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Vector valued invariants of prehomogeneous vector spaces

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0. Introduction.

0.1. Let G be a finite group acting linearly on a finite dimensional vector space V over a finite field $\boldsymbol{F}_q$. Let $\{v_0, \cdots, v_n\}$ be a complete set of representatives of V/G, $V_i = Gv_i$, $K_i = Z_G(v_i)$, $R: G \to GL(M)$ a complex representation, and M_i the set of K_i-fixed vectors in M. For each $m \in M_i$, there exists one and only one M-valued function $R_{i,m}$ on V_i such that $R_{i,m}(v_i) = m$ and $R_{i,m}(gv) = R(g)R_{i,m}(v)$ for $g \in G$ and $v \in V_i$. We extend $R_{i,m}$ by zero to the whole space V.

0.2. Our first problem is to know if the vector valued functions $R_{i,m}$ are similar in property to the complex powers of a relatively invariant polynomial function on a prehomogeneous vector space over the complex or real number field. (A rational representation of an algebraic group is called a prehomogeneous vector space, if the representation space has a Zariski open orbit.)

Let $V^{\vee}$ be the dual G-module of V, and define, in the same way as above, $\{v_0^{\vee}, \cdots, v_{n'}^{\vee}\}$, M'_i, and M-valued functions $R'_{i',m'}$ $(1 \leqq i' \leqq n',\ m' \in M'_{i'})$ such that $R'_{i',m'}(gv^{\vee}) = R(g)R'_{i',m'}(v^{\vee})$ for $g \in G$ and $v^{\vee} \in V^{\vee}$. As is easily seen, the Fourier transform of $R_{i,m}$ is a linear combination of these $R'_{i',m'}$'s. Provisionally in the introduction, let us assume that M_0 and M'_0 are one dimensional and spanned by m_0 and m'_0 respectively. Then the Fourier transform of R_{0,m_0} is a linear combination of R'_{0,m'_0} and $\{R'_{i',m'} \mid 1 \leqq i' \leqq n',\ m' \in M'_{i'}\}$. Hence if m_0 and m'_0 are given, the coefficient $c(R)$ of R'_{0,m'_0} is uniquely determined.

Our first problem is, more precisely, the evaluation of the coefficient $c(R)$. See (2.4) and (3.4) for our result, where we calculate the value of $c(R)$ for some examples. In many cases, we can say from the value of $c(R)$ that the Fourier transform of R_{0,m_0} is, in fact, equal to $c(R)R'_{0,m'_0}$. See (2.6).

0.3. Our second problem is to understand character sum analogues of the Fourier transforms of complex powers of relative invariants of non-reductive prehomogeneous vector spaces in terms of the vector valued relative invariants of reductive prehomogeneous vector spaces. Here we call a prehomogeneous vector space reductive if the algebraic group acting on it is reductive.

In [**10**; Chapter 3], two examples (examples 8 and 9) of non-reductive prehomogeneous vector spaces are studied, where explicit forms of the Fourier transforms (in the sense of tempered distribution) of products of complex powers of relative invariants are given, e.g.,

$$(0.3.1)\qquad (2\pi\sqrt{-1})^{-n^2/2}\int_{\Omega} e^{-\langle x|\sqrt{-1}y\rangle} P_1(x)^{s_1}\left(\frac{P_2(x)}{P_1(x)}\right)^{s_2}\cdots\left(\frac{P_n(x)}{P_{n-1}(x)}\right)^{s_n} dx$$
$$=\left\{\prod_{i=1}^{n}\frac{\Gamma(s_i+n+1-i)}{(2\pi\sqrt{-1})^{1/2}}\right\}\cdot\left\{\left(\frac{Q_{n-1}(\sqrt{-1}y)}{Q_n(\sqrt{-1}y)}\right)^{s_1}\cdots\right.$$
$$\left.\cdots\left(\frac{Q_1(\sqrt{-1}y)}{Q_2(\sqrt{-1}y)}\right)^{s_{n-1}}\left(\frac{1}{Q_1(\sqrt{-1}y)}\right)^{s_n}\right\}\cdot\{Q_n(\sqrt{-1}y)^{-n}\}$$

for $y\in M_n(\boldsymbol{R})$ such that $Q_i(y)>0$ $(1\leqq i\leqq n)$. Here

$$\Omega=\{x\in M_n(\boldsymbol{R})\mid P_i(x)>0\qquad(1\leqq i\leqq n)\},$$
$$dx=\prod_{i,j}dx_{ij}\,,\qquad \langle x\mid y\rangle=\mathrm{Tr}(xy)\,,$$
$$P_i(x)=\det(x_{\alpha\beta})_{1\leq\alpha,\beta\leq i},\ Q_j(y)=\det(y_{\alpha\beta})_{n-j+1\leq\alpha,\beta\leq n}\,,$$

and, $(2\pi\sqrt{-1})^s$ and $Q_j(\sqrt{-1}y)^s$ stand for $(2\pi)^s\exp(s\pi\sqrt{-1}/2)$ and $\exp(s\pi\sqrt{-1}j/2)\cdot Q_j(y)^s$ respectively. We shall show, in section 4, that a character sum analogue of (0.3.1) and a similar formula can be obtained from formulas describing the Fourier transforms of vector valued functions $R_{i,m}$ on the 'reductive prehomogeneous vector spaces' discussed in sections 2 and 3.

0.4. Analogous problem over the real or complex number field is discussed by Rubenthaler-Schiffmann [**9**] (for finite dimensional representations), and by Bopp-Rubenthaler [**2**] (for infinite dimensional representations). As is always the case for (finite dimensional) complex representations of finite reductive groups, our result can be regarded as a finite field analogue of the study of the infinite dimensional case.

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1. Preliminaries.

We keep the notations of (0.1).

LEMMA 1.1. *Assume that an M-valued function S on V satisfies*

$$S(gv)=R(g)S(v)\qquad(g\in G,\ v\in V)\,.$$

Then S is a linear combination of $\{R_{i,m} \mid 0 \leqq i \leqq n,\ m \in M_i\}$.

PROOF. It is enough to note that $S(v_i) \in M_i$.

1.2. Let $V^\vee$ be a G-module, and $\langle\ \rangle : V^\vee \times V \to \boldsymbol{F}_q$ a non-degenerate pairing such that $\langle gv^\vee | gv \rangle = \langle v^\vee | v \rangle$. Let $\psi \in \mathrm{Hom}(\boldsymbol{F}_q, \boldsymbol{C}^\times) - \{1\}$ and define the Fourier transform of a complex vector valued function φ by

$$\mathcal{F}(\varphi)(v^\vee) = \sum_{v \in V} \varphi(v) \psi(\langle v^\vee | v \rangle) \qquad (v^\vee \in V^\vee).$$

Let $\{v_0^\vee, \cdots, v_{n'}^\vee\}$ be a complete set of representatives of $V^\vee/G$, $V_i^\vee = Gv_i^\vee$, $K_i' = Z_G(v_i^\vee)$ and M_i' the set of K_i'-fixed vectors in M. Define M-valued functions $R'_{i',m'}$ $(0 \leqq i' \leqq n',\ m' \in M_i')$ in the same way as in (0.1).

