots, m_k)$ and is the elliptic surface $L_{x_1}(m_1, a_1/m_1+b_1\omega/m_1)\cdots L_{x_k}(m_k, a_k/m_k+b_k\omega/m_k)\boldsymbol{P}^1\times E$ [**11**] where $E$ is the elliptic curve of period $(1, \omega)$ and $L_x$ denotes the logarithmic transformation at $x\in\boldsymbol{P}^1$. On the other hand any Hopf surface with abelian fundamental group is diffeomorphic to (a lens space)$\times S^1$ ([**9**], [**11**]) and every Hopf surface with non-abelian fundamental group is an elliptic surface ([**11**] Theorem 32). Since every elliptic Hopf surface must be of the above form ([**11**] Theorem 27) it must coincides with one in the above list.

## § 4. Geometric 4 manifods of type $S^2\times E^2$.

In this section we discuss the structures of geometric 4 manifolds of type $S^2\times E^2$. Let $S=\Gamma\backslash S^2\times E^2$ where $\Gamma=\pi_1 S$ is a discrete subgroup of $Isom^+(S^2\times E^2)$ $=(Isom\, S^2\times Isom\, E^2)^+$ acting freely on $S^2\times E^2$. Let $\Gamma_0=\Gamma\cap$(The kernel of the projection $p: Isom(S^2\times E^2)\to Isom\, E^2$) and $\bar{\Gamma}=p(\Gamma)$. Then $\Gamma_0=1$ and $\Gamma=\bar{\Gamma}$ since $\Gamma_0\subset SO_3$ acts freely on $S^2$. Hence $S$ has a fibration over $B=\Gamma\backslash E^2$ with general fiber $S^2$ induced by the projection $S^2\times E^2\to E^2$. Furthermore any nontrivial cyclic subgroup $G_x\subset\Gamma$ fixing some point $x\in E^2$ acts on $S^2\times\{x\}$ freely. Therefore $G_x=\boldsymbol{Z}_2$ which acts on $E^2$ as a reflection and an antipodal map on $S^2\times\{x\}$. Hence the base $B$ is either $T^2$, $K$ (the Klein bottle), $A$(the Annulus), or $M$ (the Möbius band) and the fiber over a reflector point is $\boldsymbol{RP}^2$. Hereafter we will simply call $S$ an $S^2$ bundle over $B$. Let $(z, w)$ be the coordinates of $S^2\times E^2$

with $z\in \boldsymbol{C}\cup\infty$, $w\in \boldsymbol{C}$. We have just two orientable $S^2$ bundles $S_0$ and $S_1$ over $B=T^2$ or $K$ with $w_2S_0=0$, $w_2S_1\neq 0$ respectively where $w_2$ is the second Stiefel Whitney class ([**13**]). For the cases with $B=A$, $M$ we have

PROPOSITION 10. *There is only one $S^2$ bundle over $B$ up to diffeomorphisms if $B=A$ or $M$.*

PROOF. Let $S$ be such a bundle and let $I$ be an orbifold which is a unit interval with 2 reflector points. Then there is just one orientable $S^2$ bundle $N$ over $I$ which is diffeomorphic to $P^3\# P^3$ ([**19**]) and $S$ is an $N$ bundle over $S^1$. But $\pi_0(Diff^+(P^3\# P^3))=\boldsymbol{Z}_2$ whose generator interchanges 2 copies of $P^3-D^3$ in $P^3\# P^3$ ([**17**], Lemma 3.2). Hence there are just two $N$-bundles over $S^1$ one of which is the $S^2$ bundle over $A$ and the other is the $S^2$ bundle over $M$.

*Case* 1. $B^2=T^2$. In this case $\pi_1S=\boldsymbol{Z}^2$ generated by $\alpha$ and $\beta$ whose action on $S^2\times E^2$ is defined by $\alpha(z, w)=(\bar{\alpha}(z), w+a)$, $\beta(z, w)=(\bar{\beta}(z), w+b)$ where $a, b\in \boldsymbol{C}$ are linearly independent over $\boldsymbol{R}$ (we can assume that $a=1$, $b=i$) and $\bar{\alpha}, \bar{\beta}\in SO_3$. Since $\bar{\alpha}\bar{\beta}=\bar{\beta}\bar{\alpha}$ either $\bar{\alpha}$ and $\bar{\beta}$ are the rotations with the common axis or $\bar{\alpha}$ and $\bar{\beta}$ have order 2 with the mutually perpendicular axes. Then we can assume that either

(1-1) $\bar{\alpha}(z)=\gamma z$, $\bar{\beta}(z)=\delta z$ for some $\gamma, \delta\in \boldsymbol{C}$, $|\gamma|=|\delta|=1$ or

(1-2) $\bar{\alpha}(z)=-z$, $\bar{\beta}(z)=1/z$.

In either case $S$ is diffeomorphic to a ruled surface of genus 1 ([**20**]). In case (1-1) $\bar{\alpha}$ is isotopic to the identity through the isotopy commuting with $\bar{\beta}$ and $\bar{\beta}$ is also isotopic to the identity. Therefore $S=T^2\times S^2$. Suppose that $\gamma=\exp(2\pi ia/n)$, $\delta=1$ with g.c.d. $(n, a)=1$. Fix $a'\in \boldsymbol{Z}$ such that $aa'=1 \pmod n$. Then $\alpha^n$, $\alpha^{a'}$, $\beta$ generate $\pi_1S$ and yields the fibration $\{(n, a', 0), (n, -a', 0)\}$ (where $\alpha^n$ and $\beta$ form a lattice of the general fiber). Conversely any Seifert 4 manifold of the form $\{(n, a, b), (n, -a, -b)\}$ is equivalent to that of the above type. In case (1-2) $\bar{\alpha}$ and $\bar{\beta}$ generate the group $\bar{\Gamma}$ acting on $S^2$ such that $S^2/\bar{\Gamma}=S^2(2, 2, 2)$ whose cone points correspond to 0 and $\infty$, $\pm 1$, $\pm i$. Hence $S=\{(2, 0, -1), (2, -1, 0), (2, 1, 1)\}$. Here the period of the general fiber is 2 and $2i$. We have the lift $T$ of the base $B$ of the form $\gamma_{s,t}=(1-t)\exp(\pi is)+t\exp(-\pi is)$ for $0\leqq s$, $t\leqq 1$. Then we can see that $T\cdot T$ is odd and this implies that $w_2S\neq 0$ (see [**20**]).

*Case* 2. $B=K$. In this case we can assume that $\pi_1S=\pi_1K=\{\alpha, \beta\,|\,\alpha\beta\alpha^{-1}\beta=1\}$ acts on $S^2\times E^2$ by $\alpha(z, w)=(\bar{\alpha}(z), \bar{w}+1/2)$, $\beta(z, w)=(\bar{\beta}(z), w+i)$ where $\bar{\alpha}(z)=(w_1\bar{z}-\bar{w}_2)/(w_2\bar{z}+\bar{w}_1)$, $\bar{\beta}(z)=\rho z$ with $w_1, w_2\in \boldsymbol{C}$, $|w_1|^2+|w_2|^2=1$, $\rho\in \boldsymbol{C}$,, $|\rho|=1$ (take conjugation by some elements in $SO_3$). By the relation $\alpha\beta=\beta^{-1}\alpha$ we have the following 3 cases.

(2-1) $\rho=1$, $\bar{\alpha}$ is arbitrary.

(2-2) $\bar{\alpha}(z)=\rho'\bar{z}$ for $\rho'\in \boldsymbol{C}$, $|\rho'|=1$, $\rho$ is arbitrary ($w_2=0$).

(2-3) $\bar{\alpha}(z)=\rho'\bar{z}^{-1}$ for $\rho'\in \boldsymbol{C}$, $|\rho'|=1$, $\bar{\beta}(z)=-z$ ($w_1=0$).

In case (2-2) replacing $z$ by $\rho'^{-1/2}z$ we can assume that $\rho'=1$. We denote $S$ in cases (2-1), (2-2), (2-3) by $S_0(\bar{\alpha})$, $S_0'(\rho)$, $S_1(\rho')$ respectively.

PROPOSITION 11. *$w_2(S)=0$ if $S=S_0(\bar{\alpha})$, $S_0'(\rho)$ and $w_2(S)\neq 0$ if $S=S_1(\rho')$.*

PROOF. In case (2-1) $S_0(\bar{\alpha})$ is an $S^2\times S^1$ bundle over $S^1$ with monodromy represented by $\bar{\alpha}(z, \theta)=(\bar{\alpha}(z), -\theta)$ for $z\in \boldsymbol{C}\cup\infty$, $\theta\in S^1$. But $\bar{\alpha}$ is isotopic to the map $\bar{\alpha}_0(z, \theta)=(\bar{z}, -\theta)$. Then the map $w\to(r, w)$ for any $r\in \boldsymbol{R}\subset \boldsymbol{C}\cup\infty$ gives rise to a cross section $K$ of $S_0(\bar{\alpha}_0)$ satisfying $K^2=0$ as the element of $H_2(S_0(\bar{\alpha}_0), \boldsymbol{Z}_2)$ and hence $w_2(S)=0$. In case (2-2) the self-diffeomorphism $f$ of $S^2\times E^2$ defined by $f(z, w)=(\rho^{\mathrm{Im}\, w}z, w)$ commutes with $\alpha$ and $f^{-1}\beta f(z, w)=(z, w+i)$. Therefore $f$ induces the diffeomorphism between $S_0'(\rho)$ and $S_0'(1)=S_0(\bar{\alpha}_0)$. In case (2-3) putting $\rho'(t)=-\exp(it\theta)$ for $\rho'=-\exp(i\theta)$ we have the map $f(z, w)=(\rho'(2\,\mathrm{Re}\, w)z, w)$ with $f\beta=\beta f$ and $f^{-1}\alpha f(z, w)=(-\bar{z}^{-1}, \bar{w}+1/2)$ which induces the diffeomorphism between $S_1(\rho')$ and $S_1(-1)$. Let $\bar{K}=\{w=s+ti\,|\,0\leqq s\leqq 1/2,\ -1/2\leqq t\leqq 1/2\}$ be the fundamental region of $K$. Then the map from $\bar{K}$ to $S^2\times E^2$ defined by $w=s+ti\to((1-2s)\exp\pi i(t+\theta+1/2)+2s\cdot\exp\pi i(-t+\theta-1/2), w)$ gives rise to a cross section $K_\theta$ of the $S^2$ bundle $S_1(-1)$. We can see that $K_0$ and $K_{1/2}$ intersect exactly when $t=0$, $s=1/4$ at one point $0\in \boldsymbol{C}\cup\infty$. Since $K_{1/2}$ is homotopic to $K_0$, the self-intersection number $K_0^2$ of $K_0$ in $H_2(S, \boldsymbol{Z}_2)$ is equal to $K_0\cdot K_{1/2}\neq 0$. Therefore $w_2(S_1(-1))\neq 0$.

