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Homogeneity and complete decomposability of torsion free knot modules

Dedicated to Professor Fujitsugu Hosokawa on his 60th birthday

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Let Λ be the integral group ring of the infinite cyclic group $\langle t: \rangle$. A Λ-module A is called a *knot module* if A is finitely generated over Λ and $t-1$ induces an automorphism of A. The purpose of this paper is to generalize results of E. S. Rapaport [8], R. H. Crowell [2] and D. W. Sumners [9] on knot modules. M. Kervaire [5] showed that the Z-torsion part T of a knot module A is a finite Λ-submodule. It follows from [3, vol. 2, p. 187] that A splits as an abelian group, i.e., $A \cong_Z F \oplus T$, where $F=A/T$. The Z-torsion part of a knot module has been completely determined. That is, a finite abelian group T is isomorphic to the Z-torsion part of some knot module if and only if the number of factors isomorphic to Z_{2^i} in the 2-primary component of T is not equal to one for any positive integer i (cf. [4], [6]). On the other hand, it still remains open to characterize the Z-structure of a Z-torsion free knot module. In this paper, we investigate two classes of Z-torsion free knot modules; one is homogeneous and the other is completely decomposable. Using our result, we can find an answer to Sumners's question [9, p. 84] for models of Z-torsion free knot modules.

Throughout this paper (unless otherwise specified), all groups will be Z-torsion free abelian and all Λ-modules will be Z-torsion free knot modules.

1. Introduction.

A polynomial $f(t)$ of Λ is *primitive* if all its coefficients are relatively prime. Let A be a Λ-module. Then $A \otimes_Z Q$ is a finitely generated Γ-module, where $\Gamma = \Lambda \otimes_Z Q$. Therefore, since Γ is a principal ideal domain, we have

$$A \otimes_Z Q \cong_\Gamma \Gamma/(\lambda_1) \oplus \cdots \oplus \Gamma/(\lambda_k).$$

In the above decomposition, one can take the λ_i to be primitive elements of Λ such that $\lambda_{i+1} | \lambda_i$ in Λ, $i=1, \cdots, k-1$. We call $\{\lambda_i\}_{i=1}^k$ the *(rational) polynomial* *invariants* of A and $\Delta(A)=\lambda_1 \cdots \lambda_k$ the (*rational*) *polynomial* of A.

By the *rank* $r(G)$ of an abelian group G, we mean the dimension of the vector space $G\otimes_Z Q$ over Q. It is well-known that $r(A)=\deg \Delta(A)$ for a Λ-module A. An abelian group G is said to be *completely decomposable* if it is a direct sum of rank one abelian groups. Let G be an abelian group. If S is a subset of G, then S_* denotes the smallest pure subgroup of G containing S. (A subgroup H is called a *pure* subgroup of G if $H\cap nG=nH$ for all $n\in Z$.) An isomorphism class of abelian groups of rank one is called a *type*. The type determined by an abelian group G_1 of rank one is denoted by $\boldsymbol{t}(G_1)$. If x $(\neq 0)$ $\in G$, then $\boldsymbol{t}(\{x\}_*)$ is called the *type* of x and denoted by $\boldsymbol{t}_G(x)$. We say that G is *homogeneous* of type τ if $\boldsymbol{t}_G(x)=\tau$ for any non-zero element x of G. (In Section 2, types will be explained in detail.) Let $Z[1/m]$, $m\in Z-\{0\}$, denote the additive group consisting of all rationals of the form $r=i/m^j$ $(i, j\in Z)$.

Let $f(t)=c_0t^i+c_1t^{i+1}+\cdots+c_dt^{i+d}\in\Lambda$, where $c_0c_d\neq 0$. Then let $\mu(f(t))$ denote the integer c_0c_d. Let $\Lambda_p=\Lambda\otimes_Z Z_p$, where p is a prime, and $\varepsilon_p: \Lambda\to\Lambda_p$ be the homomorphism given by $\varepsilon_p(f)=f\otimes 1$. Then the units of Λ_p are of the form $t^i\otimes\alpha$, $\alpha(\neq 0)\in Z_p$. For a polynomial $f(t)$ of Λ, $\varepsilon_p(f(t))$ is a unit in Λ_p if and only if p divides all the coeffïcents of $f(t)$ except one.

In [8], E. S. Rapaport showed that a Λ-module A which has a square presentation matrix is finitely generated as an abelian group (i.e., $A\cong_Z\oplus_{i=1}^d Z$, where $d=\deg\Delta(A)$) if and only if $\mu(\Delta(A))=\pm 1$. More generally, by [2] and [9], the following holds:

THEOREM 1.1 (Crowell [2]; Sumners [9]). *A Λ-module A with polynomial $f(t)$ is homogeneous of type $\boldsymbol{t}(Z[1/\mu(f)])$ and completely decomposable of rank d, i.e., $A\cong_Z\oplus_{i=1}^d Z[1/\mu(f)]$, where $d=\deg f(t)$, if and only if*

(1.1) *for each prime p which divides $\mu(f(t))$, $\varepsilon_p(f(t))$ is a unit in Λ_p.*

In Section 3, we generalize this theorem as follows:

THEOREM 1.2. *A Λ-module A is homogeneous of type τ if and only if each irreducible factor of $\Delta(A)$ is of type τ.*