LEMMA 1.3. *The Fourier transform of* $R_{i,m}$ *is a linear combination of* $\{R'_{i',m'} \mid 0 \leqq i' \leqq n',\ m' \in M'_{i'}\}$.

PROOF. By (1.1), it is enough to note that

$$\mathcal{F}(R_{i,m})(gv^\vee) = R(g)\mathcal{F}(R_{i,m})(v^\vee) \qquad (g \in G,\ v^\vee \in V^\vee).$$

LEMMA 1.4. *Let* $X(g) = \mathrm{Tr}\, R(g)$. *Then*

$$|K_i|^{-1} \sum_{k \in K_i} X(k) = \dim M_i.$$

Here and below $|-|$ means the cardinality.

1.5. Let $(R^\vee, M^\vee)$ be the contragradient representation of (R, M) and $\langle\ \rangle : M^\vee \times M \to \boldsymbol{C}$ the natural pairing. By (1.4), the dimension of the space $M_0^\vee$ of K_0-fixed vectors in $M^\vee$ is equal to $\dim M_0$. Moreover, as is easily seen, $\langle\ \rangle|_{M_0^\vee \times M_0}$ is non-degenerate. Henceforth we assume that

$$K_0 = K_0'. \tag{1.5.1}$$

Let $K = K_0 = K_0'$. By (1.3), we can define a linear endomorphism $C(R)$ of $M_0 = M_0'$ by

$$\mathcal{F}(R_{0,m})(v_0^\vee) = C(R)m \qquad (m \in M_0).$$

Then

$$|K|^{-1} \sum_{g \in G} \langle m^\vee | g | m \rangle \psi(\langle v_0^\vee | g | v_0 \rangle) = \langle m^\vee | C(R) | m \rangle \tag{1.5.2}$$

for $m \in M_0$ and $m^\vee \in M_0^\vee$. Here $g|m = gm$, i.e., $m^\vee | g = g^{-1}m^\vee$ etc. Let I be the complete set of representatives of $K \backslash G/K$ and, for $x \in G$, define an element $[x]$ of the group ring $\boldsymbol{C}G$ by

$$[x] = |K|^{-1} \sum_{g \in KxK} g.$$

Then (1.5.2) can be written as

$$\sum_{g \in I} \langle m^\vee | [g] | m \rangle \psi(\langle v_0^\vee | g | v_0 \rangle) = \langle m^\vee | C(R) | m \rangle .$$

Hence

(1.5.3) $$\sum_{g \in I} \mathrm{Tr}(R([g]))\psi(\langle v_0^\vee | g | v_0 \rangle) = \mathrm{Tr}(C(R) | M_0) .$$

Note that here naturally appears a representation of the Hecke algebra

$$H(G, K) = \{ x \in \boldsymbol{C}G \mid k_1 x k_2 = x \quad (k_1, k_2 \in K) \}.$$

1.6. Let B be a subgroup of G, $\theta : B \to GL(N)$ a complex representation of B such that N has a non-zero $B \cap K$-fixed vector, and $M = \mathrm{ind}(\theta | B \to G)$ the set of N-valued function m on G such that $m(bx) = \theta(b)m(x)$ for any $b \in B$ and $x \in G$. Define a G-action R on M by $(R(g)m)(x) = m(xg)$. For a $B \cap K$-fixed vector n_0 in N, let m_0 be the element of M such the $m_0(bk) = \theta(b)n_0$ for $b \in B$ and $k \in K$, and $m_0 \equiv 0$ on $G - BK$. Then m_0 is a non-zero K-fixed vector of the representation space of (R, M). By the definition of the endomorphism $C(R)$ of M_0,

$$C(R)m_0 = \sum_{v \in V} R_{0, m_0}(v)\psi(\langle v_0^\vee | v \rangle)$$
$$= |K|^{-1} \sum_{g \in G} (R(g)m_0)\psi(\langle v_0^\vee | g | v_0 \rangle) .$$

Define N-valued functions f_0 and $f_0^\vee$ on V_0 and $V_0^\vee$ by

$$f_0(xv_0) = f_0^\vee(xv_0^\vee) = m_0(x)$$

for $x \in G$, and denote their zero extensions to the whole spaces by the same letters. Assume that $C(R)$ is a homothetic transformation, i.e.,

(1.6.1) $$C(R) = c(R) \times (\text{identity})$$

with a scalar $c(R)$. Then

$$c(R) f_0^\vee(xv_0^\vee) = c(R)m_0(x) = (C(R)m_0)(x) = |K|^{-1} \sum_{g \in G} m_0(xg)\psi(\langle v_0^\vee | g | v_0 \rangle)$$
$$= |K|^{-1} \sum_{g \in G} m_0(g)\psi(\langle v_0^\vee | x^{-1} | g v_0 \rangle) = \sum_{v \in V_0} f_0(v)\psi(\langle x v_0^\vee | v \rangle) .$$

Thus we get the following lemma.

LEMMA 1.7. *With notations and assumptions as above,*

$$\mathcal{F}(f_0) = c(R) f_0^\vee \qquad \text{on } V_0^\vee .$$

REMARK 1.7.1. Even without assuming that $K_0 = K_0'$, we can obtain (1.7). This assumption is used to define the algebra structure of $H(G, K)$.

REMARK 1.7.2. If $\dim M_0 = \dim M_0' = 1$, then assumption (1.6.1) is automatically satisfied. More generally, if the following conditions are satisfied, then (1.6.1) becomes valid:

(1) For any irreducible constituent (R_1, M_1) of (R, M), the space M_1^K of K-fixed vectors in M_1 is at most one-dimensional.

(2) Moreover, $c(R_1)$ is the same for any irreducible constituent such that $\dim M_1^K=1$.

REMARK 1.7.3. If (1.6.1) is satisfied, then by the definition of $c(R)$, we get

$$\mathcal{F}(R_{0,m}) = c(R)R'_{0,m} \qquad \text{on } V_0^\vee .$$

Hence $c(R)$ is a generalization of 'the coefficient $c(R)$ of $R'_{0,m}$ in $\mathcal{F}(R_{0,m})$' considered in (0.2).