*Case* 3. $B=A$. In this case we can assume that $\pi_1 S=\pi_1^{orb}A=\{\alpha, \beta, \iota\,|\,\iota^2=1, [\alpha, \beta]=1, \iota\alpha\iota^{-1}=\alpha, \iota\beta\iota^{-1}=\beta^{-1}\}$ acts on $S^2\times E^2$ by $\iota(z, w)=(-\bar{z}^{-1}, \bar{w})$, $\alpha(z, w)=(\bar{\alpha}(z), w+1)$, $\beta(z, w)=(\bar{\beta}(z), w+i)$. By the relation $\iota\beta\iota^{-1}=\beta^{-1}$ and the fact that $\beta\iota$ must induce the free involution on $S^2$ the action of $\pi_1 S$ is reduced to that satisfying $\bar{\alpha}(z)=\rho z$, $\bar{\beta}(z)=z$ for $\rho\in S^1\subset \boldsymbol{C}$. The diffeomorphism type of $S$ does not depend on $\rho$.

*Case* 4. $B=M$. In this case $\pi_1 S=\pi_1^{orb}M=\{\alpha, \beta, \iota\,|\,\alpha\beta\alpha^{-1}\beta=1, \iota\alpha\iota^{-1}=\beta\alpha, \iota\beta\iota^{-1}=\beta^{-1}, \iota^2=1\}$. Since the antipodal map commutes with every element of $O_3$ we can assume that $\iota(z, w)=(-\bar{z}^{-1}, \bar{w}+i/2)$, $\alpha(z, w)=(\bar{\alpha}(z), \bar{w}+1/2)$, $\beta(z, w)=(z, w+i)$ where $\bar{\alpha}\in O_3\setminus SO_3$. The diffeomorphism type of $S$ does not depend on the choice of $\bar{\alpha}$.

Now we describe the structures of the corresponding Seifert 4 manifolds for each $S$.

THEOREM 12. *Let $S$ be an $S^2$ bundle over $B$ with $B=T^2, K, A, M$. Then $S$ has the structures of Seifert 4 manifolds as follows.*

LIST A-II

(I) $B=T^2$, $w_2S=0$.

(1) $\{(n, a, b), (n, -a, -b)\}$ over $S^2(n, n)$.

(II) $B=T^2$, $w_2S\neq 0$.

(1) $\{(2, 0, -1), (2, -1, 0), (2, 1, 1)\}$ over $S^2(2, 2, 2)$.

(III) $B=K$, $w_2S=0$.

(1) $\left\{\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, (n, k, 0)\right\}$ over $P^2(n)$ with $k$ odd.

(2) $\{(n, k, 0)\}$ over $D^2(n)$ with $n$ odd.

(3) $\{(\overline{n, 0, k}), (\overline{n, 0, -k})\}$ over $D^2(\bar{n}, \bar{n})$.

(IV) $B=K$, $w_2(S)\neq 0$.

(1) $\{(\overline{1, 0, -1}), (2n, 2a, n)\}$ over $D^2(2n)$ with $n$ odd.

(2) $\left\{\begin{pmatrix}1 & 1\\0 & -1\end{pmatrix}, (2n, k, 0)\right\}$ over $P^2(2n)$.

(V) $B=A$.

(1) $\{(n, k, 0)\}$ over $D^2(n)$ with $n$ even.

(2) $\left\{\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, (n, 2a, 0)\right\}$ over $P^2(n)$.

(VI) $B=M$.

(1) $\left\{\begin{pmatrix}1 & 1\\0 & -1\end{pmatrix}, (n, k, 0)\right\}$ over $P^2(n)$ with $n$ odd.

(2) $\{(\overline{1, 0, -1}), (2n, k, n)\}$ over $D^2(2n)$ with $k$ odd.

PROOF. Cases (I) and (II) were already treaded. First consider Case (III). Choose the action of the generators $\alpha$, $\beta$ of $\pi_1 S$ defined by $\alpha(z, w)=(\rho\bar{z}^{-1}, \bar{w}+1/2)$, $\beta(z, w)=(z, w+i)$ where $\rho=\exp(2\pi ik/n)$ with g.c.d. $(n, k)=1$.

*Case* (i). $n=2m$, $m\in \boldsymbol{Z}$. Choose $a, b\in \boldsymbol{Z}$ such that $ak+bn=1$ ($k$ and $a$ are odd). Then $\pi_1 S$ is generated by $\alpha^a$, $l=\alpha^{2m}$, $h=\beta$ where $l$ and $h$ form a lattice of $T^2$ of period $m$ and $i$ and $\alpha^a(z, w)=(\exp(2\pi i/2m)\bar{z}^{-1}, \bar{w}+a/2)$. Then $\alpha^{2a}$ induces the $\boldsymbol{Z}_m$ action on $S^2\times T^2$ such that $S^2\times T^2/\boldsymbol{Z}_m=\{(m, a, 0), (m, -a, 0)\}$. Furthermore $\alpha^a$ induces the involution on $S^2\times T^2/\boldsymbol{Z}_m$ acting on $S^2(m, m)$ as the antipodal map and preserving (resp. reversing) the orientation of $l$ (resp. $h$). Thus we have $S=\left\{\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, (m, a, 0)\right\}$. Conversely every such Seifert 4 manifold for $m, a\in \boldsymbol{Z}$, g.c.d. $(m, a)=1$ with $a$ odd is induced by the above process since there exist $b, k\in \boldsymbol{Z}$ such that $ak+2bm=1$. This proves III (1).

*Case* (ii). $n$ is odd. There exist $a, b\in \boldsymbol{Z}$ such that $2ak+bn=1$ with $a$ even. Then $\pi_1 S$ generated by $\iota=\alpha^{-n}$, $q=\alpha^{2a}$ and $\beta=h$ with $\iota(z, w)=(\bar{z}^{-1}, \bar{w}-n/2)$, $q(z, w)=(\exp(2\pi i/n)z, w+a)$, $h(z, w)=(z, w+i)$. This yields the Seifert fibration $\{(n, a, 0)\}$ over $D^2(n)$ with general fiber $l=\iota^2$, $h$. Every Seifert 4 manifold $\{(n, a, 0)\}$ with $n$ odd is induced by the above process. This proves III (2). Consider the action of $\alpha$ and $\beta$ of the form $\alpha(z, w)=(\bar{z}, \bar{w}+1/2)$, $\beta(z, w)=(\rho z, w+i)$ where $\rho=\exp(2\pi ik/n)$ with g.c.d. $(n, k)=1$. For some $a, b\in \boldsymbol{Z}$ with $ak+bn=1$, $\pi_1 S$ is generated by $\iota=\alpha$, $q=\beta^a$, $h=\beta^n$ with $q(z, w)=(\exp(2\pi i/n)z,$

$w+ai$) such that $l=\iota^2$ and $h$ form a lattice of $T^2$ of period 1 and $ni$. Considering the induced action of $\iota$ on $S^2\times T^2$ divided by $q$, we have $S=\{(\overline{n, 0, a}), (\overline{n, 0, -a})\}$. Here $a$ attains an arbitrary integer such that g.c.d. $(n, a)=1$. This proves III (3).

The other cases can be proved similarly as follows. In either case below we take the new generators $(q, \iota, l, h, \sigma, \cdots)$ of $\pi_1 S$ in which the elements $l$ and $h$ form a lattice of the general fiber $T^2$. Futhermore we can see that every Seifert manifold in IV~VI is obtained by one of the processes below. By $G\langle a\rangle$ we mean the group $G$ generated by $a$.

Case IV, $B=K$, $w_2S\neq 0$ with the action of $\pi_1 S$ defined by $\alpha(z, w)=(\rho\bar{z}^{-1}, \bar{w}+1/2)$, $\beta(z, w)=(-z, w+i)$ with $\rho=\exp(2\pi ik/n)$, g.c.d. $(n, k)=1$.