We say that a polynomial $f(t)$ of Λ is *of type* τ if there exist primes $p_1, \cdots, p_m$ such that $\tau=\boldsymbol{t}(Z[1/p_1\cdots p_m])$ and $\varepsilon_{p_i}(f(t))$ is a unit in Λ_{p_i}, $i=1, \cdots, m$, but $\varepsilon_p(f(t))$ is not a unit in Λ_p, $p\neq p_1, \cdots, p_m$. (If $m=0$, then we set $\tau=\boldsymbol{t}(Z)$.) Let $\boldsymbol{t}(f)$ denote the type of $f(t)$. Note that a polynomial $f(t)$ of Λ satisfies (1.1) if and only if

(1.1)$'$ *each irreducible factor of $f(t)$ is of type $\boldsymbol{t}(Z[1/\mu(f)])$.*

In Section 4, we obtain the following result:

THEOREM 1.3. *If a Λ-module A is completely decomposable, then each irreducible factor of $\Delta(A)$ satisfies* (1.1) *in Theorem* 1.1. *Conversely, given a polynomial $f(t)$ of Λ such that* (i) $f(1)=\pm 1$ *and* (ii) *each irreducible factor of $f(t)$ satisfies* (1.1), *there exists a completely decomposable Λ-module of rank d with polynomial $f(t)$, where $d=\deg f(t)$.*

In Section 5, we consider cyclic Λ-modules and give a criterion for a cyclic Λ-module to be completely decomposable (Theorem 5.1). D. W. Sumners [**9**] has defined the model of a Λ-module A as follows: Let $\{\lambda_i\}_{i=1}^k$ be the polynomial invariants of A. Then we call the Λ-module M defined by

$$M=\Lambda/(\lambda_1)\oplus\cdots\oplus\Lambda/(\lambda_k)$$

the *model* of A. (Note that M is also Z-torsion free because λ_i, $i=1,\cdots,k$, are primitive.) Sumners [**9**] proved that if A is a Λ-module, then there exists an exact sequence of Λ-modules

$$0\longrightarrow M\xrightarrow{\phi}A\longrightarrow G\longrightarrow 0,$$

where G is a finite Λ-module. Thus the monomorphic image $\phi(M)$ of the model M has finite index in A. Then, Sumners asks the following question:

QUESTION 1.4 ([**9**, p. 84]). *Is the model M of any Λ-module A Z-isomorphic to A?*

In Section 5, we can answer in the negative as follows:

THEOREM 1.5. *There exists a Λ-module A whose model is not Z-isomorphic to A.*

2. Preliminaries.

Let x be an element of an abelian group G. We say that x is *divisible* by a positive integer n in G if there exists an element y of G such that $ny=x$. Given a prime p, the largest integer k such that x is divisible by p^k in G is called the *p-height* $h_p(x)$ of x; if no such maximal integer k exists, then we set $h_p(x)=\infty$. Let $p_1, p_2, \cdots, p_n, \cdots$ be the sequence of all primes of Z in increasing order of magnitude, i.e., $0<p_1<p_2<\cdots<p_n<\cdots$. Then, the sequence of p-heights

$$\chi_G(x) \text{ (or simply } \chi(x))=(h_{p_1}(x), h_{p_2}(x), \cdots)$$

is called the *characteristic* or the *height-sequence* of x. Let $(k_1,\cdots,k_n,\cdots)$ and $(l_1,\cdots,l_n,\cdots)$ be two characteristics. Then by $(k_1,\cdots,k_n,\cdots)\geqq(l_1,\cdots,l_n,\cdots)$, we mean that $k_n\geqq l_n$ for all n.

The following are easily obtained:

PROPOSITION 2.1 ([**3**, vol. 2, p. 108]). *Let G be an abelian group. Then the following hold;*

(a) $\chi(-a)=\chi(a)$ *for all* $a\in G$,

(b) $\chi_A(a)\leqq\chi_G(a)$ *for any element* a *of a subgroup* A *of* G*; and* A *is pure in* G *if and only if equality holds for all* $a\in A$,

(c) $\chi(a+b)\geqq\chi(a)\cap\chi(b)$ *for all* $a, b\in G$, *where* $\chi(a)\cap\chi(b)=(\min\{h_{p_1}(a), h_{p_1}(b)\}, \cdots)$,

(d) *if* $G=A\oplus B$ *and* $a\in A$, $b\in B$, *then* $\chi(a+b)=\chi(a)\cap\chi(b)$,

(e) *for any homomorphism* α *of* G *to an abelian group* H *and for any* $a\in G$, $\chi_G(a)\leqq\chi_H(\alpha(a))$.

We say that two characteristics $(k_1, \cdots, k_n, \cdots)$ and $(l_1, \cdots, l_n, \cdots)$ are *equivalent* if $\sum_{n=1}^{\infty}|k_n-l_n|$ is finite. (Set $\infty-\infty=0$.) An equivalence class of characteristics is called a *type*, Let x be an element of an abelian group G. If $\chi_G(x)$ belongs to a type τ, then we say that x is *of type* τ and write $\boldsymbol{t}_G(x)=\tau$ (or simply $\boldsymbol{t}(x)=\tau$). Let τ_1 and τ_2 be types. Then by $\tau_1\geqq\tau_2$, we mean that there are characteristics $\chi_1\in\tau_1$ and $\chi_2\in\tau_2$ such that $\chi_1\geqq\chi_2$. Two types τ_1 and τ_2 are *comparable* if $\tau_1\geqq\tau_2$ or $\tau_1\leqq\tau_2$. A type τ of a subset S of the set of types is said to be *maximal* in S if $\tau\not<\tau_S$ for any $\tau_S\in S$.