2. Fourier transform of $R_{i,m}$ (first example).

2.1. Let $\boldsymbol{A}=\boldsymbol{GL}_n$ (with a split $\boldsymbol{F}_q$-structure), $\boldsymbol{G}=\boldsymbol{A}\times\boldsymbol{A}$, $\boldsymbol{V}=\boldsymbol{V}^\vee=\boldsymbol{M}_n$ and $\langle v^\vee | v\rangle=\mathrm{Tr}(v^\vee\cdot v)$. Here and below, we denote algebraic varieties defined over a finite field $\boldsymbol{F}_q$ by a boldfaced letter and the set of its rational points by the same lightfaced letter unless otherwise stated. Define $\boldsymbol{G}$-actions on $\boldsymbol{V}$ and $\boldsymbol{V}^\vee$ by $(g_1, g_2)v=g_1vg_2^{-1}$ and $(g_1, g_2)v^\vee=g_2v^\vee g_1^{-1}$ respectively. Let $v_i=v_i^\vee=\mathrm{diag}(0, \cdots, 0, 1, \cdots, 1)$, where zeros appear i times. Then $V/G=V^\vee/G=\{v_0, v_1, \cdots, v_n\}$ and $K=K_0=K_0'$ is the diagonal of $A\times A$. Hence by (1.4), an irreducible $\boldsymbol{C}G$-module M has a non-zero K-fixed vector if and only if it is of the form

$$M = N\otimes N^\vee(\cong \mathrm{End}(N)),$$

where N is an irreducible $\boldsymbol{C}A$-module and $N^\vee$ its dual. If M is of this form, M_0 is one-dimensional and spanned by the identity in $\mathrm{End}(N)$, which we shall denote by m_0. If the matrix representation $R: G\to GL(M)$ is given by

$$R((g_1, g_2)) = S(g_1)\otimes S^\vee(g_2) \quad (S(g_1)\in GL(N), \quad S^\vee(g_2)\in GL(N^\vee)),$$

then, for $g\in V$ such that $\det g\neq 0$,

$$R_{0,m_0}(g) = R_{0,m_0}((g, 1)v_0) = R(g, 1)m_0 = S(g), \tag{2.1.1}$$

and

$$R'_{0,m_0}(g) = R'_{0,m_0}((g^{-1}, 1)v_0^\vee)= R(g^{-1}, 1)m_0 = S(g^{-1}). \tag{2.1.2}$$

Denote the zero extension of S to the whole space V by the same letter. By (2.1.1) and (2.1.2), the coefficient $c(R)=c(S\otimes S^\vee)$ of R'_{0,m_0} in $\mathcal{F}(R_{0,m_0})$ satisfies

$$\mathcal{F}(S)(g) = c(S\otimes S^\vee)S(g^{-1}) \qquad (g\in Gv_0 = GL_n(\boldsymbol{F}_q)). \tag{2.1.3}$$

Let Y be the character of S. Considering the trace of (2.1.3) at $g=v_0(=1_n)$, we get

$$\sum_{g\in A} Y(g)\psi(\mathrm{Tr}(g)) = c(R)Y(1). \tag{2.1.4}$$

The left hand side of (2.1.4) is evaluated by T. Kondo [7]. Let us recall his result.

2.2. First, we need to review the character theory of $GL_n(\boldsymbol{F}_q)$ due to J.A. Green [5].

2.2.1. For a prime power q, let $\boldsymbol{Q}_{q'}$ be the set of rational numbers whose denominators are relatively prime to q, and $\langle q\rangle$ the multiplicative group generated by q. Then $\langle q\rangle$ acts on $(\boldsymbol{Q}/\boldsymbol{Z})_{q'}=\boldsymbol{Q}_{q'}/\boldsymbol{Z}$ so that $q\cdot(x \bmod \boldsymbol{Z})=(qx \bmod \boldsymbol{Z})$. Denote by $f(\alpha)$ the length of a $\langle q\rangle$-orbit α in $(\boldsymbol{Q}/\boldsymbol{Z})_{q'}$.

A partition is a non-increasing sequence $\lambda=(\lambda_1, \lambda_2, \cdots)$ of non-negative integers λ_i such that almost all λ_i's are zero. Let P be the set of partitions, $|\lambda|=\sum_i\lambda_i$ for $\lambda\in P$, and $P(n, q)$ the set of P-valued functions Λ on $O(q)=(\boldsymbol{Q}/\boldsymbol{Z})_{q'}/\langle q\rangle$ such that

$$\sum_{\alpha\in O(q)} f(\alpha)|\Lambda(\alpha)| = n\,.$$

2.2.2. Let A be GL_n with a split $\boldsymbol{F}_q$-structure. Let $\bar{\boldsymbol{F}}_q$ be an algebraic closure of $\boldsymbol{F}_q$ and fix an isomorphism $(\boldsymbol{Q}/\boldsymbol{Z})_{q'}\cong\bar{\boldsymbol{F}}_q^\times$. Then the conjugacy classes $\{\mathrm{cl}_A(x) \mid x\in A\}$ of A are in one-to-one correspondence with $P(n, q)$. For $\Lambda\in P(n, q)$, let a_Λ be a representative of the corresponding conjugacy class.

2.2.3. If $\boldsymbol{F}_{q^{f'}}$ is an extension of $\boldsymbol{F}_{q^f}$, the norm mapping induces an injection $\mathrm{Hom}(\boldsymbol{F}_{q^f}^\times, \boldsymbol{C}^\times)\to\mathrm{Hom}(\boldsymbol{F}_{q^{f'}}^\times, \boldsymbol{C}^\times)$. Fix an isomorphism $(\boldsymbol{Q}/\boldsymbol{Z})_{q'}\cong\varinjlim\mathrm{Hom}(\boldsymbol{F}_{q^f}, \boldsymbol{C}^\times)$. Then the irreducible character of A are in natural one-to-one correspondence with $P(n, q)$. For $\Psi\in P(n, q)$, let $R(\Psi)\colon A\to GL(M(\Psi))$ be a corresponding irreducible representation. An element α of $\varinjlim\mathrm{Hom}(\boldsymbol{F}_{q^f}^\times, \boldsymbol{C}^\times)$ can be regarded as an element of $\mathrm{Hom}(\boldsymbol{F}_{q^{f(\alpha)}}^\times, \boldsymbol{C}^\times)$, which we shall denote by θ_α.

2.2.4. For $\Psi\in P(n, q)$, let $\boldsymbol{C}=\boldsymbol{C}(\Psi)$ be the centralizer in $\boldsymbol{A}$ of the semisimple part of a_Ψ, $W(\Psi)$ the Weyl group of $C(\Psi)$, $\boldsymbol{T}(w)$ a maximal torus of $C(\Psi)$ corresponding to $w\in W(\Psi)$, and $\theta(\Psi)$ the (one dimensional) character

$$\theta(\Psi)=\prod_{\alpha\in O(q)}\theta_\alpha\circ\det$$

of

$$C(\Psi)\cong\prod_{\alpha\in O(q)} GL_{|\Psi(\alpha)|}(\boldsymbol{F}_{q^{f(\alpha)}})\,.$$

The Weyl group $W(\Psi)$ is isomorphic to

$$\prod_{\alpha\in O(q)}\mathfrak{S}_{|\Psi(\alpha)|}\,,$$

where $\mathfrak{S}_p$ is the p-th symmetric group. For a partition λ of p, Z_λ denotes the irreducible character of $\mathfrak{S}_p$ corresponding to λ, e.g., $Z_{(p)}$ is the trivial character and $Z_{(1^p)}$ is the signature. Let

$$Z_{\Psi} = \prod_{\alpha \in O(q)} Z_{\Psi(\alpha)}.$$

2.2.5 ([8]). For $\Psi \in P(n, q)$, the character of $R(\Psi)$ is

$$(-1)^{s(A)-s(C(\Psi))}|W(\Psi)|^{-1} \sum_{w \in W(\Psi)} Z_{\Psi}(w) R_{T(w)}^{\theta(\Psi)|T(w)},$$

where $R^{\theta}_{T(w)}$ denotes the Deligne-Lusztig (virtual) character [4].