*Case* (IV(1)). $n$ is odd. Take $a, b\in \boldsymbol{Z}$ such that $4ka+nb=1$. Put $q=\alpha^{2a}\beta^b$, $\iota=\alpha^n$, $l=\alpha^{2n}$, $h=\beta^2$ with $q(z, w)=(\exp(2\pi i/2n)z, w+a+bi)$, $\iota(z, w)=(\bar{z}^{-1}, \bar{w}+n/2)$. Then $S^2\times T^2/\boldsymbol{Z}_{2n}\langle q\rangle=\{(2n, -2a, -nb), (2n, 2a, nb)\}$ on which $\iota$ acts as the involution descending to a reflection of $S^2(2n, 2n)$ which interchanges two cone points. Thus we have $S=\{(\overline{1, 0, b}), (2n, -2a, -nb)\}$ ($q\iota q^{-1}\iota^{-1}=h^b$ in $\pi_1 S$) which is equivalent to $\{(\overline{1, 0, -1}), (2n, 2a, n)\}$ since $b$ is odd (replace $q$),

*Case* (IV(2)). $n=2m$ for some $m\in \boldsymbol{Z}$. Suppose that $m=2m'$. (If $m$ is odd we have the Seifert 4 manifold of the same type as in case (IV(1)).) Put $\sigma=\alpha^{k'}$, $l=\alpha^{4m'}$ and $h=\alpha^{2m'}\beta^{-1}$ where $k'\in \boldsymbol{Z}$ such that $kk'\equiv 1 \pmod{4m'}$ with $\sigma^2(z, w)=(\exp(2\pi i/2m')z, w+k')$, $\sigma(z, w)=(\exp(2\pi i/4m')\bar{z}^{-1}, \bar{w}+k'/2)$. Then $S^2\times T^2/\boldsymbol{Z}_{2m'}\langle\sigma^2\rangle=\{(2m', k', 0), (2m', -k', 0)\}$ on which the involution induced by $\sigma$ acts so that it yields $S=\left\{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, (2m', k', 0)\right\}$ (use $\sigma(l, h)\sigma^{-1}=(l, lh^{-1})$).

Case V. $B=A$ with the action of $\pi_1 S$ defined by $\iota(z, w)=(-\bar{z}^{-1}, \bar{w})$, $\alpha(z, w)=(\rho z, w+1)$, $\beta(z, w)=(z, w+i)$ with $\rho=\exp(2\pi ik/n)$, g.c.d. $(n, k)=1$.

*Case* (V(1)). $n=2m$ for $m\in \boldsymbol{Z}$. Take $k'\in \boldsymbol{Z}$ such that $kk'\equiv 1 \pmod{2m}$. Put $\sigma=\iota\alpha^m$, $q=\alpha^{k'}$, $h=\beta$ and also $l=\alpha^{2m}=\sigma^2$. Then $S^2\times T^2/\boldsymbol{Z}_{2m}\langle q\rangle=\{(2m, k', 0), (2m, -k', 0)\}$ on which $\sigma$ acts as the involution and yields $S=\{(2m, k', 0)\}$ on $D^2(2m)$.

*Case* (V(2)). $n$ is odd. Choose $a$ such that $2ak+n\equiv 1$ (mod. $2n$). Put $\sigma=\alpha^a\iota$, $l=\alpha^n$, $h=\beta$ with $\sigma(z, w)=(\exp(2\pi i/2n)\bar{z}^{-1}, \bar{w}+a)$. Then $S^2\times T^2/\boldsymbol{Z}_n\langle\sigma^2\rangle=\{(n, 2a, 0), (n, -2a, 0)\}$ on which $\sigma$ acts as the involution and yields $S=\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (n, 2a)\right\}$ over $P^2(n)$ (use $\sigma(l, h)\sigma^{-1}=(l, h^{-1})$).

Case VI. $B=M$ with the action of $\pi_1 S$ defined by $\iota(z, w)=(-\bar{z}^{-1}, \bar{w}+i/2)$, $\alpha(z, w)=(-\rho\bar{z}^{-1}, \bar{w}+1/2)$, $\beta(z, w)=(z, w+i)$ with $\rho=\exp(2\pi ik/n)$, g.c.d. $(n, k)=1$.

*Case* (VI(1)). $n$ is odd. Take an odd integer $a$ such that $2ka+n=1$ (mod $2n$). Put $\sigma=\alpha^a$, $l=\alpha^{2n}$, $h=\alpha^n\iota$ ($\beta=\iota\alpha\iota^{-1}\alpha^{-1}$) with $\sigma(z, w)=(\exp(2\pi i/2n)\bar{z}^{-1},$

$\bar{w}+a/2)$. Hence $S^2\times T^2/\boldsymbol{Z}_n\langle\sigma^2\rangle=\{(n, a, 0), (n, -a, 0)\}$ on which $\sigma$ acts as the involution and yields $S=\left\{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, (n, a, 0)\right\}$ (use $\sigma(l, h)\sigma^{-1}=(l, lh^{-1})$ which comes from $[\iota, \alpha^2]=\iota^2=1$) which is equivalent to $\left\{\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, (n, a-n, 0)\right\}$ (replace $\sigma$ by $\sigma h$).

*Case* (VI(2)). $n=2m$ for $m\in\boldsymbol{Z}$. First preform the parallel translation $w\to w+(m-1)i/4$ to get the new representation of $\pi_1 S$ defined by $\iota(z, w)=(-\bar{z}^{-1}, \bar{w}+mi/2)$, $\alpha(z, w)=(-\rho\bar{z}^{-1}, \bar{w}+(1+(m-1)i)/2)$, $\beta(z, w)=(z, w+i)$ where $\rho=\exp(2\pi ik/2m)$, g. c. d. $(2m, k)=1$. Then for an odd integer $a$ with $ak\equiv 1$ (mod. $2m$) $\pi_1 S$ is generated by $\sigma=\iota(\alpha\iota)^{-m}$, $q=(\alpha\iota)^a$, $l=\sigma^2$, $h=\beta$. (From the relations $\iota^2=\alpha\beta\alpha^{-1}\beta=1$, $\beta=\iota\alpha\iota^{-1}\alpha^{-1}$, we have $l=\alpha^{-2m}$ and $\beta^{-m}\alpha^{2m}=(\alpha\iota)^{2m}$.) Here $q(z, w)=(\exp(2\pi i/2m)z, w+(1-i)a/2)$, $\sigma(z, w)=(\bar{z}^{-1}, \bar{w}-m/2)$. Then $S^2\times T^2/\boldsymbol{Z}_{2m}\langle q\rangle=\{(2m, a, am), (2m, -a, -am)\}$ on which $\sigma$ acts as the involution and yields $S=\{\overline{(1, 0, -a)}, (2m, a, am)\}$ which is equivalent to $\{\overline{(1, 0, -1)}, (2m, a, m)\}$ since $a$ is odd.

It is easy to check that every Seifert 4 manifold over the orientable bad or spherical 2 orbifold with $e=0$ appeared in List A-II in Theorem 12 where we have just six manifolds up to diffeomorphisms distinguished by $w_2$ and the fundamental groups. This class coincides with the class of the closed orientable geometric 4 manifolds of type $S^2\times E^2$. Thus we have proved Theorem A for the cases of type $S^2\times E^2$.

COROLLARY 13. *A Seifert 4-manifold S in List* A-II *is diffeomorphic to a complex surface if and only if S is an $S^2$ bundle over $T^2$.* (*They are the ruled surfaces of genus* 1.)

See [**23**] for the complex structures compatible with $S^2\times E^2$. The first betti number $b_1 S$ of any $S^2$-bundle $S$ over $K$, $A$ or $M$ is 1. On the other hand $S$ has an unbranced covering $\tilde{S}$ which is an $S^2$ bundle over $T^2$ with $b_1=2$. But $b_1 S\equiv b_1\tilde{S}$ (mod 2) if $S$ is a complex surface ([**23**]). Thus $S$ does not have a complex structure. We also note that every member in List A-II satisfies Wu's criterion and has an almost complex structure.

## §5. Geometric 4 manifolds of type $H^2\times E^2$ and $\widetilde{SL}_2\times E$.

First we describe the invariants of a Seifert 4 manifold $S$ over a hyperbolic 2 orbifold $B$. Suppose that $B$ has no reflectors. Denote $B$ by $\Sigma_g(m_1, \cdots, m_k)$ (resp. $\Sigma'_g(m_1, \cdots, m_k)$) if the underlying space $|B|$ of $B$ is orientable (resp. non-orientable) of genus $g$ with $k$ cone points whose cone angles are $2\pi/m_1, \cdots, 2\pi/m_k$. Then if we fix the lattice $(l, h)$ of the general fiber of $S$ we have the following representation of $\pi_1 S$. The identity matrix is denoted by I.

$$\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, q_1, \cdots, q_k, l, h \mid [l, h] = [q_i, l] = [q_i, h] = 1, \alpha_i(l, h)\alpha_i^{-1} = (l, h)A_i, \beta_i(l, h)\beta_i^{-1} = (l, h)B_i, q_j^{m_j} l^{a_j} h^{b_j} = 1, \Pi_i[\alpha_i, \beta_i]\Pi_j q_j = l^a h^b\}$$

if $B=\Sigma_g(m_1, \cdots, m_k)$,

$$\{v_1, \cdots, v_g, q_1, \cdots, q_k, l, h \mid [l, h] = [q_i, l] = [q_i, h] = 1, v_i(l, h)v_i^{-1} = (l, h)A_i', q_j^{m_j} l^{a_j} h^{b_j} = 1, \Pi_i v_i^2 \Pi_j q_j = l^a h^b\}$$ if $B=\Sigma_g'(m_1, \cdots, m_k)$.