The following are well known results:

PROPOSITION 2.2 ([**3**, vol. 2, p. 109]). *Let G be an abelian group.*

(A) *If* $ma=rb$ $(a, b\in G,\ m, r\in Z-\{0\})$, *then* $\boldsymbol{t}(a)=\boldsymbol{t}(b)$,

(B) $\boldsymbol{t}_A(a)\leqq\boldsymbol{t}_G(a)$ *for any element* a *of a subgroup* A *of* G,

(C) $\boldsymbol{t}(a+b)\geqq\boldsymbol{t}(a)\cap\boldsymbol{t}(b)$ *for all* $a, b\in G$, *where* $\boldsymbol{t}(a)\cap\boldsymbol{t}(b)$ *is the type represented by* $\chi_1\cap\chi_2$ $(\chi_1\in\boldsymbol{t}(a), \chi_2\in\boldsymbol{t}(b))$,

(D) *if* $G=A\oplus B$ *and* $a\in A$, $b\in B$, *then* $\boldsymbol{t}(a+b)=\boldsymbol{t}(a)\cap\boldsymbol{t}(b)$ *and* $\boldsymbol{t}_A(a)=\boldsymbol{t}_G(a)$,

(E) *for any homomorphism* α *of* G *to an abelian group* H *and for any* $a\in G$, $\boldsymbol{t}_G(a)\leqq\boldsymbol{t}_H(\alpha(a))$.

If all the nonzero elements of an abelian group G are of the same type τ, we say that G is *homogeneous of type* τ.

REMARK 2.3. From Proposition 2.2(A), all the nonzero elements of an abelian group of rank one are of the same type, that is, it is homogeneous. Moreover, it is known that two abelian groups of rank one are isomorphic if and only if they are of the same type [**1**], [**3**, vol. 2]. Therefore the above definition of types agrees with that in Section 1. Concerning completely decomposable abelian groups, it is known that any direct summand of a completely decomposable abelian group is also completely decomposable, and the decomposition of a completely decomposable abelian group A into a direct sum of rank one abelian groups is unique in the sense that, if $A=\oplus_i B_i=\oplus_j C_j$, where $r(B_i)=r(C_j)=1$, then one can find a one-to-one correspondence between the two sets $\{B_i\}$ and $\{C_j\}$ of components such that corresponding components are isomorphic [**1**], [**3**, vol. 2].

LEMMA 2.4. *Let A be a Λ-module. If $h(t)$ is a factor of $\Delta(A)$ such that* g.c.d.$(h(t), \Delta(A)/h(t))=1$, *then hA and $\{hA\}_*$ are Λ-submodules with polynomial* $\Delta(A)/h(t)$.

PROOF. The proof is elementary.

PROPOSITION 2.5. *If a is a nonzero element of a Λ-module A, then we have $\boldsymbol{t}_A(a) \leqq \boldsymbol{t}(Z[1/\mu(\Delta(A))])$, i.e., there exists a positive integer k such that $\boldsymbol{t}_A(a)=\boldsymbol{t}(Z[1/k])$ and $k \mid \mu(\Delta(A))$.*

PROOF. By [**2**] or [**9**], there exists a monomorphism $\phi: A \to B$, where $B=\bigoplus_{i=1}^{d} Z[1/\mu(\Delta(A))]$ and $d=\deg \Delta(A)$. Therefore, from Proposition 2.2(E), we obtain

$$\boldsymbol{t}_A(a) \leqq \boldsymbol{t}_B(\phi(a)) = \boldsymbol{t}(Z[1/\mu(\Delta(A))])$$

for each $a\ (\neq 0) \in A$.

LEMMA 2.6. *Let A be a Λ-module and $A(\tau)=\{x \in A : \boldsymbol{t}_A(x) \geqq \tau\}$, where τ is a type. Then*

(1) *$A(\tau)$ is a pure Λ-submodule of A, and $\boldsymbol{t}_{A(\tau)}(a)=\boldsymbol{t}_A(a)$ for any element a of $A(\tau)$, and*

(2) *if A is completely decomposable, then $A(\tau)$ is a completely decomposable direct summand of A as an abelian group.*

PROOF. By [**3**, vol. 2, p. 109] or Propositions 2.1(b), 2.2(A) and (C), $A(\tau)$ is a pure subgroup of A and $\boldsymbol{t}_{A(\tau)}(a)=\boldsymbol{t}_A(a)$ for any element a of $A(\tau)$. Since $t^{\pm 1}$ induces an automorphism of A, it follows from Proposition 2.2(E) that $\boldsymbol{t}_A(t^{\pm 1}a)=\boldsymbol{t}_A(a) \geqq \tau$ for any element of $A(\tau)$. Hence $A(\tau)$ is a Λ-submodule, and so (1) holds. Suppose that A is completely decomposable and $A=A_1 \oplus \cdots \oplus A_n$, where $r(A_i)=1$, $i=1, \cdots, n$. We may assume that $\boldsymbol{t}(A_i) \geqq \tau$ and $\boldsymbol{t}(A_j) \ngeqq \tau$, $0<i \leqq r<j \leqq n$ for some integer r. Then, from Proposition 2.2(D), it is easily seen that $A(\tau)=A_1 \oplus \cdots \oplus A_r$.

3. Homogeneous modules.

In this section, we give the proof of Theorem 1.2. To prove the theorem, we will prove some lemmas.