2.2.6 ([4; 7.1]). In our case ($A = GL_n(\boldsymbol{F}_q)$),

$$R^{\theta}_{T(w)}(1) = (-1)^{s(A)-s(T(w))}|A| q^{-n(n-1)/2}|T(w)|^{-1}.$$

2.3. For a finite extension k' of $k = \boldsymbol{F}_q$ and for a multiplicative character θ' of $k'^{\times}$, let $G(\theta') = \sum_{x \in k'^{\times}} \theta'(x) \psi(\mathrm{Tr}_{k'/k}(x))$. Let us describe the value of the character sum

$$\sum_{t \in T(w)} \theta(\Psi)(t) \psi(\mathrm{Tr}(t)) \tag{2.3.1}$$

in terms of $G(\theta_{\alpha})$. Assume that $T(w) = \prod_{\alpha} T_{\alpha}$, where $T_{\alpha} \subset GL_{|\Psi(\alpha)|}(\boldsymbol{F}_{q^{f(\alpha)}})$ and $T_{\alpha} \cong \prod_i \boldsymbol{F}^{\times}_{q^{f(\alpha)c(\alpha,i)}}$ with $\sum_i c(\alpha, i) = |\Psi(\alpha)|$. Then by a formula of Davenport-Hasse [3; (0.8)], (2.3.1) is equal to

$$\prod_{\alpha, i} G(\theta_{\alpha} \circ N_{\alpha,i}) = \prod_{\alpha, i} (-1)^{c(\alpha,i)-1} G(\theta_{\alpha})^{c(\alpha,i)},$$

where $N_{\alpha,i}$ is the norm mapping from $\boldsymbol{F}_{q^{f(\alpha)c(\alpha,i)}}$ to $\boldsymbol{F}_{q^{f(\alpha)}}$. Hence

$$\sum_{t \in T(w)} \theta(\Psi)(t) \psi(\mathrm{Tr}(t)) = (-1)^{s(T(w))} \prod_{\alpha} (-G(\theta_{\alpha}))^{|\Psi(\alpha)|}. \tag{2.3.2}$$

THEOREM 2.4 (Kondo [7]). *For $G = GL_n \times GL_n$ and $V = V^{\vee} = M_n$,*

$$q^{-\dim V/2} c(R(\Psi) \otimes R(\Psi)^{\vee}) = (-1)^n \prod_{\alpha \in O(q)} (-q^{-f(\alpha)/2} G(\theta_{\alpha}))^{|\Psi(\alpha)|}$$

(See (2.2.1) for $O(q)$, $f(\alpha)$ and $|-|$, (2.2.3) for $R(\Psi)$ and θ_{α}, and (2.3) for $G(\theta_{\alpha})$.)

2.5. Here we prove (2.4) by using the results of [4].

Let $Y(\Psi)$ be the character of $R(\Psi)$. For $Y = Y(\Psi)$, the left hand side of (2.1.4) is equal to

$$\begin{aligned}
&|A| \langle Y(\Psi) | \psi^{-1} \circ \mathrm{Tr} \rangle_A \\
&= |A|(-1)^{s(A)-s(C(\Psi))}|W(\Psi)|^{-1} \sum_{w \in W(\Psi)} Z_{\Psi}(w) \langle R_{T(w)}^{\theta(\Psi)|T(w)} | \psi^{-1} \circ \mathrm{Tr} \rangle_A \\
&= |A|(-1)^{s(A)-s(C(\Psi))}|W(\Psi)|^{-1} \sum_{w \in W(\Psi)} Z_{\Psi}(w) \langle \theta(\Psi) | \psi^{-1} \circ \mathrm{Tr} \rangle_{T(w)}
\end{aligned}$$

by (2.2.5) and [**4**; Proposition 7.11]. Here $\langle\,\rangle_A$ and $\langle\,\rangle_{T(w)}$ denote the usual inner product of class functions. By (2.3.2), this is equal to

$$|A|(-1)^{s(A)-s(C(\Psi))}|W(\Psi)|^{-1}\sum_{w\in W(\Psi)}Z_\Psi(w)|T(w)|^{-1}(-1)^{s(T(w))}\prod_\alpha(-G(\theta_\alpha))^{|\Psi(\alpha)|}.$$

On the other hand, by (2.2.6),

$$Y(\Psi)(1)=(-1)^{s(A)-s(C(\Psi))}|W(\Psi)|^{-1}\sum_{w\in W(\Psi)}Z_\Psi(w)|A|q^{-n(n-1)/2}|T(w)|^{-1}\times(-1)^{s(A)-s(T(w))}.$$

Hence by (2.1.4),

$$\begin{aligned}c(R(\Psi)\otimes R(\Psi)^\vee)&=(-1)^{s(A)}q^{n(n-1)/2}\prod_{\alpha\in O(q)}(-G(\theta_\alpha))^{|\Psi(\alpha)|}\\&=q^{n^2/2}(-1)^n\prod_{\alpha\in O(q)}(-q^{-f(\alpha)/2}G(\theta_\alpha))^{|\Psi(\alpha)|}.\end{aligned}$$

COROLLARY 2.6. *The Fourier transform of the matrix valued function* $S=R(\Psi)$ *vanishes identically on* $\{x\in V\mid\det x=0\}$ *if and only if* $\theta_\alpha\neq 1$ *for any* α *such that* $|\Psi(\alpha)|>0$.

PROOF. Since $q^{-\dim V/2}\mathfrak{F}$ preserves the L^2-norm, $S\equiv 0$ on $\{x\in V\mid\det x=0\}$ if and only if

$$\Big|\prod_{\alpha\in O(q)}(-q^{-f(\alpha)/2}G(\theta_\alpha))^{|\Psi(\alpha)|}\Big|=1.$$

This condition is equivalent to

$$\theta_\alpha\neq 1,\qquad\text{if }|\Psi(\alpha)|>0.$$

REMARK 2.7. Let R_T^θ be a fixed Deligne-Lusztig character of G. For any irreducible constituent R of R_T^θ of the form $R=R(\Psi)\otimes R(\Psi)^\vee$, the value of $c(R)$ is the same.