Here $\{\alpha_i, \beta_i\}$ and $\{v_i\}$ are the sets of the oriented loops projecting to the standard generators $\{\bar{\alpha}_i, \bar{\beta}_i\}$ of $H_1\Sigma_g$ and $\{\bar{v}_i\}$ of $H_1\Sigma_g'$ respectively, $q_i$ is the lift of the meridian $\bar{q}_i$ of the $i$-th cone point, $A_i$, $B_i$ (resp. $A_i'$) are the monodromy matrices along $\bar{\alpha}_i$, $\bar{\beta}_i$ (resp. $\bar{v}_i$) with $A_i, B_i \in SL_2\boldsymbol{Z}$, $A_i' \in GL_2\boldsymbol{Z} \setminus SL_2\boldsymbol{Z}$, $\Pi_i[A_i, B_i] = \Pi_i A_i'^2 = I$, $(m_i, a_i, b_i)$ is the type of the $i$-th multiple torus and $(a, b)$ is the obstruction to extending $q_i'$s to the cross section over $B-\bigcup_i$ (the meridian disk of the $i$-th cone point). If $B$ has reflectors we have extra invariants ([22]). Let $C_1, \cdots, C_s$ be the reflector circles of $B$, $N_t$ be the annular neighborhood of $C_t$ in $B$, and $B_0=B-\bigcup_t \operatorname{int} N_t$. Put $\bar{\tau}_t=\partial N_t - C_t$ oriented as in Fig. 5. Over $B_0$ there are the monodromy matrices and the types of the multiple tori with respect to $(l, h)$ as before: $A_j, B_j \in SL_2\boldsymbol{Z}$ (resp. $A_j' \in GL_2\boldsymbol{Z} \setminus SL_2\boldsymbol{Z}$), along $\bar{\alpha}_j$, $\bar{\beta}_j$ (resp. $\bar{v}_j$) which are the standard generators other than $\bar{\tau}_t$'s of $H_1B_0$ when $B_0$ is orientable (resp. non-orientable). Over $N_t$ choose the lattice $(l_t, h_t)$ of $T^2$ so that the reflection $\iota_t$ along $C_t$ is represented by $J=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to $(l_t, h_t)$. Fix the lift $\tau_t$ of $\bar{\tau}_t$ which are the boundaries of the cross section over $B_0-\bigcup$ (the meridian disk for the cone point) extending the lifts $q_j$'s of the meridian circles $\bar{q}_j$'s for the cone points. Then we have the types $(\overline{m_{tk}', 0, b_{tk}'})$ of the multiple Klein bottles with respect to the lifts $q_{tk}$ of the meridians $\bar{q}_{tk}$ of the cone points on the double cover $\tilde{B}$ of $B$ projecting to the corner reflectors on $C_t$ and the euler class$(\overline{a_t'', b_t''})$ of $C_t$ defined with respect to $\tau_t$ as in §1. The monodromy $E_t$ along $C_t$ is $\pm I$ and $a_t''=0$ if $E_t=I$ and $a_t''=-1$ if $E_t=-I$ (cf. [22]). Finally we have the coordinate transformations of the general fibers at $\bar{\tau}_t$'s: $(l_t, h_t)=(l, h)P_t$ for $P_t \in SL_2\boldsymbol{Z}$. If we take the curves $\bar{\sigma}_1, \cdots, \bar{\sigma}_s$ in $\tilde{B}$ as in Figure 5 then the monodromy along $\bar{\sigma}_t$ is $P_t J P_t^{-1} J$ with respect to $(l, h)$. We note that $P_t$ can be replaced by $-P_t$ and so only the value of $P_t$ in $PSL_2\boldsymbol{Z}$ makes sense. (Replace $l_t$, $h_t$ and the lift $\sigma_t$ of $\bar{\sigma}_t$ if necessary.) We can assume that $P_1=I$. If $A_t=B_t=P_t=I$ or $A_t'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $P_t=I$ for each $t$, or $A_t'=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $P_t=I$ for each $t$ we can define the rational euler class $e(S)$ of $S$ by $e=(a+\Sigma(a_i/m_i),\ b+\Sigma(b_i/m_i))$ (if $B$ is orientable) or $e=b+\Sigma_i(b_i/m_i)+\Sigma_t(b_t''+\Sigma_k b_{tk}'/m_{tk}')/2$ (for the other cases). If $B$ has reflectors the above invariants are defined so that $b=0$. The following proposition which claims the uniqueness of the Seifert fibrations over the hyperbolic orbifolds is proved in [**25**] when the bases have no reflectors. The proof for the general cases is almost the same and is omitted (use the fact that for a closed hyperbolic 2-orbifold $B$ $\pi_1^{orb}B$ has no nontrivial normal abelian subgroup and any automorphism of $\pi_1^{orb}B$ is induced by a self-isomorphism of $B$ even if $B$ has reflectors ([**12**], [**26**], corollary 6.6.10)).

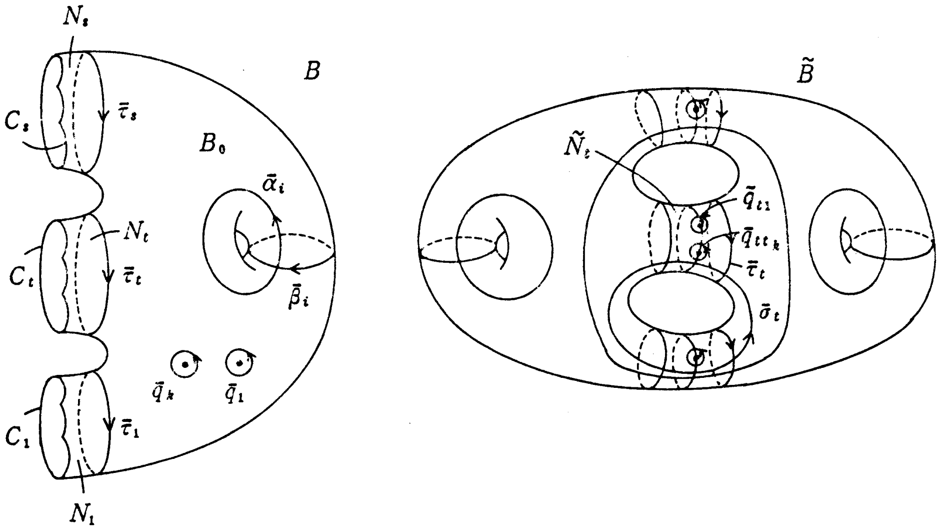


Figure 5.

PROPOSITION 14. *Let* $p_i: S_i \to B_i$ $(i=1, 2)$ *be the closed orientable Seifert* 4 *manifolds over the hyperbolic* 2 *orbifolds with* $\pi_1 S_1 = \pi_1 S_2$. *Then there is a fiber preserving diffeomorphism between* $S_1$ *and* $S_2$.

Next consider $S=\Gamma\backslash X$ where $\Gamma$ is a discrete subgroup of $G_X=Isom^+X$ acting freely on $X=H^2\times E^2$ or $\widetilde{SL}_2\times E$. If $X=H^2\times E^2$ then the identity component $G_X^0$ of $G_X=(Isom H^2\times Isom E^2)^+$ is $Isom^0H^2\times Isom^0E^2$, and $G_X/G_X^0=\boldsymbol{Z}_2$. If $X=\widetilde{SL}_2\times E$ then $G_X=Isom\widetilde{SL}_2\times \boldsymbol{R}$, $G_X^0=Isom^0\widetilde{SL}_2\times \boldsymbol{R}$ and $G_X/G_X^0=\boldsymbol{Z}_2$ with $Isom^0\widetilde{SL}_2 =\boldsymbol{R}\times_{\boldsymbol{Z}}\widetilde{SL}_2\boldsymbol{R}$. Here $\boldsymbol{Z}$ is the center of $\widetilde{SL}_2\boldsymbol{R}$ and the $\boldsymbol{R}$ factor acts as the translations of the fiber of the fibration $\boldsymbol{R}\to\widetilde{SL}_2\to H^2$. In either case $G_X$ preserves the fibration $\boldsymbol{R}^2\to X\to H^2$ so that the generator of $G_X/G_X^0$ reverses the orientations of the fiber and the base. The action of $\Gamma$ induces the Seifert fibration $\Gamma_0\backslash\boldsymbol{R}^2\to\Gamma\backslash X\to\bar{\Gamma}\backslash H^2$ where $\Gamma_0=\Gamma\cap$(The kernel of the projection $p: Isom X\to Isom H^2$) and $\bar{\Gamma}=p(\Gamma)$ ([**23**], [**24**], § 3). Next we determine the Seifert invariants of the fibration $\pi: S\to B$ with $B=\bar{\Gamma}\backslash H^2$ of $S$ induced by the the above process.

THEOREM B. (1) *Let* $S$ *be a geometric* 4 *manifold of type* $H^2\times E^2$ *or* $\widetilde{SL}_2\times E$.

*Then the Seifert fibration of S over the hyperbolic 2 orbifold B induced by the above process satisfies one of the following conditions up to fiber preserving diffeomorphisms.*

(I) *B is orientable.*

(i) $X=H^2\times E^2$. *All the monodromy matrices are some powers of the common matrix Q, where Q is conjugate in* $SL_2\boldsymbol{Z}$ *to* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. *If all the monodromies are trivial then the rational euler class e is zero.*

(ii) $X=\widetilde{SL_2}\times R$. *All the monodromies are trivial and e is nonzero.*

(II) *B is non-orientable without reflectors.*

(i) $X=H^2\times E^2$. *The monodromy matrix* $A_j'$ $(j=1, \cdots, g)$ *along the loop* $\bar{v}_j$ *is either* $\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *for every j or* $A_j$ *is either* $\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\pm\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, *or* $\pm\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ *for every j (up to conjugation by a common matrix in* $GL_2\boldsymbol{Z}$*). If* $A_j'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *for every j or* $A_j'=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ *for every j(up to conjugation by a common matrix) then e is zero.*

(ii) $X=\widetilde{SL_2}\times E$. $A_j'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *for every j or* $A_j'=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ *for every j (up to conjugation by a common matrix) and e is nonzero.*

(III) *B has reflectors. Let* $P_t$, $A_j$, $B_j$ *(resp.* $A_j'$*) be the matrices explained above.*

(i) $X=H^2\times E^2$. $A_j$, $B_j$ *are either* $\pm I$, $\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *for every j (or* $A_j'$ *is either* $\pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *or* $\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *if* $B_0$ *is non-orientable) and* $P_t=I$ *or* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *(up to sign) for every t. If* $A_j=B_j=P_t=I$ *for every j, t, (resp* $A_j'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $P_t=I$ *for every j, t if* $B_0$ *is non-orientable) then* $e=0$.