LEMMA 3.1. *Let p be a prime. Then a Λ-module A is p-divisible, i.e., $\boldsymbol{t}_A(a) \geqq \boldsymbol{t}(Z[1/p])$ for any $a \in A$, if and only if $\varepsilon_p(\Delta(A))$ is a unit in Λ_p.*

PROOF. Suppose that A is p-divisible. Then, for any $a \in A$, there is an element b of A such that $a=pb$. Therefore we have $a\otimes 1=pb\otimes 1=b\otimes p=0$ in the vector space $A\otimes_Z Z_p$ over Z_p. Hence $\deg \varepsilon_p(\varDelta(A))=\dim_{Z_p} A\otimes_Z Z_p=0$, and so $\varepsilon_p(\varDelta(A))$ is a unit in $\varLambda_p$. Conversely, suppose that $\varepsilon_p(\varDelta(A))$ is a unit in $\varLambda_p$. Then there exists a polynomial $h(t)$ of $\varLambda$ such that

$$1=q\cdot t^i\varDelta(A)+p\cdot h(t), \qquad q\in Z.$$

Therefore, for any element a of A, we have $a=(q\cdot t^i\varDelta(A)+p\cdot h(t))a=p(h(t)a)$. This shows that A is p-divisible.

LEMMA 3.2. *Let A and B be abelian groups of finite rank with a monomorphism $\phi: A\to B$ such that $|B/\phi(A)|$ is finite. Then for any element a of A, we have $\boldsymbol{t}_A(a)=\boldsymbol{t}_B(\phi(a))$.*

PROOF. Let $n=|B/\phi(A)|$. Since nB is contained in the image $\phi(A)$ and ϕ is one-one, we can define a homomorphism $\psi: B\to A$ by $\psi(b)=\phi^{-1}(nb)$. Then we see that

$$\psi(\phi(a))=\phi^{-1}(n(\phi(a)))=na$$

for any element a of A. Therefore, by Proposition 2.2(A) and (E), we obtain

$$\boldsymbol{t}_A(a)\leqq \boldsymbol{t}_B(\phi(a))\leqq \boldsymbol{t}_A(\psi(\phi(a)))=\boldsymbol{t}_A(na)=\boldsymbol{t}_A(a).$$

The proof is completed.

LEMMA 3.3. *Let A be a cyclic $\varLambda$-module and $\tau=\boldsymbol{t}(Z[1/p_1\cdots p_m])$, where $p_1,\cdots,p_m$ are primes. Then there exists an element x of A whose type is τ if and only if there exists a factor $g(t)$ of $\varDelta(A)$ whose type is τ. (If $m=0$, then we set $\tau=\boldsymbol{t}(Z)$.)*

PROOF. Suppose that there exists $x\in A$ such that $\boldsymbol{t}_A(x)=\tau$. Let $A(\tau)=\{b\in A: \boldsymbol{t}_A(b)\geqq\tau\}$. Then, by Lemma 2.6, $A(\tau)$ is a pure $\varLambda$-submodule of A and, for any element b of $A(\tau)$,

$$\boldsymbol{t}_{A(\tau)}(b)=\boldsymbol{t}_A(b)\geqq\tau\geqq\boldsymbol{t}(Z[1/p_i]), \qquad i=1,\cdots,m.$$

Hence by Lemma 3.1, $\varepsilon_{p_i}(\varDelta(A(\tau)))$ is a unit in $\varLambda_{p_i}$, $i=1,\cdots,m$. Moreover, since $\boldsymbol{t}_{A(\tau)}(x)=\tau\ngeqq\boldsymbol{t}(Z[1/p])$ for any prime $p\neq p_1,\cdots,p_m$, it follows from Lemma 3.1 that $\varepsilon_p(\varDelta(A(\tau)))$ is not a unit in $\varLambda_p$. Thus, since $\varDelta(A(\tau))|\varDelta(A)$, we complete the proof of the sufficiency, i.e., we get a factor $g(t)=\varDelta(A(\tau))$.

Conversely, suppose that there exists a factor $g(t)$ of $\varDelta(A)$ such that

(3.1) $\varepsilon_{p_i}(g(t))$ is a unit in $\varLambda_{p_i}$, $i=1,\cdots,m$, but $\varepsilon_p(g(t))$ is not a unit in $\varLambda_p$, $p\neq p_1,\cdots,p_m$.

Let $g^*(t)=\varDelta(A)/g(t)$ and $A^*=g^*A$. Then A^* is a cyclic $\varLambda$-submodule of A with $\varDelta(A^*)=g(t)$ and is generated by g^*a, where a is a generator of A as a $\varLambda$-module. Therefore, from (3.1) and Lemma 3.1, we see that

$$\boldsymbol{t}_{A^*}(g^*a) \geqq \boldsymbol{t}(Z[1/p_i]), \qquad i=1, \cdots, m, \text{ and}$$
$$\boldsymbol{t}_{A^*}(g^*a) \ngeqq \boldsymbol{t}(Z[1/p]), \qquad p \neq p_1, \cdots, p_m.$$

Thus, by Proposition 2.5, we have

$$\boldsymbol{t}_{A^*}(g^*a) = \tau.$$

On the other hand, since A/A^* ($\cong_{\Lambda} \Lambda/(g^*(t))$) is Z-torsion free, A^* is pure in A. It follows from Proposition 2.1(b) that

$$\boldsymbol{t}_A(g^*a) = \boldsymbol{t}_{A^*}(g^*a) = \tau.$$

Thus we complete the proof.