3. Fourier transform of $R_{i,m}$ (second example).

3.1. Let $\boldsymbol{G}=GL_{2n}$ (with a split $\boldsymbol{F}_q$-structure), $\boldsymbol{V}=\boldsymbol{V}^\vee=\{v\in M_{2n}\mid v^*+v=0\}$, and $\langle v^\vee|v\rangle=-\mathrm{Tr}(v^\vee\cdot v)/2$. Here $*$ means the transposition. Define $\boldsymbol{G}$-actions on $\boldsymbol{V}$ and $\boldsymbol{V}^\vee$ by $g\cdot v=gvg^*$ and $g\cdot v^\vee=(g^*)^{-1}vg^{-1}$ respectively. Let

$$v_0=v_0^\vee=\begin{pmatrix}0&1_n\\-1_n&0\end{pmatrix}$$

with 1_n the identity matrix of degree n. Then the isotropy groups $\boldsymbol{K}_0$ and $\boldsymbol{K}_0^\vee$ at v_0 and $v_0^\vee$ are both equal to the symplectic group $\boldsymbol{K}=Sp_{2n}$.

3.2. In order to calculate the constant $c(R)$ in the above case, we need a result of Bannai-Kawanaka-Song [**1**]. Here we shall review their result to the extent of our direct concern, using the same notations as in (2.2).

3.2.1. Let CG be the group ring of G, $\varepsilon=|K|^{-1}\sum_{k\in K}k$, $H(G, K)$ the subalgebra $\varepsilon(CG)\varepsilon$ of CG, and $[x]=|K|^{-1}\sum_{y\in KxK}y$ for $x\in G$.

3.2.2 ([**6**]). Consider $A=GL_n(\boldsymbol{F}_q)$ as a subgroup of $G=GL_{2n}(\boldsymbol{F}_q)$ by

$$a \longrightarrow \begin{pmatrix} 1_n & 0 \\ 0 & a \end{pmatrix}$$

Then G is a disjoint union of $Ka_\Lambda K(\Lambda\in P(n, q))$, i.e., $\{[a_\Lambda] \mid \Lambda\in P(n, q)\}$ is a linear basis of $H(G, K)$. (See [**1**; 2.3.4].)

3.2.3 ([**4**; Theorem 4.2]). Let $R_{\boldsymbol{T}}^{\theta}$ be the Deligue-Lusztig (virtual) character of A associated to a maximal torus $\boldsymbol{T}$ of $\boldsymbol{A}$ and $\theta\in\mathrm{Hom}(T, \bar{\boldsymbol{Q}}_l^{\times})(\cong\mathrm{Hom}(T, \boldsymbol{C}^{\times}))$. Let s and u be the semisimple and unipotent part of an element x of A. Then

$$R_{\boldsymbol{T}}^{\theta}(x)=\sum_{\substack{y\in A/Z(s)\\ ysy^{-1}\in T}} Q_{y^{-1}\boldsymbol{T}y}^{\boldsymbol{Z}(s)}(u)\theta(ysy^{-1}),$$

where $Q_{y^{-1}\boldsymbol{T}y}^{\boldsymbol{Z}(s)}(u)$ are Green plynomials. (See [**1**; 1.2] for Green polynomials.)

3.2.4 ([**1**; 5.3.2]). Let $x=su$ and $(\boldsymbol{T}, \theta)$ be as in (3.2.3), and

$$\{R_{\boldsymbol{T}}^{\theta}(x)\}_{q\to q^2}=\sum_{\substack{y\in A/Z(s)\\ ysy^{-1}\in T}} \{Q_{y^{-1}\boldsymbol{T}y}^{\boldsymbol{Z}(s)}(u)\}_{q\to q^2}\theta(ysy^{-1}).$$

Define a function $\chi_{\boldsymbol{T}}^{\theta}$ on $H(G, K)$ by

$$\chi_{\boldsymbol{T}}^{\theta}([a_\Lambda])=\{|\mathrm{cl}_A(a_\Lambda)|R_{\boldsymbol{T}}^{\theta}(a_\Lambda)/R_{\boldsymbol{T}}^{\theta}(1)\}_{q\to q^2},$$

which is called the basic function. (See [**1**; 3.2]).

3.2.5 ([**1**; 4.1.1]). For a partition $\lambda=(\lambda_1, \lambda_2, \cdots)$, 2λ denotes the partition $(2\lambda_1, 2\lambda_2, \cdots)$. For $\Psi\in P(n, q)$, define $2\Psi\in P(2n, q)$ by $(2\Psi)(\alpha)=2(\Psi(\alpha))$. The G-module $M(\Omega)$ $(\Omega\in P(2n, q))$ has a non-zero K-fixed vector if and only if $\Omega=2\Psi$ for some $\Psi\in P(n, q)$. (See (2.2.3) for $M(\Omega)$.)

3.2.6 ([**6**]). The set of K-fixed vectors in $M(2\Psi)$ is a one-dimensional vector space. (See [**1**; 2.3.5].)

3.2.7 ([**1**; 3.4]). The character of $(R(2\Psi)|_{H(G,K)}, M(2\Psi))$ is given by

$$\mathrm{Tr}(R(2\Psi)|H(G, K))=\sum_{w\in W(\Psi)} e(w)\chi_{\boldsymbol{T}(w)}^{\theta(\Psi)}$$

with some rational coefficients $e(w)$. (Here we have written $\theta(\Psi)$ for $\theta(\Psi)|T(w)$.) Although an algorithm for determining these coefficients $e(w)$ is given in section 6 of [**1**], all that we need is that they satisfy

$$\sum_{w\in W(\Psi)} e(w)=1$$

[**1**; 6.2.2].

3.3. Now let us calculate the constant $c(R(2\Psi))$. By (1.5.3), (3.2.2) and (3.2.7),

$$c(R(2\Psi))=\sum_{\Lambda\in P(n,q)}\mathrm{Tr}(R(2\Psi)([a_\Lambda]))\psi(\langle v_0^\vee|a_\Lambda|v_0\rangle) \tag{3.3.1}$$

$$=\sum_{w\in W(\Psi)}e(w)\sum_{\Lambda\in P(n,q)}\chi_{T(w)}^{\theta(\Psi)}([a_\Lambda])\psi(\mathrm{Tr}(a_\Lambda)).$$