(ii) $X=\widetilde{SL_2}\times E$. $A_j=B_j=P_t=I$ *for every j, t (resp.* $A_j'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $P_t=I$ *for every j, t) and e is nonzero.*

*Conversely in either case every Seifert 4 manifold satisfying the above condition has a geometric structure of type* $H^2\times E^2$ *or* $\widetilde{SL_2}\times E$.

(2) *A Seifert 4 manifold S over a hyperbolic 2 orbifold does not admit a geometric structure if S is not one of the classes listed in* (1).

**Proof of Theorem B (1) for** $X=H^2\times E^2$.

Let $(z, w)$ for $z\in\boldsymbol{C}$, $\operatorname{Im} z>0$, $w\in\boldsymbol{C}$ be the coordinate of $H^2\times E^2$. The action of $\Gamma$ for $S=\Gamma\backslash H^2\times E^2$ induces the faithful discrete representations of $\pi_1^{orb}B$ to *Isom* $H^2$ and we have the images $\bar{\alpha}_j$, $\bar{\beta}_j$, $\bar{v}_j$, $\bar{q}_j$, $\bar{q}_{t,k}$, $\bar{\tau}_t$, $\bar{\sigma}_t$, $\bar{\iota}$ of the generators of

$\pi_1^{orb}B$ in *Isom* $H^2$ (Fig. 5). Here $\bar{\alpha}_j$, $\bar{\beta}_j$, $\bar{v}_j$, $\bar{\tau}_t$, $\bar{\sigma}_t$, $\bar{q}_j$, $\bar{q}_{t,k}$ correspond to the curves with the same symbols explained as before and $\bar{\iota}$ is the reflection along $C_1$. Then to determine the action of $\Gamma$ we need to give the action of the lift $\gamma$ of each generator $\bar{\gamma}$ and the lattice $l$, $h$ of the general fiber on $H^2\times E^2$.

*Case* (I). $B$ is orientable without reflectors (cf. [23] for this case). The action of $\Gamma$ must be given as follows.

$$l(z, w)=(z, w+c),\quad h(z, w)=(z, w+d) \quad \text{where } c=u+u'i \text{ and } d=v+v'i$$

$(u, u', v, v'\in \boldsymbol{R})$ are linearly independent on $\boldsymbol{R}$. $\alpha_j(z, w)=(\bar{\alpha}_j(z), \rho_j w+w_j)$, $\beta_j(z, w)=(\bar{\beta}_j(z), \rho'_j w+w'_j)$ where $\rho_j, \rho'_j\in S^1\subset \boldsymbol{C}$, $w_j, w'_j\in \boldsymbol{C}$. We also deduce $q_j(z, w)=(\bar{q}_j(z), w-(a_jc+b_jd)/m_j)$ from $[q_j, l]=[q_j, h]=1$ and $q_j^{m_j}l^{a_j}h^{b_j}=1$.

Putting $R(\rho)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ for $\rho=\exp(i\theta)$ we have

$$(1)\qquad R(\rho_j)=PA_jP^{-1},\quad R(\rho'_j)=PB_jP^{-1} \quad \text{for } P=\begin{pmatrix}u & v\\ u' & v'\end{pmatrix},$$

from the relations $\alpha_j(l, h)\alpha_j^{-1}=(l, h)A_j$, $\beta_j(l, h)\beta_j^{-1}=(l, h)B_j$. Furthermore we duduce from $\Pi[\alpha_j, \beta_j]\Pi q_j=l^a h^b$

$$(2)\qquad \sum_j(\rho_j-1)w'_j-\sum_j(\rho'_j-1)w_j=(a+\sum_j a_j/m_j)c+(b+\sum_j b_j/m_j)d\,.$$

From (1) we see that $A_j$, $B_j$ are commutative, $|trA_j|$, $|trB_j|\leqq 1$ if $A_j, B_j\neq \pm I$ and hence $A_j$, $B_j$ are periodic. Therefore either $\rho_j$, $\rho'_j$ are the powers of $i$ for every $j$ or $\rho_j$, $\rho'_j$ are the powers of $\exp(\pi i/3)$ for every $j$. This implies that all of $A_j$, $B_j$ are some powers of a common matrix $Q=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ or $\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$ up to conjugation in $SL_2\boldsymbol{Z}$. Fix $c=u+u'i$, $d=v+v'i$ satisfying (1). If all the monodromies are trivial ($\rho_j=\rho'_j=1$ for every $j$) we must have $e=(a+\sum a_j/m_j, b+\sum b_j/m_j)=(0, 0)$. Conversely any Seifert 4-manifold $S$ over $B$ satisfying the above conditions have the desired action of $\pi_1 S$ on $H^2\times E^2$ by reversing the above process. For, if $\rho_j\neq 1$ or $\rho'_j\neq 1$ for some $j$, we can put $w_k=w'_k=0$ except for $w'_j$ (if $\rho_j\neq 1$) or $w_j$ (if $\rho'_j\neq 1$) and arrange $w'_j$ or $w_j$ so that it satisfies (2).

*Case* (II). $B$ is non-orientable without reflectors. In this case the action of $\Gamma$ is represented by $l(z, w)=(z, w+c)$, $h(z, w)=(z, w+d)$, $v_j(z, w)=(\bar{v}_j(z), \rho_j\bar{w}+w_j)$, $q_j(z, w)=(\bar{q}_j(z), w-(a_jc+b_jd)/m_j)$ where $c=u+u'i$, $d=v+v'i$ are linearly independent over $\boldsymbol{R}$, $\rho_i\in S^1\subset\boldsymbol{C}$, $w_i\in\boldsymbol{C}$. Then from $v_j(l, h)v_j^{-1}=(l, h)A'_j$ we have

$$(1)\qquad PA'_jP^{-1}=R(\rho_j)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{for } P=\begin{pmatrix}u & v\\ u' & v'\end{pmatrix}.$$

Therefore $A'_j$ has order 2 and is conjugate in $GL_2\boldsymbol{Z}$ to $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ $\left(\text{or equivalently } \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$. Performing the coordinate change $w\to\rho_1^{-1/2}w$ we can assume that $\rho_1=1$. Furthermore by some coordinate transformation of $(l, h)$ we can assume that $A_1'=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Suppose that $A_1'=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Then by (1) $P=\begin{pmatrix}u & 0\\ 0 & v'\end{pmatrix}$. Since $A_j'\in GL_2\boldsymbol{Z}$ and by (1) we have $A_j'=\pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ (and $\rho_j=\pm 1$) or $A_j'=\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ (and $\rho_j=\pm i$, $u=\pm v'$). Next suppose $A_1'=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Then $P=\begin{pmatrix}t & t\\ 1 & -1\end{pmatrix}$ for some $t\in\boldsymbol{R}$ up to scalar multiplication. Again by (1) and since $A_j'\in GL_2\boldsymbol{Z}$ we have the following possibilities.

(a) $A_j'=\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ for arbitrary $t$, $\rho_j=\pm 1$,

(b) $A_j'=\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ for $t=\pm 1$, $\rho_j=\pm 1, \pm i$.

(c) $A_j'=\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ for $t=\pm 1/\sqrt{3}$, $\rho_j=\pm 1$, $\pm\exp(\pm\pi i/3)$,

(d) $A_j'=\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \pm\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, \pm\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ for $t=\pm\sqrt{3}$, $\rho_j=\pm 1$, $\pm\exp(\pm\pi i/3)$.

But (c) is reduced to (d) by the conjugation by $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. On the other hand from the relation $\Pi v_j^2\Pi q_j=l^a h^b$ we have

(2) $$\sum(\rho_j\bar{w}_j+w_j)=(a+\sum a_j/m_j)c+(b+\sum b_j/m_j)d \qquad (\rho_1=1).$$

Suppose $\rho_j=1$ for every $j$. Then we can assume that $A_j'=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ for every $j$ ($c\in\boldsymbol{R}$, $d\in\boldsymbol{R}i$) or $=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ for every $j$ ($c=2v$, $d=v+v'i$ for $v, v'\in\boldsymbol{R}$). In either case there are $w_j$'s satisfying (2) if and only if $b+\sum b_j/m_j=0$. This implies that the rational euler class is zero. Suppose that $\rho_j\neq 1$ for $j\geqq 2$ ($\rho_j=-1$, $\pm i$, $\pm\exp(\pm\pi i/3)$). Then putting $w_k=0$ for $k\neq 1, j$, we can arrange $w_1$, $w_j$ so that $w_1+\bar{w}_1+\rho_j\bar{w}_j+w_j$ is an arbitrary complex number. This proves the claim for Case (II).

*Case* (III) *B has reflectors.* In this case $\pi_1 S$ is generated by $l$, $h$ (the lattice of the general fiber), $\tau_t$ ($t=1, \cdots, s$), $q_j$ ($j=1, \cdots, k$), $\alpha_i$, $\beta_i$ (or $v_i$) ($i=1, \cdots, g$), $\sigma_t$ ($t=1, \cdots, s$), $q_{tk}$ ($k=1, \cdots, s_t$, $t=1, \cdots, s$), $\iota$ where $\gamma$ is the lift of $\bar{\gamma}$ for any $\gamma$ satisfying the following relations (cf. [22]).