More generally, concerning the type of an element of a Λ-module, we have the following:

THEOREM 3.4. *Let A be a Λ-module and $\tau=\boldsymbol{t}(Z[1/p_1 \cdots p_m])$, where $p_1, \cdots, p_m$ are primes. Then there exists an element x of A whose type is τ if and only if there exists a factor $g(t)$ of $\Delta(A)$ whose type is τ.*

PROOF. The sufficiency is proved in the same way as in the proof of Lemma 3.3.

We will prove the necessity. Let $M=\Lambda/(\lambda_1)\oplus \cdots \oplus \Lambda/(\lambda_k)$ be the model of A. Since $\lambda_{i+1} | \lambda_i$, $i=1, \cdots, k-1$, we may assume that $g(t)$ is a factor of λ_1. Therefore, by Lemma 3.3, there exists an element x of $\Lambda/(\lambda_1)$ such that $\boldsymbol{t}_{\Lambda/(\lambda_1)}(x)=\tau$. Since $\Lambda/(\lambda_1)$ is a direct summand of M, it follows from Proposition 2.2(D) that $\boldsymbol{t}_M(x)=\boldsymbol{t}_{\Lambda/(\lambda_1)}(x)$. Therefore, since $|A/M|$ is finite (see Section 1 or [**9**]), Lemma 3.2 shows that $\boldsymbol{t}_A(x)=\boldsymbol{t}_M(x)=\tau$.

PROOF OF THEOREM 1.2. By the definition of homogeneity, A is homogeneous of type τ if and only if $\boldsymbol{t}_A(a)=\tau$ for any a $(\neq 0) \in A$. Therefore, from Theorem 3.4, we obtain the theorem.

COROLLARY 3.5. *If a Λ-module A is π-primary, i.e., $\Delta(A)$ is a power of a single irreducible polynomial of Λ, then A is homogeneous.*

REMARK 3.6. In [7, p. 32], J. Levine defined 'homogeneity' for Λ-modules. Since his 'homogeneity' implies π-primary, every 'homogeneous' Λ-module in Levine's sense is homogeneous as an abelian group in our sense.

In case of a homogeneous Λ-module A of type $\boldsymbol{t}(Z[1/\mu(\Delta(A))])$, we obtain the following:

PROPOSITION 3.7. *Let A be a Λ-module. Then, the following four statements are equivalent:*

(1) *A is completely decomposable and homogeneous,*

(2) *A is homogeneous of type $\boldsymbol{t}(Z[1/m])$, where $m=\mu(\Delta(A))$,*

(3) $\Delta(A)$ *satisfies the condition* (1.1) *in Theorem* 1.1,

(4) $A\cong_Z\bigoplus_{i=1}^d Z[1/m]$, *where* $d=r(A)$.

PROOF. First we show that (1) implies (2). By Remark 2.3 and Proposition 2.5, we have

$$A\cong_Z \bigoplus_{i=1}^{d} Z[1/k],$$

where k is a positive integer such that $k|m$. Let p be any prime such that $p|m$. Then, since $\dim_{Z_p} A\otimes_Z Z_p=\deg \varepsilon_p(\Delta(A))<\deg \Delta(A)=d$, it follows that $p|k$, otherwise $A\otimes_Z Z_p\cong\bigoplus_{i=1}^d Z[1/k]\otimes_Z Z_p\cong\bigoplus_{i=1}^d Z_p$. Therefore we see that $Z[1/k]=Z[1/m]$. Thus A is homogeneous of type $\boldsymbol{t}(Z[1/m])$. Next, Theorem 1.2 shows that (2) implies (3). Therefore, by Theorem 1.1, we complete the proof.

4. Completely decomposable modules.

In this section, we study completely decomposable Λ-modules. First we prove the following:

THEOREM 4.1. *Let A be a Λ-module with polynomial $f(t)$. Then, if A is completely decomposable,*

(1) $A\cong_Z Z[1/m_1]\oplus\cdots\oplus Z[1/m_d]$, *where* $d=\deg f(t)$,

(2) *there exists a decomposition* $f(t)=f_1(t)\cdots f_n(t)$ *into non-unit factors such that each factor* $f_i(t)$, $i=1,\cdots,n$, *satisfies* (1.1) *in Theorem* 1.1, *and* $\boldsymbol{t}(f_i)\neq\boldsymbol{t}(f_j)$, $i\neq j$, *and*

(3) $\deg f_i(t)$ *equals the number of rank one components isomorphic to* $Z[1/\mu(f_i)]$, $i=1,\cdots,n$.

PROOF. Let $A=B_1\oplus\cdots\oplus B_n$, where each B_i $(\neq 0)$, $i=1,\cdots,n$, is completely decomposable and homogeneous, and $\boldsymbol{t}(B_i)\neq\boldsymbol{t}(B_j)$, $i\neq j$. Then, by Proposition 2.5, there exists a positive integer m_i, $i=1,\cdots,n$, such that $\boldsymbol{t}(B_i)=\boldsymbol{t}(Z[1/m_i])\leqq \boldsymbol{t}(Z[1/\mu(\Delta(A))])$. Therefore, by Remark 2.3, we see that