(Note that, by (3.2.6), $\dim M_0=1$ and $C(R)=c(R)$ in our case.) For $\Lambda\in P(n,q)$, let s_Λ and u_Λ be the semisimple and unipotent part of a_Λ respectively. Let $P_s(n,q)$ be the set of $\Lambda\in P(n,q)$ such that a_Λ is semisimple and, for $\Xi\in P_s(n,q)$, (Ξ) the set of $\Lambda\in P(n,q)$ such that s_Λ is conjugate to a_Ξ. Choose the representatives a_Λ so that $s_\Lambda=a_\Xi$ for $\Lambda\in(\Xi)$. Then, writing $\boldsymbol{\theta}$ and $\boldsymbol{T}$ for $\theta(\Psi)|T(w)$ and $\boldsymbol{T}(w)$,

$$\sum_{\Lambda\in P(n,q)}\chi_{\boldsymbol{T}}^{\boldsymbol{\theta}}([a_\Lambda])\psi(\mathrm{Tr}(a_\Lambda)) \tag{3.3.2}$$

$$=\sum_{\Xi\in P_s(n,q)}\sum_{\Lambda\in(\Xi)}\{(-1)^{s(A)-s(T)}q^{n(n-1)/2}|T|\cdot|Z_A(a_\Lambda)|^{-1}R_{\boldsymbol{T}}^{\boldsymbol{\theta}}(a_\Lambda)\}_{q\to q^2}\psi(\mathrm{Tr}(a_\Xi))$$

$$=(-1)^{s(A)-s(T)}q^{n(n-1)}|T|_{q\to q^2}$$

$$\times\sum_{\Xi\in P_s(n,q)}\sum_{\Lambda\in(\Xi)}\sum_{\substack{y\in A/C(\Xi)\\ ya_\Xi y^{-1}\in\boldsymbol{T}}}\{|Z_{C(\Xi)}(u_\Lambda)|^{-1}Q_{y^{-1}\boldsymbol{T}y}^{C(\Xi)}(u_\Lambda)\}_{q\to q^2}\theta(ya_\Xi y^{-1})\psi(\mathrm{Tr}(a_\Xi)).$$

Here recall that $C(\Xi)=Z_A(a_\Xi)=Z_A(s_\Lambda)$ and note that $Z_A(a_\Lambda)=Z_{Z(s_\Lambda)}(u_\Lambda)=Z_{C(\Xi)}(u_\Lambda)$. By [**4**; (7.11.4)],

$$\sum_{\Lambda\in(\Xi)}\{|Z_{C(\Xi)}(u_\Lambda)|^{-1}Q_{y^{-1}\boldsymbol{T}y}^{C(\Xi)}(u_\Lambda)\}_{q\to q^2}=|T|_{q\to q^2}^{-1}. \tag{3.3.3}$$

By (2.3.2), (3.3.2) and (3.3.3),

$$\sum_{\Lambda}\chi_{\boldsymbol{T}}^{\boldsymbol{\theta}}([a_\Lambda])\psi(\mathrm{Tr}(a_\Lambda))=(-1)^{s(A)-s(T)}q^{n(n-1)}\sum_{\Xi\in P_s(n,q)}\sum_{\substack{y\in A/C(\Xi)\\ ya_\Xi y^{-1}\in\boldsymbol{T}}}\theta(ya_\Xi y^{-1})\psi(\mathrm{Tr}(a_\Xi)) \tag{3.3.4}$$

$$=(-1)^{s(A)-s(T)}q^{n(n-1)}\sum_{t\in\boldsymbol{T}}\theta(t)\psi(\mathrm{Tr}(t))=(-1)^n q^{n(n-1)}\prod_{\alpha\in O(q)}(-G(\theta_\alpha))^{|\Psi(\alpha)|}.$$

By (3.3.1) and (3.3.4), we get the following theorem.

THEOREM 3.4. *For* $\boldsymbol{G}=GL_{2n}$ *and* $\boldsymbol{V}=\boldsymbol{V}^\vee=\{v\in M_{2n}|v^*+v=0\}$,

$$q^{-\dim V/2}c(R(2\Psi))=(-1)^n\prod_{\alpha\in O(q)}(-q^{-f(\alpha)/2}G(\theta_\alpha))^{|\Psi(\alpha)|}.$$

(See (2.2.1) for $O(q)$, $f(\alpha)$ and $|-|$, (2.2.3) for $R(-)$ and θ_α, (2.3) for $G(\theta_\alpha)$, and (3.2.5) for 2Ψ.)

REMARK 3.5. We can deduce from (3.4) a consequence analogous to (2.6).

REMARK 3.6. Let R_T^θ be a fixed Deligne-Lusztig character of G. For any irreducible constituent R of R_T^θ of the form $R=R(2\Psi)$, the value of $c(R)$ is the same.

4. Non-reductive prehomogeneous vector spaces.

In this section, we give explicit forms of f_0 and $f_0^\vee$ for two cases. (See (1.6) for f_0 and $f_0^\vee$.) The explicit forms of f_0 and $f_0^\vee$ in each case, combined with (1.7), (2.4) and (3.4), give a character sum analogue (4.1.3) of (0.3.1) or a similar formula (4.2.3).

4.1. First, let us consider the case where $A=GL_n(\boldsymbol{F}_q)$, $G=A\times A$, $V=V^\vee=M_n(\boldsymbol{F}_q)$, $v_0=v_0^\vee=1_n$, K is the diagonal of $A\times A$, B' (resp. B'') is the group of lower (resp. upper) triangular matrices in A, and $B=B'\times B''$. The G-actions on V and $V^\vee$ are given by $(a', a'')v=a'va''^{-1}$ and $(a', a'')v^\vee=a''v^\vee a'^{-1}$ respectively. Let $\theta_i\in\mathrm{Hom}(\boldsymbol{F}_q^\times, \boldsymbol{C}^\times)$ and define linear characters θ' and θ'' of B' and B'' by

$$\theta'((b'_{ij})_{1\leq i,j\leq n})=\prod_{i=1}^{n}\theta_i(b'_{ii})$$

and

$$\theta''((b''_{ij})_{1\leq i,j\leq n})=\prod_{i=1}^{n}\theta_i(b''^{-1}_{ii}).$$

As usual, we define $\theta_i(0)=0$. Let $\theta=\theta'\otimes\theta''$ and $(R, M)=\mathrm{ind}(\theta\,|\,B\to G)$ be the representation of G defined in (1.6). Since $\ker\theta$ contains $B\cap K$, we can define $\boldsymbol{C}$-valued functions f_0 and $f_0^\vee$ as in (1.6) by taking $1\in\boldsymbol{C}$ as n_0. Let $P_i(x)=\det(x_{\alpha\beta})_{1\leq\alpha,\beta\leq i}$ and $Q_j(y)=\det(y_{\alpha\beta})_{n-j+1\leq\alpha,\beta\leq n}$. Then

$$f_0=\theta_1(P_1)\theta_2(P_2/P_1)\cdots\theta_n(P_n/P_{n-1}) \tag{4.1.1}$$

and

$$f_0^\vee=\theta_1(Q_{n-1}/Q_n)\cdots\theta_{n-1}(Q_1/Q_2)\theta_n(1/Q_1). \tag{4.1.2}$$