(a) $[l, h]=[l, q_i]=[h, q_i]=[l, q_{tk}]=[h, q_{tk}]=1$,

(b) $q_j^{m_j}l^{a_j}h^{b_j}=1$,

(c) $\alpha_j(l, h)\alpha_j^{-1}=(l, h)A_j$, $\beta_j(l, h)\beta_j^{-1}=(l, h)B_j$ or $v_j(l, h)v_j^{-1}=(l, h)A_j'$,

$\tau_t(l, h)\tau_t^{-1}=(l, h)E_t$, $\sigma_t(l, h)\sigma_t^{-1}=(l, h)P_tJP_t^{-1}J$, $\Pi[\alpha_i, \beta_i]\Pi q_j=\Pi\tau_t$

or $\Pi v_i^2\Pi q_j=\Pi\tau_t$

where $A_j$, $B_j \in SL_2\boldsymbol{Z}$, $A'_j \in GL_2\boldsymbol{Z} \setminus SL_2\boldsymbol{Z}$, $E_j = \pm I$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $P_t \in SL_2\boldsymbol{Z}$ with $\Pi[A_j, B_j] = \Pi E_t$ or $\Pi A_j'^2 = \Pi E_t$ $(P_1 = I)$.

(d) $q_{tk}^{m'_{tk}} h_t^{b'_{tk}} = 1$ where $(l_t, h_t) = (l, h)P_t$,

(e) $\iota^2 = l$, $\iota h \iota^{-1} = h^{-1}$, $\iota_t^2 = l_t$, $\iota_t h_t \iota_t^{-1} = h_t^{-1}$ where $\iota_t = \sigma_t \iota$,

(f) $\iota_t q_{t1} \iota_t^{-1} = q_{tt_k}^{-1} \cdots q_{t2}^{-1} q_{t1}^{-1} q_{t2} \cdots q_{tt_k}, \cdots, \iota_t q_{tt_k} \iota_t^{-1} = q_{tt_k}^{-1}$,

(g) $\tau_t q_{t1} \cdots q_{tt_k} \iota_t \tau_t^{-1} \iota_t^{-1} = l_t^{a''_t} h_t^{b''_t}$ where $a''_t = 0$ if $E_t = I$, $a''_t = -1$ if $E_t = -I$.

Fix the representation of $\pi_1^{orb} B$ as before. Then the actions of the generators of $\pi_1 S$ must be of the following form.

$\iota(z, w) = (\iota(z), \bar{w} + w_0)$ (replace the coordinate $w$ if necessary),

$l(z, w) = (z, w + c)$, $h(z, w) = (z, w + d)$, $\alpha_j(z, w) = (\bar{\alpha}_j(z), \rho_j w + w_j)$,

$\beta_j(z, w) = (\bar{\beta}_j(z), \rho'_j w + w'_j)$ or $v_j(z, w) = (\bar{v}_j(z), \rho_j \bar{w} + w_j)$

where $\rho_j$, $\rho'_j \in S^1$, $w_j$, $w'_j \in \boldsymbol{C}$, $q_j(z, w) = (\bar{q}_j(z), w - (a_j c + b_j d)/m_j)$,

$\tau_t(z, w) = (\bar{\tau}_t(z), \varepsilon_t w + \tilde{w}_t)$ where $\varepsilon_t = 1$, (if $E_t = I$) or $= -1$ (if $E_t = -I$), $\tilde{w}_t \in \boldsymbol{C}$,

$q_{tk}(z, w) = (\bar{q}_{tk}(z), w + y_{tk})$, $\sigma_t(z, w) = (\bar{\sigma}_t(z), \lambda_t w + \tilde{w}'_t)$ with $\lambda_t \in S^1$, $y_{tk'}$, $\tilde{w}'_t \in \boldsymbol{C}$.

Next we determine the monodromies. Replacing the coordinate $w$ if necessary we can assume that $\iota(z, w) = (\bar{\iota}(z), \bar{w} + u/2)$, $l(z, w) = (z, w + u)$, $h(z, w) = (z, w + vi)$ for $u, v \in \boldsymbol{R}$ from (e). Hence the element $\gamma$ of the form $\gamma(z, w) = (\bar{\gamma} z, \rho w + w_0)$, $\rho \in S^1$, $w_0 \in \boldsymbol{C}$ with $\gamma(l, h)\gamma^{-1} = (l, h)Q$ for $Q \in SL_2\boldsymbol{Z}$ $(\gamma = \alpha_j, \beta_j, \tau_\tau, \sigma_\tau)$ must satisfy $\rho = \pm 1$ and $Q = \pm I$, $\rho = i$ and $Q = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ according to $u = \pm v$ or $\rho = -i$ and $Q = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ according to $u = -v$ or $v$. For $v_j(z, w) = (\bar{v}_j(z), \rho_j \bar{w} + w_j)$ either $\rho_j = \pm 1$ and $A'_j = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\rho_j = i$ and $A'_j = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ according to $u = \pm v'$, or $\rho_j = -i$ and $A'_j = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ according to $u = -v'$ or $v'$. In particular for $\sigma_t$ we can see that only the following cases can occur: $P_t = \pm I$ and $P_t J P_t^{-1} J = I$ $(\lambda_t = 1)$ or $P_t = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $P_t J P_t^{-1} J = -I$ $(\lambda_t = -1)$. We can assume that $P_t = I$ in the first case and $P_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the second case (replace $\iota_t$, $l_t$, $h_t$, $\sigma_t$ if necessary). Now we determine the other parameters. If $\lambda_\tau = 1$, $P_\tau = I$, we have

$$(l_t, h_t) = (l, h), \ \sigma_t(z, w) = (\bar{\sigma}_t(z), w + \tilde{w}'_t), \ \iota_t(z, w) = (\bar{\sigma}_t \bar{\iota}(z), \bar{w} + u/2 + \tilde{w}'_t).$$

From (d) and (e) we see that

(i) $\tilde{w}'_t = s_t i$ for some $s_t \in \boldsymbol{R}$.

(ii) $y_{tk} = -(b'_{tk}/m'_{tk}) vi$. ((f) is automatically satisfied.)

On the other hand from (g) we have

(iii) $u(\varepsilon_t-1)/2+\tilde{w}_t'(\varepsilon_t-1)+\tilde{w}_t-\bar{\tilde{w}}_t-\varepsilon_t\sum(b_{tk}'/m_{tk}')vi=a_t''u+b_t''vi.$

$(a_t''=0$ if $\varepsilon_t=1$ and $a_t''=-1$ if $\varepsilon_t=-1.)$

If $\lambda_t=-1$, $P_t=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, we have $l_t(z, w)=(z, w+vi)$, $h_t(z, w)=(z, w-u)$, $\iota_t(z, w)=(\bar{\sigma}_t\bar{\iota}(z), -\bar{w}-u/2+\tilde{w}_t')$. From (d), (e), (g) we deduce

(i′) $\tilde{w}_t'=e_t'+vi/2$ for some $e_t'\in\boldsymbol{R}$,

(ii′) $y_{tk}=ub_{tk}'/m_{tk}'$,

(iii′) $u(1-\varepsilon_t)/2+\varepsilon_t\tilde{w}_t'-\bar{\tilde{w}}_t'+\tilde{w}_t+\bar{\tilde{w}}_t+\varepsilon_t(\sum b_{tk}'/m_{tk}')u=a_t''vi-b_t''u.$

Therefore from (i), (i′), (iii), (iii′) we deduce

(iv) $\tilde{w}_t=e_t+(b_t''+\sum b_{tk}'/m_{tk}')vi/2$ for some $e_t\in\boldsymbol{R}$ if $\lambda_t=\varepsilon_t=1$,

$\tilde{w}_t=e_t+s_ti+(b_t''-\sum b_{tk}'/m_{tk}')vi/2$ for some $e_t$, $s_t\in\boldsymbol{R}$ if $\lambda_t=1$, $\varepsilon_t=-1$,

$\tilde{w}_t=s_t'i-(b_t''+\sum b_{tk}'/m_{tk})u/2$ for some $s_t'\in\boldsymbol{R}$ if $\lambda_t=-1$, $\varepsilon_t=1$,

$\tilde{w}_t=e_t'+s_t'i-(b_t''-\sum b_{tk}'/m_{tk}')u/2-u/2$ for some $e_t'$, $s_t'\in\boldsymbol{R}$ if $\lambda_t=-1$, $\varepsilon_t=-1$.

We note that we can arrange $e_t$, $s_t$, $e_t'$, $s_t'$ arbitrarily by changing $\tilde{w}_t$ and $\tilde{w}_t'$.

Finally we need to examine the relation (c).

*Case* (i). $|B|$ is orientable. From $\Pi[\alpha_j, \beta_j]\Pi q_j=\Pi\tau_t$ we have

(v) $\sum(\rho_j-1)w_j'-\sum(\rho_j'-1)w_j-\sum(a_ju+b_jvi)/m_j=\varepsilon_1\cdots\varepsilon_{s-1}\tilde{w}_s+\varepsilon_1\cdots\varepsilon_{s-2}\tilde{w}_{s-1}+\cdots+\tilde{w}_1$

where $\tilde{w}_t$ is given by (iv).

First suppose that $\rho_j=\rho_j'=\lambda_t=\varepsilon_t=1$ for every $j$, $t$. Then (v) is equivalent to

(v′) $\sum\limits_t e_t+(\sum\limits_j a_j/m_j)u+(\sum\limits_j b_j/m_j+\sum\limits_t(b_t''+\sum\limits_k b_{tk}'/m_{tk}')/2)vi = 0.$

Hence all the parameters can be well determined if and only if the rational euler class $e$ is zero. For the other cases we can find the parameters satisfying (v) as follows. Suppose that $\rho_j\neq 1$ for some $j$. Then putting $w_k=w_k'=0$ except for $w_j'$ we can arrange $w_j'$ so that (v) holds for any fixed $\tilde{w}_t$ and $\tilde{w}_t'$. If $\rho_j'\neq 1$ exchange the roles of $w_k$ and $w_k'$ to get the desired result. Suppose that $\varepsilon_t=-1$ for some $t$. Then putting $w_j=w_j'=0$, $\tilde{w}_k=0$ for $k\neq t$, we have $\tilde{w}_t$ satisfying $-\sum(a_ju+b_jvi)/n_j=\varepsilon_1\cdots\varepsilon_{t-1}\tilde{w}_t$. Finally suppose that $\rho_j=\rho_j'=\varepsilon_t=1$ for every $j$ and $t$ but $\lambda_t=-1$ for some $t$. Since $\lambda_1=1$ (we assumed that $P_1=I$) by putting $\tilde{w}_k=0$ for $k\neq 1, t$, we can reduce (v) to

$$\sum(a_ju+b_jvi)/n_j = e_1+(b_1''+\sum b_{1k}'/m_{1k}')vi/2+s_t'i-(b_t''+\sum b_{tk}'/m_{tk}')u/2\,.$$

Hence we can determine $e_1$, $s_t'$satisfying (v).