$$B_i\cong_Z \bigoplus_{j=1}^{d_i} Z[1/m_i],$$

where $d_i=r(B_i)$, $i=1,\cdots,n$, and so

$$A\cong_Z \bigoplus_{i=1}^{n}\left\{\bigoplus_{j=1}^{d_i} Z[1/m_i]\right\}.$$

To complete the proof, we use induction on the number n of the components B_i. When $n=1$, the assertions follow from Proposition 3.7. Suppose that $n>1$. We may assume that $\tau=\boldsymbol{t}(B_1)$ is maximal in $\{\boldsymbol{t}(B_1),\cdots,\boldsymbol{t}(B_n)\}$. Then, by Lemma 2.6, B_1 $(=\{x\in A:\boldsymbol{t}_A(x)\geqq\tau\})$ is a Λ-submodule of A. Since B_1 is completely decomposable and homogeneous of type τ, it follows from Theorem 1.1 that $f_1(t)=\Delta(B_1)$ satisfies (1.1) and $\deg f_1(t)=r(B_1)$, $\boldsymbol{t}(f_1)=\tau$. On the other hand, $A/B_1=B_2\oplus\cdots\oplus B_n$ is also a completely decomposable Λ-module. Thus, by the inductive hypothesis, we see that there exists a decomposition $\Delta(A/B_1)=f_2(t)\cdots f_n(t)$ into non-unit factors such that

(2′) each factor $f_i(t)$, $i=2, \cdots, n$, satisfies (1.1), and $\boldsymbol{t}(f_i)\neq\boldsymbol{t}(f_j)$, $i\neq j$,

(3′) $\deg f_i(t)$ equals the number of rank one components isomorphic to $Z[1/\mu(f_i)]$, $i=2, \cdots, n$.

Therefore, since $\Delta(A)=\Delta(A/B_1)\cdot\Delta(B_1)=f_1(t)\cdots f_n(t)$, we establish the theorem. (Note that, since τ is maximal, A/B_1 has no component of type τ, and so $\boldsymbol{t}(f_i)\neq\tau=\boldsymbol{t}(f_1)$, $i=2, \cdots, n$.)

Using Theorem 4.1, we can prove Theorem 1.3.

PROOF OF THEOREM 1.3. The first assertion immediately follows from Theorem 4.1. To prove the second, we give the following example:

$$A=\Lambda/(g_1(t)^{k_1})\oplus\cdots\oplus\Lambda/(g_r(t)^{k_r}),$$

where $g_i(t)$, $i=1, \cdots, r$, are irreducible and $g_1(t)^{k_1}\cdots g_r(t)^{k_r}=f(t)$, $k_i>0$. Then, by Theorem 1.1, each component $\Lambda/(g_i(t)^{k_i})$, $i=1, \cdots, r$, is completely decomposable. Thus, since $\Delta(A)=f(t)$, this completes the proof.

REMARK 4.2. There exists a Λ-module whose polynomial satisfies (i) and (ii), but which is not completely decomposable (see the proof of Theorem 1.5 in Section 5).

COROLLARY 4.3. *Let* $\{\lambda_i(t)\}_{i=1}^k$ *be a family of non-unit polynomials of* Λ *such that* (1) $\lambda_{i+1}\mid\lambda_i$, $\lambda_i(1)=\pm1$ *and* (2) *each irreducible factor of* λ_i *satisfies* (1.1). *Then there exists a completely decomposable* Λ*-module whose polynomial invariants are* $\{\lambda_i(t)\}_{i=1}^k$.

COROLLARY 4.4. *If* Λ*-modules* A_1 *and* A_2 *are completely decomposable and* $\Delta(A_1)=\Delta(A_2)$, *then* $A_1\cong_Z A_2$.

Let $(g_1(t), \cdots, g_n(t))$ denote the ideal of Λ generated by polynomials $g_1(t), \cdots, g_n(t)\in\Lambda$.

REMARK 4.5. If $(g_1(t), \cdots, g_n(t))=(1)$, then $\text{g.c.d.}(g_1(t), \cdots, g_n(t))=1$ and $(g_1'(t), \cdots, g_n'(t))=(1)$, where $g_i'(t)$ is any factor of $g_i(t)$, $i=1, \cdots n$. Moreover $(g_1(t), \cdots, g_n(t))=(1)$ if and only if there exist polynomials $h_1(t), \cdots, h_n(t)$ of Λ such that $h_1(t)g_1(t)+\cdots+h_n(t)g_n(t)=1$.

Though we do not know a necessary and sufficient condition for a Λ-module to be completely decomposable, we can give the following sufficient condition:

THEOREM 4.6. *Let A be a Λ-module with polynomial $f(t)$. Suppose that*

(1) *each irreducible factor of $f(t)$ satisfies* (1.1) *in Theorem* 1.1, *and*

(2) *if $h_1(t)$, $h_2(t)$ are factors of $f(t)$ such that arbitrary irreducible factors $g_1(t)$, $g_2(t)$ of $h_1(t)$, $h_2(t)$ are of incomparable types, i.e.,*

$$t(g_1) \ngeqq t(g_2), \qquad t(g_2) \ngeqq t(g_1), \tag{4.1}$$

then $(h_1(t), h_2(t))=(1)$. Then A is completely decomposable of rank d, where $d=\deg f(t)$.

Before proving the theorem, we will prove some lemmas.

LEMMA 4.7. *Let A be a completely decomposable Λ-module and $\Delta(A)=f_1(t)\cdots f_m(t)$ be a decomposition into non-unit factors such that each factor $f_i(t)$, $i=1, \cdots, m$, satisfies* (1.1) *in Theorem* 1.1 *and $t(f_i)\neq t(f_j)$, $i\neq j$. If $h_1(t)=f_1(t)\cdots f_r(t)$ and $h_2(t)=f_{r+1}(t)\cdots f_m(t)$ satisfy the condition*

$$t(f_i) \ngeqq t(f_j), \qquad 0<i\leqq r<j\leqq m, \tag{4.2}$$

then the pure Λ-submodule $\{h_1A\}_*$ is a direct summand of A as an abelian group. Moreover, if $A={}_ZA_1\oplus\cdots\oplus A_m$, where $A_i\cong_Z\oplus_{j=1}^{d_i}Z[1/\mu(f_i)]$ and $d_i=\deg f_i(t)$, is a direct decomposition of A, then we have

$$\{h_1A\}_* ={}_Z A_{r+1}\oplus\cdots\oplus A_m$$

and so $A={}_ZA_1\oplus\cdots\oplus A_r\oplus\{h_1A\}_*$.