By (1.7.2) and (2.7), the condition (1.6.1) is satisfied. From (1.7), (2.4), (4.1.1) and (4.1.2), we get the following character sum analogue of (0.3.1);

$$\begin{aligned} &q^{-\dim V/2}\sum_{x\in V}\theta_1(P_1(x))\theta_2\Big(\frac{P_2(x)}{P_1(x)}\Big)\cdots\theta_n\Big(\frac{P_n(x)}{P_{n-1}(x)}\Big)\psi(\langle y\,|\,x\rangle) \\ &=(-1)^n\prod_{i=1}^{n}(-q^{-1/2}G(\theta_i))\cdot\theta_1\Big(\frac{Q_{n-1}(y)}{Q_n(y)}\Big)\cdots\theta_{n-1}\Big(\frac{Q_1(y)}{Q_2(y)}\Big)\theta_n\Big(\frac{1}{Q_1(y)}\Big)\end{aligned} \tag{4.1.3}$$

for $y\in V_0^\vee(=GL_n(\boldsymbol{F}_q))$.

4.2. Next, let us consider the case where $G=GL_{2n}(\boldsymbol{F}_q)$, $V=V^\vee=\{x\in M_{2n}(\boldsymbol{F}_q)\,|\,x+x^*=0\}$, $\langle x\,|\,y\rangle=\langle y\,|\,x\rangle=-\mathrm{Tr}(xy)/2$ for $x\in V$ and $y^\vee\in V^\vee$,

$$J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\in M_2(\boldsymbol{F}_q),$$

$v_0=v_0^\vee=\mathrm{diag}(J,\cdots,J)\in V$, $K=Sp_{2n}(\boldsymbol{F}_q)$, and B is the group of lower triangular matrices in G. Here * means the transposition. The G-actions on V and $V^{\vee}$ are given by $g\cdot v=gvg^*$ and $g\cdot v^{\vee}=(g^*)^{-1}vg^{-1}$. Let $\theta_i\in\mathrm{Hom}(\boldsymbol{F}_q^{\times}, \boldsymbol{C}^{\times})$ $(1\leqq i\leqq n)$ and define a linear character of B by

$$\theta((b_{ij})_{1\leq i,j\leq 2n})=\prod_{i=1}^{n}\theta_i(b_{2i-1,2i-1}b_{2i,2i}).$$

Let $(R, M)=\mathrm{ind}(\theta\,|\,B\to G)$. In the present case, the functions f_0 and $f_0^{\vee}$ given in (1.6) can be written as follows: For a skew-symmetric matrix $x=(x_{\alpha,\beta})_{1\leq\alpha,\beta\leq 2n}$, let

$$P_i(x)=\mathrm{Pf}((x_{\alpha\beta})_{1\leq\alpha,\beta\leq 2i})$$

and

$$Q_j(x)=\mathrm{Pf}((x_{\alpha\beta})_{2(n-j+1)\leq\alpha,\beta\leq 2n}),$$

where Pf denotes the Pfaffian. Then

$$f_0=\theta_1(P_1)\theta_2(P_2/P_1)\cdots\theta_n(P_n/P_{n-1}) \tag{4.2.1}$$

and

$$f_0^{\vee}=\theta_1(Q_{n-1}/Q_n)\cdots\theta_{n-1}(Q_1/Q_2)\theta_n(1/Q_1). \tag{4.2.2}$$

By (1.7.2) and (3.6), the condition (1.6.1) is satisfied. Combining (1.7), (3.4), (4.2.1) and (4.2.2), we get

$$\begin{aligned}&q^{-\dim V/2}\sum_{x\in V}\theta_1(P_1(x))\theta_2\Big(\frac{P_2(x)}{P_1(x)}\Big)\cdots\theta_n\Big(\frac{P_n(x)}{P_{n-1}(x)}\Big)\psi(\langle y\,|\,x\rangle)\\&=(-1)^n\prod_{i=1}^{n}(-q^{-1/2}G(\theta_i))\cdot\theta_1\Big(\frac{Q_{n-1}(y)}{Q_n(y)}\Big)\cdots\theta_{n-1}\Big(\frac{Q_1(y)}{Q_2(y)}\Big)\theta_n\Big(\frac{1}{Q_1(y)}\Big)\end{aligned} \tag{4.2.3}$$

for $y\in V_0^{\vee}(=V^{\vee}\cap GL_{2n}(\boldsymbol{F}_q))$.

4.3. In conclusion, let us consider an analogue of (4.2.3) over the real number field $\boldsymbol{R}$. The result of this number is the formula (4.3.4), whose generalization is announced in [2]. Our argument here is similar to that of [**10**; Example 8].

Let P_i and Q_j be the polynomials given in (4.2), $V(\boldsymbol{R})=V^{\vee}(\boldsymbol{R})=\{x\in M_{2n}(\boldsymbol{R})\,|\,x^*+x=0\}$, $\langle x\,|\,y\rangle=\langle y\,|\,x\rangle=-\mathrm{Tr}(xy)/2$ for $x\in V(\boldsymbol{R})$ and $y\in V^{\vee}(\boldsymbol{R})$,

$$\Omega=\{x\in V(\boldsymbol{R})\mid P_1(x)>0,\cdots,P_n(x)>0\},$$

and

$$\Omega^{\vee}=\{y\in V^{\vee}(\boldsymbol{R})\mid Q_1(y)>0,\cdots,Q_n(y)>0\}.$$

(In fact, $\Omega=\Omega^{\vee}$.) Let φ be a compactly supported C^{∞}-function on $\Omega^{\vee}$, $s=(s_1,\cdots,s_n)$ complex numbers such that $\mathrm{Re}(s_1)>\cdots>\mathrm{Re}(s_n)>0$, and

$$I(s,\varphi)=\int_{\Omega}\mathcal{F}(\varphi)(x)P_1(x)^{s_1}\Big(\frac{P_2(x)}{P_1(x)}\Big)^{s_2}\cdots\Big(\frac{P_n(x)}{P_{n-1}(x)}\Big)^{s_n}dx,$$

where the measure dx on Ω is given by $dx=\prod_{i>j}dx_{ij}$, and

$$\mathcal{F}(\varphi)(x)=\int_{\Omega^\vee}e^{-\sqrt{-1}\langle x|y\rangle}\varphi(y)dy$$

with $dy=\prod_{i>j}dy_{ij}$. Let U be the group of $2n$ by $2n$ lower triangular matrices $u=(u_{ij})$ such that $u_{ii}=1$ $(1\leqq i\leqq 2n)$ and $u_{2i,2i-1}=0$ $(1\leqq i\leqq n)$. Let D be the group of matrices

$$t=\operatorname{diag}(t_1, 1, t_2, 1, \cdots, t_n, 1)$$

with $t_i>0$. Note that D normalizes U. Define the invariant measures on U and D by

$$du=\prod_{i\geqq j+2}du_{ij}\wedge\prod_{i=1}^{n-1}du_{2i+1,2i} \qquad \text{and} \qquad d^\times t=\prod_{i=1}^{n}d\log t_i\,.$$