*Case* (ii). $|B|$ is non-orientable. From $\Pi v_j^2 \Pi q_j^2 = \Pi \tau_t$ we deduce

(vi) $\sum(\rho_j \bar{w}_j + w_j) - \sum(a_j u + b_j v i)/m_j = \varepsilon_1 \cdots \varepsilon_{s-1} \tilde{w}_s + \cdots + \tilde{w}_1$.

First suppose that $\rho_j = \varepsilon_t = \lambda_t = 1$ for any $j, t$. Then we can arrange $\tilde{w}_t$ to get the well determined representation if and only if $e = \sum b_j/n_j + \sum(b_t'' + \sum b_{tk}'/n_{tk}')/2 = 0$. Next suppose that $\varepsilon_t = \lambda = 1$ for every $t$ but $\rho_k \neq 1$ for some $k$. In this case $\rho_k = -1$ or $\pm i$. Then for any $r \in \boldsymbol{R}$ there is a $w_k \in \boldsymbol{C}$ satisfying $\mathrm{Im}(\rho_k \bar{w}_k + w_k) = ri$. Hence putting $w_j = 0$ for $j \neq k$ we can obtain $\tilde{w}_t$, $w_k$ which satisfy (v). If $\lambda_t = -1$ or $\varepsilon_t = -1$ for some $t$ the argument goes as in Case (i). This proves the claim of Theorem B (1) for the cases with $X = H^2 \times E^2$.

**Proof of Theorem B (1) for** $X = \widetilde{SL}_2 \times E$.

According to [**23**], §2 we identify $\widetilde{SL}_2 \times E$ with $H^2 \times \boldsymbol{C}$ with coordinates $(z, w)$ $(z, w \in \boldsymbol{C}, \mathrm{Im}\, z > 0)$. The $\boldsymbol{R}$-factor of $Isom^0 \widetilde{SL}_2 = \widetilde{SL}_2 \times_{\boldsymbol{Z}} \boldsymbol{R}$ acts on $X$ by translating $w$ by a pure imaginary (the generator of $\boldsymbol{Z}$ acts on $X$ by $(z, w) \to (z, w + 2\pi i)$) and the other $\boldsymbol{R}$-factor of $Isom^0 X = Isom^0 \widetilde{SL}_2 \times R$ acts on $X$ by translating $w$ by a real number. Moreover $Isom^+ X / Isom^0 X = Isom^+ \widetilde{SL}_2 / Isom^0 \widetilde{SL}_2 = \boldsymbol{Z}_2$ which is generated by $(z, w) \to (-\bar{z}, \bar{w})$ ([**23**], [**19**]). The action of the element $\bar{\gamma}: z \to (az+b)/(cz+d)$ in $PSL_2\boldsymbol{R}$ lifts to the action $\tilde{\gamma}(z, w) = (\bar{\gamma}(z), w - 2\log(cz+d))$. Here if $\bar{\gamma}$ is a hyperbolic element the imaginary part of the second factor is defined by the parallel translation of the unit tangent vector along the axis of $\bar{\gamma}$. If $\bar{\gamma}$ is an elliptic element of order $m$ choose the lift $\tilde{\gamma}$ of $\bar{\gamma}$ so that $\tilde{\gamma}^m = 1$ in *Isom* $X$ (arrange the translation part of $\tilde{\gamma}$ if necessary). Now we describe the action of $\pi_1 S = \Gamma$ on $X = \widetilde{SL}_2 \times E$ for $S = \Gamma \backslash X$. We use the same notations for generators of $\pi_1 S$ and $\pi_1^{orb} B$ as in the case $X = H^2 \times E^2$. Then the action of $\pi_1 S$ induces the images of the generators of $\pi_1^{orb} B$ in *Isom* $H^2$ as before.

*Case* I *B is orientable.* (cf. [**23**], theorem 7.4.)

The action of $\pi_1 S$ is given by $l(z, w) = (z, w + c)$, $h(z, w) = (z, w + d)$ where $c = u + u'i$, $d = v + v'i$ are linearly independent over $\boldsymbol{R}$ $(u, u', v, v' \in \boldsymbol{R})$, $\alpha_j(z, w) = \tilde{\alpha}_j(z, w) + (0, w_j)$, $\beta_j(z, w) = \tilde{\beta}_j(z, w) + (0, w_j')$, $q_j(z, w) = \tilde{q}_j(z, w) + (0, y_j)$ for $w_j, w_j', y_j \in \boldsymbol{C}$. Then the monodromies $A_j, B_j$ are trivial and $y_j = -(a_j c + b_j d)/m_j$ since $q_j^{m_j}(z, w) = (z, w + n_j y_j)$ and $q_j^{m_j} l^{a_j} h^{b_j} = 1$. We can see that $\Pi[\tilde{\alpha}_j, \tilde{\beta}_j] \Pi \tilde{q}_j$ coincides with the translation of the $w$ coorinate by $2\pi i \mathfrak{X}^{orb} B$ where $\mathfrak{X}^{orb}$ denotes the euler characteristic of the orbifold. This follows from the fact the holonomy angle along the geodesic triangle $\Delta$ is given by the area of $\Delta$ ([**19**]. §4). Since the translation along $\boldsymbol{C}$ commutes with the action of the lift of $PSL_2\boldsymbol{R}$, we deduce from $\Pi[\alpha_j, \beta_j] \Pi q_j = l^a h^b$

(1) $$2\pi i \mathfrak{X}^{orb} B = (a + \sum a_j/m_j) c + (b + \sum b_j/m_j) d\,.$$

Since $\mathfrak{X}^{orb}B\neq 0$ we have $e=(a+\sum a_j/m_j,\ b+\sum b_j/m_j)\neq(0,0)$. Conversely if $e(S)\neq(0,0)$ for the Seifert 4 manifold $S$ we can arrange $l$, $h$ so that $a+\sum a_j/m_j=0$. Then putting $c=u$, $d=vi$, $u$, $v\in\boldsymbol{R}$, we can determine the representation of $\pi_1 S$ satisfying (1). Then $S$ has a geometric structure of type $\widetilde{SL_2}\times E$.

*Case* II *B is non-orientable without reflectors.*

In this case we can assume that the action of $v_j$ is the composition of $(z, w)\to(-\bar{z}, \bar{w})$, $(z, w)\to\tilde{\delta}_j(z, w)$ for $\tilde{\delta}_j\in PSL_2\boldsymbol{R}$, and the translation along the $w$-coordinate. Then $v_j(z, w)=\tilde{\delta}_j(-\bar{z}, \bar{w})+(0, w_j')$ and $v_j^2(z, w)=\tilde{v}_j^2(z, w)+(0, w_j'+\bar{w}_j')$ where $\tilde{v}_j^2$ is the lift of $\bar{v}_j^2\in PSL_2\boldsymbol{R}$ given as above. On the other hand we have $v_j l v_j^{-1}(z, w)=(z, w+\bar{c})$, $v_j h v_j^{-1}(z, w)=(z, w+\bar{d})$. Then from $v_j(l, h)v_j^{-1}=(l, h)A_j'$ we deduce

$$(1)\qquad PA_j'P^{-1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad\text{for } P=\begin{pmatrix}u & v\\ u' & v'\end{pmatrix}.$$

Since $\Pi\tilde{v}_j^2\Pi\tilde{q}_j$ is the translation $w\to w+2\pi i\mathfrak{X}^{orb}B$ by the same reason as in Case I, we deduce

$$(2)\qquad \sum(w_j'+\bar{w}_j')+2\pi i\mathfrak{X}^{orb}B=(a+\sum a_j/m_j)c+(b+\sum b_j/m_j)d\,.$$

Therefore we can assume that either $A_j'=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ for every $j$ (and $c=u$, $d=v'i$, $u$, $v'\in\boldsymbol{R}$) or $A_j'=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ for every $j$ (and $c=2v$, $d=v+v'i$, $v$, $v'\in\boldsymbol{R}$). In either case we can define the well determined representation of $\pi_1 S$ if and only if $b+\sum b_j/n_j\neq 0$.