PROOF. Let $B=A_{r+1}\oplus\cdots\oplus A_m$, $A'=A/\{h_1A\}_*$ and $\phi: A\to A'$ be the canonical Λ-homomorphism of A onto A'. Then we have $\Delta(A')=\Delta(A)/\Delta(\{h_1A\}_*)=h_1(t)$. Therefore, by Proposition 2.2(E), we see that, for any non-zero element a_j of A_j, $j=r+1, \cdots, m$,

$$t(f_j) = t(A_j) = t_A(a_j) \leqq t_{A'}(\phi(a_j)). \tag{4.3}$$

If $\phi(a_j)\neq 0$, it follows from Theorem 3.4 that there exists a non-unit factor $h_1'(t)$ of $h_1(t)$ such that $t_{A'}(\phi(a_j))=t(h_1')$. Therefore we have

$$t_{A'}(\phi(a_j)) = t(h_1') \leqq t(f_i)$$

for some factor $f_i(t)$, $0<i\leqq r$. Thus, from (4.3), we see that

$$t(f_j) \leqq t(f_i).$$

However this contradicts (4.2). Hence we obtain $\phi(a_j)=0$ and so $B\subset\{h_1A\}_*$. On the other hand, since B is pure in A, B is also pure in $\{h_1A\}_*$. Therefore, the quotient group $\{h_1A\}_*/B$ is Z-torsion free. Moreover, since $r(\{h_1A\}_*)=\deg\Delta(\{h_1A\}_*)=\deg h_2(t)=\sum_{j=r+1}^{m}\deg f_j(t)=r(B)$, we have $r(\{h_1A\}_*/B)=r(\{h_1A\}_*)-r(B)=0$, and so $\{h_1A\}_*=B$. This completes the proof.

LEMMA 4.8. *Let H and B be abelian groups of finite rank and $\phi: H\to B$ an epimorphism such that $t_H(c)\geqq t_B(b)$ for any $c\in\operatorname{Ker}\phi$ and for any $b\,(\neq 0)\in B$. If B is completely decomposable and each component B_i of rank* 1 *is isomorphic to*

$Z[1/m_i]$, *then* H *is isomorphic to* $\mathrm{Ker}\,\psi\oplus B$.

PROOF. Let $B=B_1\oplus\cdots\oplus B_n$, where $B_i\cong Z[1/m_i]$. Let b_i, $i=1,\cdots,n$, be a non-zero element of B_i such that if the p-height $h_p(b_i)$ is finite, then $h_p(b_i)=0$. (If the p-height $h_p(u)$ of an element u of B_i is k $(\neq\infty)$, then there exists $v\in B_i$ such that $p^k v=u$, and the p-height $h_p(v)$ of v is zero. Hence such an element b_i of B_i exists.) Then, since $B_i\cong Z[1/m_i]$, for each $x\in B_i$, there exist integers k and l such that

$$m_i^k x = lb_i\,, \qquad k\geqq 0\,. \tag{4.4}$$

Let a_i and y be elements of H such that $\psi(a_i)=b_i$, $\psi(y)=x$, and let $c=la_i-m_i^k y$. Then we have $c\in\mathrm{Ker}\,\psi$. Therefore, since $\boldsymbol{t}_H(c)\geqq\boldsymbol{t}_B(b_i)=\boldsymbol{t}(Z[1/m_i])$, there exists $c'\in H$ such that $m_i^k c'=c$. Hence, we have

$$m_i^k\bar{x} = la_i\,,$$

where $\bar{x}=y+c'$. Then $\bar{x}$ is uniquely determined by x and does not depend on a choice of k and l in (4.4). (Note that H and B are Z-torsion free.) Therefore we can define a mapping β_i of B_i to H by $\beta_i(x)=\bar{x}$. We will show that β_i is a homomorphism. Let $m_i^{k_1}x_1=l_1 b_i$ and $m_i^{k_2}x_2=l_2 b_i$ $(k_1,\ k_2\geqq 0)$. Then, since $m_i^{k_1+k_2}(x_1+x_2)=(m_i^{k_2}l_1+m_i^{k_1}l_2)b_i$, we see that

$$\begin{aligned} m_i^{k_1+k_2}\beta_i(x_1+x_2) &= m_i^{k_1+k_2}\overline{(x_1+x_2)} = (m_i^{k_2}l_1+m_i^{k_1}l_2)a_i = m_i^{k_2}l_1 a_i+m_i^{k_1}l_2 a_i \\ &= m_i^{k_2}(m_i^{k_1}\bar{x}_1)+m_i^{k_1}(m_i^{k_2}\bar{x}_2) = m_i^{k_1+k_2}(\bar{x}_1+\bar{x}_2) \\ &= m_i^{k_1+k_2}(\beta_i(x_1)+\beta_i(x_2))\,. \end{aligned}$$

Therefore we have $\beta_i(x_1+x_2)=\beta_i(x_1)+\beta_i(x_2)$ and so β_i is a homomorphism, $i=1,\cdots,n$.