If we define elements v_0 and $v_0^\vee$ of $M_{2n}(\boldsymbol{R})$ in the same way as in (4.2), then $v_0\in\Omega$ and $v_0^\vee\in\Omega^\vee$. Define isomorphisms $DU\cong\Omega$ and $UD\cong\Omega^\vee$ by $tu\to(tu)v_0(tu)^*$ and $u't'\to(u't')^*v_0^\vee(u't')$. Then

$$\begin{aligned}(4.3.1)\quad I(s,\varphi)&=\int_{DU}\mathcal{F}(\varphi)(tu)t_1^{s_1}t_2^{s_2}\cdots t_n^{s_n}(t_1\cdots t_n)^{2n-1}d^\times t du\\&=\int_{DU}t_1^{s_1+2n-1}\cdots t_n^{s_n+2n-1}d^\times tdu\\&\quad\times\int_{UD}\varphi((u't')^*v_0^\vee(u't'))\exp(-\sqrt{-1}\langle(tu)v_0(tu)^*|(u't')^*v_0^\vee(u't')\rangle)\\&\quad\times(t_1'\cdots t_n')^{2n-1}d^\times t'du'\,.\end{aligned}$$

By using repeatedly the formula

$$\int_{\boldsymbol{R}}e^{\sqrt{-1}xy}dy=2\pi\delta(y)\,.$$

where $\delta(x)$ is the Dirac's delta function, we get

$$\begin{aligned}(4.3.2)\quad &\int_U\exp(-\sqrt{-1}\langle(tu)v_0(tu)^*|(u't')^*v_0^\vee(u't')\rangle)du\\&=(2\pi)^{n(n-1)}\exp(-\sqrt{-1}\sum_{i=1}^{n}t_it_i')\prod_{i=1}^{n}(t_it_i')^{-2i+2}\,.\end{aligned}$$

(Here we omit the details of the calculation, which is elementary but complicated.) By (4.3.1) and (4.3.2), we get

$$\begin{aligned}I(s,\varphi)&=(2\pi)^{n(n-1)}\int_D t_1^{s_1+2n-1}\cdots t_n^{s_n+2n-1}d^\times t\\&\quad\times\int_{UD}\varphi((u't')^*v_0^\vee(u't'))\exp(-\sqrt{-1}\sum_{i=1}^{n}t_it_i')\\&\quad\left(\prod_{i=1}^{n}(t_it_i')^{-2i+2}\right)(t_1'\cdots t_n')^{2n-1}d^\times t'du'\,.\end{aligned}$$

Since

$$\int_0^\infty t^s e^{-\sqrt{-1}tt'} d\log t = \Gamma(s)t'^{-s}e^{-\pi\sqrt{-1}s/2},$$

we get

$$\begin{aligned} I(s,\varphi) &= (2\pi)^{n(n-1)}\int_{UD}\varphi((u't')^* v_0^\vee(u't'))\prod_{i=1}^n t_i'^{-s_i}d^\times t' du' \\ &\times \prod_{i=1}^n (\Gamma(s_i+2n-2i+1)\exp(-\pi\sqrt{-1}(s_i+2n-2i+1)/2)) \\ &= (2\pi)^{n(n-1)}\int_{\Omega^\vee}\varphi(y)\Big(\frac{Q_{n-1}(y)}{Q_n(y)}\Big)^{s_1}\cdots\Big(\frac{Q_1(y)}{Q_2(y)}\Big)^{s_{n-1}}\Big(\frac{1}{Q_1(y)}\Big)^{s_n}Q_n(y)^{-2n+1}dy \\ &\times \prod_{i=1}^n (\Gamma(s_i+2n-2i+1)\exp(-\pi\sqrt{-1}(s_i+2n-2i+1)/2)). \end{aligned}$$

Hence as tempered distributions

$$\begin{aligned} (4.3.3)\qquad &\int_\Omega e^{-\sqrt{-1}\langle x|y\rangle}P_1(x)^{s_1}\Big(\frac{P_2(x)}{P_1(x)}\Big)^{s_2}\cdots\Big(\frac{P_n(x)}{P_{n-1}(x)}\Big)^{s_n}dx \\ &= (2\pi)^{n(n-1)}\prod_{i=1}^n(\Gamma(s_i+2n-2i+1)\exp(-\pi\sqrt{-1}(s_i+2n-2i+1)/2))) \\ &\times\Big(\frac{Q_{n-1}(y)}{Q_n(y)}\Big)^{s_1}\cdots\Big(\frac{Q_1(y)}{Q_2(y)}\Big)^{s_{n-1}}\Big(\frac{1}{Q_1(y)}\Big)^{s_n}Q_n(y)^{-2n+1} \end{aligned}$$

for $y\in\Omega^\vee$. (Since $\mathrm{Re}(s_1)>\cdots>\mathrm{Re}(s_n)>0$, the zero extension of $P_1^{s_1}(P_2/P_1)^{s_2}\cdots(P_n/P_{n-1})^{s_n}|_\Omega$ is a continuous function. Hence the left hand side has a meaning as a Fourier transform of a tempered distribution. The equality means that the support of the difference of the both sides is contained in the complement of $\Omega^\vee$. Of course, by the analytic continuation, we may drop the condition on s.) The above formula can be written more neatly as follows;

$$\begin{aligned} (4.3.4)\qquad &(2\pi\sqrt{-1})^{-n(2n-1)/2}\int_\Omega e^{-\langle x|\sqrt{-1}y\rangle}P_1(x)^{s_1}\Big(\frac{P_2(x)}{P_1(x)}\Big)^{s_2}\cdots\Big(\frac{P_n(x)}{P_{n-1}(x)}\Big)^{s_n}dx \\ &= \Big(\prod_{i=1}^n\frac{\Gamma(s_i+2n-2i+1)}{(2\pi\sqrt{-1})^{1/2}}\Big)\cdot\Big(\frac{Q_{n-1}(\sqrt{-1}y)}{Q_n(\sqrt{-1}y)}\Big)^{s_1}\cdots \\ &\cdots\Big(\frac{Q_1(\sqrt{-1}y)}{Q_2(\sqrt{-1}y)}\Big)^{s_{n-1}}\Big(\frac{1}{Q_1(\sqrt{-1}y)}\Big)^{s_n}\cdot(Q_n(\sqrt{-1}y)^{-2n+1}). \end{aligned}$$

Here $(2\pi\sqrt{-1})^s$ and $Q_j(\sqrt{-1}y)^s$ stand for $(2\pi)^s\exp(s\pi\sqrt{-1}/2)$ and $\exp(s\pi\sqrt{-1}j/2)\cdot Q_j(y)^s$ respectively.

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