*Case* III *B has reflectors.*

Performing some coordinate transformation we can assume that $\iota(z, w)=(-\bar{z}, \bar{w}+u/2)$, $l(z, w)=(z, w+u)$, $h(z, w)=(z, w+vi)$ for $u$, $v\in\boldsymbol{R}$ from $\iota^2=l$, $\iota h\iota^{-1}=h^{-1}$. Define the actions of $\alpha_j$, $\beta_j$ (or $v_j$), $q_j$ as in Cases I and II. Then as before we have $A_j=B_j=I$ for every $j$ or $A_j'=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ for every $j$. (The case $A_j'=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$ cannot occur since $c=u$, $d=vi$.) The action of $\tau_t$, $\sigma_t$ are defined by $\tau_t(z, w)=\tilde{\tau}_t(z, w)+(0, \tilde{w}_t)$, $\sigma_t(z, w)=\tilde{\sigma}_t(z, w)+(0, \tilde{w}_t')$, $\tilde{w}_t$, $\tilde{w}_t'\in\boldsymbol{C}$. Then again all the monodromies along $\tau_t$, $\sigma_t$ must be trivial and hence $P_t=E_t=I$ and $(l_t, h_t)=(l, h)$ for every $t$. The actions of $\iota_t$ and $q_{tk}$ are given by $\iota_t(z, w)=\tilde{\sigma}_t(-\bar{z}, \bar{w})+(0, \tilde{w}_t'+u/2)$, $q_{tk}(z, w)=\tilde{q}_{tk}(z, w)+(0, -(b_{tk}'/m_{tk}')vi)$, where $\tilde{q}_{tk}$ is arranged so that $\tilde{q}_{tk}^{m_{tk}'}=1$. Consider the decomposition $B=B_0\cup\bigcup_t N_t$ as in Figure 5. Let $\tilde{N}_t$ be the standard double covering of $N_t$ without reflectors. Then over $\tilde{N}_t$ we have $\tau_{tk}\Pi q_{tk}\iota_t\tau_t^{-1}\iota_t^{-1}=h^{b''t}$ and $\iota_t\tau_t^{-1}\iota_t^{-1}(z, w)=(\widetilde{\iota_t\tau_t^{-1}\iota_t^{-1}})(z, w)+(0, -\bar{\tilde{w}}_t)$. Then over $\tilde{N}_t$ we have

$$(1)\qquad \tilde{w}_t-\bar{\tilde{w}}_t+2\pi i\mathfrak{X}^{orb}(\tilde{N}_t)=(b_t''+\sum_k b_{tk}'/m_{tk}')vi$$

where $\tilde{N}_t$ is considered as an orbifold with two boundaries. Since $2\mathcal{X}^{orb}N_t=\mathcal{X}^{orb}\tilde{N}_t$ (1) is equivalent to

$$(1') \qquad \tilde{w}_t=e_t-2\pi i\mathcal{X}^{orb}N_t+(b_t''+\sum b_{tk}'/m_{tk}')vi/2 \quad \text{for some } e_t\in\boldsymbol{R}.$$

On the other hand from $\prod[\alpha_j, \beta_j]\prod q_j=\prod\tau_t$ or $\prod v_j^2\prod q_j=\prod\tau_t$ we have

$$(2) \qquad 2\pi i\mathcal{X}^{orb}B_0-(\sum a_j/m_j)u-(\sum b_j/m_j)vi=\sum\tilde{w}_t \quad \text{or}$$

$$2\pi i\mathcal{X}^{orb}B_0+\sum(w_j'+\bar{w}_j')-(\sum a_j/m_j)u-(\sum b_j/m_j)vi=\sum\tilde{w}_t.$$

Then from (1′) and (2) we deduce

$$\sum e_t+(\sum a_j/m_j)u+\{\sum b_j/m_j+\sum(b_t''+\sum b_{tk}'/m_{tk}')/2\}vi=2\pi i\mathcal{X}^{orb}B \quad \text{or}$$

$$\sum e_t+(\sum a_j/m_j)u-2\sum(\mathrm{Re}\,w_j')+\{\sum b_j/m_j+\sum(b_t''+\sum b_{tk}'/m_{tk}')/2\}vi=2\pi i\mathcal{X}^{orb}B$$

since $\mathcal{X}^{orb}B=\mathcal{X}^{orb}B_0+\sum\mathcal{X}^{orb}N_t$. Then we have the well defined representation of $\pi_1S$ if and only if $e=\sum b_j/m_j+\sum(b_t''+\sum b_{tk}'/m_{tk}')/2\neq 0$ since $\mathcal{X}^{orb}B\neq 0$.

The proof of Theorem B is completed by the following proposition.

PROPOSITION 15. *Suppose that a Seifert 4 manifold $\pi: S\to B$ over a hyperbolic 2 orbifold $B$ admits a geometric structure of type $X$. Then $X=H^2\times E^2$ or $\widetilde{SL}_2\times E$.*

PROOF. It is easy to see that $S$ is not diffeomorphic to a Seifert 4 manifold over a euclidean 2 orbifold and hence $X\neq E^4$ (with one exception), $Nil^3\times E$, $Nil^4$, $Sol^3\times E$ ([**22**]). If $S$ has a structure of type $E^4$ then $\pi_1S$ contains an free abelian normal subgroup of rank 4. But the subgroup $H$ of $\pi_1S$ generated by the lattice $l$, $h$ of the general fiber of $S$ is the unique maximal normal abelian subgroup of $\pi_1S$. Therefore the case $X=E^4$ is also excluded. If $S=\Gamma\backslash X$ with $X=H^3\times E$ then $Isom^0X=Isom^0H^3\times\boldsymbol{R}$ and $\Gamma_0=\Gamma\cap\boldsymbol{R}$ is the lattice of the radical $\boldsymbol{R}$ of $Isom^0X$ ([**23**], § 2). ($Isom^+X/Isom^0X=\boldsymbol{Z}_2$ which acts as the reflections both on $H^3$ and $E$.) Hence $S$ is a fibration over a closed hyperbolic 3-orbifold $\bar{\Gamma}\backslash H^3$ with general fiber $S^1=\Gamma_0\backslash\boldsymbol{R}$. In this case $\bar{\Gamma}_0=\boldsymbol{Z}$ is the unique maximal normal abelian subgroup of $\Gamma$ since $\bar{\Gamma}$ has no such subgroup. Hence we also have $X\neq H^3\times E$. On the other hand $S$ is spherical and the euler number of $S$ is zero. Therefore if $X\neq H^2\times E^2$, $\widetilde{SL}_2\times E$ the remaining possible cases are $X=Sol^4_{m,n}$, $Sol^4_0$, $Sol^4_1$ ([**23**], Theorem 6.1. Here we use the notations in [**23**]. The geometry $F^4$ does not admit a compact model.) Suppose that $X=Sol^4_{m,n}$. Since $Sol^4_{n,n}=Sol^3\times E$ we can assume that $m\neq n$. In this case $Isom^0Sol^4_{m,n}=Sol^4_{m,n}$ (left multiplication). $Sol^4_{m,n}$ is the semi-direct product of $\boldsymbol{R}^3$ and $\boldsymbol{R}$ such that $t\in\boldsymbol{R}$ acts on $\boldsymbol{R}^3$ by $\gamma_{m,n}(t)=\exp t\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ where $a, b, c\in\boldsymbol{R}$ $a>b>c$, $a+b+c=0$, and $e^a$, $e^b$, $e^c$ are the roots of $\lambda^3-m\lambda^2+n\lambda-1=0$. Then if we put $\Gamma_0=\Gamma\cap Isom^0X$, $\Gamma_0\cap\boldsymbol{R}^3$ is a lattice of $\boldsymbol{R}^3$ ([**23**], § 2) and $\Gamma_0\backslash X$ has the structure of a $T^3$ bundle over

$S^1$ whose monodromy is conjugate in $SL_3\boldsymbol{R}$ to the matrix of the form $\gamma_{m,n}(t)$ above. On the other hand consider the exact sequence $1 \to H \to \pi_1 S \xrightarrow{p} \pi_1^{orb} B \to 1$ induced by the Seifert fibration of $S$. Then $\bar{\Gamma}_0 = p(\Gamma_0)$ is a subgroup of $\pi_1^{orb} B$ of finite index and hence contains a torsion free subgroup $\bar{\Gamma}'$ which is a fundamental group of a closed hyperbolic surface. Put $\Gamma' = p^{-1}(\bar{\Gamma}') \cap \Gamma_0$. Then $\Gamma' \subset Isom^0 X$ and hence again $\Gamma' \backslash X$ is a $T^3$ bundle over $S^1$ whose monodromy matrix $A$ is of the above form. Since any eigenvalue of $A$ is not 1, we have $b_1(\Gamma' \backslash X) = 1$ by the Wang sequence. But the projection $p: \Gamma' \to \bar{\Gamma}'$ induces the epimorphism $H_1 \Gamma' \to H_1 \bar{\Gamma}'$ and the rank of $H_1 \bar{\Gamma}'$ is greater than 1. This is a contradiction and hence $X \neq Sol^4_{m,n}$. Suppose that $X = Sol^4_0$ or $Sol^4_1$. If we put $\Gamma_0 = \Gamma \cap Isom^0 X$ and define the subgroup $\Gamma'$ of $\Gamma_0$ so that $p(\Gamma')$ is a surface group then we can see that $\Gamma' \backslash X$ is an Inoue surface by [**23**], Proposition 9.1 and hence $b_1(\Gamma' \backslash X) = 1$. This leads us to the same contradiction. This completes the proof.

COROLLARY 16. *A Seifert 4 manifold $S$ over a hyperbolic 2 orbifold $B$ is diffeomorphic to a complex surface if and only if $S$ is one of the classes in Theorem B*(I) *with an orientable base orbifold.*

PROOF. If $S$ is one of the list in Theorem B(I) $S$ has a geometric structure of type $X = H^2 \times E^2$ or $\widetilde{SL}_2 \times E$ and $\pi_1 S \subset Isom^0 X$. Then $S$ has a compatible complex structure and is an elliptic surface with Kodaira dimension 1 ([**23**], Theorem 1.1, 7.4). Conversely suppose that $S$ has a complex structure. $S$ has a $T^2$ bundle over a hyperbolic surface as its finite unbranched covering. Then the Enriques-Kodaira classification of the complex surfaces ([**11**]) shows that $S$ is an elliptic surface with Kodaira dimension 1 with no singular fibers other than multiple tori. Then $S$ has a compatible geometric structure ([**23**], Theorem 7.4) and is one of the classes in Theorem B (I).

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