Since B is the direct sum of $B_1,\cdots,B_n$, there is a homomorphism β of B to H such that $\beta|B_i=\beta_i$, $i=1,\cdots,n$. It is easy to see that $\psi\beta=1_B$. Therefore β is one-one. Hence, by [**3**, vol. 1, Lemma 9.1], $H=\mathrm{Ker}\,\psi\oplus\beta(B)\cong_Z \mathrm{Ker}\,\psi\oplus B$.

LEMMA 4.9. *Let* G, A *and* B *be abelian groups of finite rank and* $\phi: G\to A\oplus B$ *an epimorphism such that* $\boldsymbol{t}_G(c)\geqq\boldsymbol{t}_B(b)$ *for any* $c\in\mathrm{Ker}\,\phi$ *and for any* $b\,(\neq 0)\in B$. *Suppose that* B *is completely decomposable and each component* B_i *of rank* 1 *is isomorphic to* $Z[1/m_i]$. *If there exists a subgroup* A' *of* G *such that the restriction* $\phi|A': A'\to A$ *is an isomorphism of* A' *onto* A, *then* G *is isomorphic to* $A\oplus B\oplus\mathrm{Ker}\,\phi$.

PROOF. Let $H=\phi^{-1}(B)$. Then we have

$$\phi(A'\cap H)\subset\phi(A')\cap\phi(H) = A\cap B = 0\,.$$

Therefore $A'\cap H\subset\mathrm{Ker}\,\phi\cap A'$. Since $\phi|A'$ is one-one, it follows that $A'\cap H=0$. Thus, since $G=A'+H$, we get

(4.5) $$G=A'\oplus H.$$

Let $\psi: H\to B$ be the epimorphism of H onto B defined by $\psi(x)=\phi(x)$, $x\in H$. Then, from (4.5) and Proposition 2.2, we see that $\boldsymbol{t}_H(x)=\boldsymbol{t}_G(x)$, $x\in H$. Therefore, we have

$$\boldsymbol{t}_H(c)=\boldsymbol{t}_G(c)\geqq \boldsymbol{t}_B(b)$$

for any $c\in \mathrm{Ker}\,\psi$ ($\subset \mathrm{Ker}\,\phi$) and for any $b\,(\neq 0)\in B$. Thus, from Lemma 4.8, we obtain

(4.6) $$H\cong \mathrm{Ker}\,\psi\oplus B.$$

On the other hand, since $\mathrm{Ker}\,\psi=\mathrm{Ker}\,\phi\cap H=\mathrm{Ker}\,\phi$, it follows from (4.5) and (4.6) that

$$G\cong A'\oplus(\mathrm{Ker}\,\phi\oplus B)\cong A\oplus B\oplus \mathrm{Ker}\,\phi.$$

PROOF OF THEOREM 4.6. By (1), there exists a decomposition $f(t)=f_1(t)\cdots f_n(t)$ into non-unit factors such that each factor $f_i(t)$, $i=1,\cdots,n$, satisfies (1.1) and $\boldsymbol{t}(f_i)\neq \boldsymbol{t}(f_j)$, $i\neq j$. We may asumme that $\boldsymbol{t}(f_1)$ is maximal in $\{\boldsymbol{t}(f_1),\cdots,\boldsymbol{t}(f_n)\}$. We use induction on n. If $n=1$, then the assertion follows from Theorem 1.1. Suppose that $n>1$. Without loss of generality, we may assume that, for some integer r $(1\leqq r\leqq n)$

(4.7) $$\boldsymbol{t}(f_i)\leqq \boldsymbol{t}(f_1),\qquad \boldsymbol{t}(f_j)\nleqq \boldsymbol{t}(f_1)\qquad 2\leqq i\leqq r<j\leqq n.$$

(Note that $\boldsymbol{t}(f_j)\ngeqq \boldsymbol{t}(f_1)$ because $\boldsymbol{t}(f_1)$ is maximal.) Therefore we have

(4.8) $$\boldsymbol{t}(f_i)\ngeqq \boldsymbol{t}(f_j),\qquad 2\leqq i\leqq r<j\leqq n.$$

Let $h_1(t)=f_2(t)\cdots f_r(t)$, $h_2(t)=f_{r+1}(t)\cdots f_n(t)$, $C=\{(f/f_1)A\}_*$ and $H=A/C$. Then C and H are Λ-modules with $\Delta(C)=f_1(t)$ and $\Delta(H)=\Delta(A)/\Delta(C)=h_1(t)h_2(t)$. Therefore, by the inductive hypothesis, C and H are completely decomposable. Furthermore, by (4.8) and Lemma 4.7, we have

(4.9) $$H=_Z\{h_1H\}_*\oplus B,$$

where $\{h_1H\}_*$ is a completely decomposable Λ-submodule of H with $\Delta(\{h_1H\}_*)=h_2(t)$, and B is a completely decomposable subgroup of H Z-isomorphic to the quotient Λ-module $H/\{h_1H\}_*$ with $\Delta(H/\{h_1H\}_*)=h_1(t)$.

Let $\phi: A\to H$ be the canonical Λ-epimorphism of A onto H, and let $D=\phi^{-1}(\{h_1H\}_*)$. Then, since $A/D\cong_\Lambda H/\{h_1H\}_*\cong_Z B$, D is a pure Λ-submodule of A with $\Delta(D)=\Delta(A)/\Delta(H/\{h_1H\}_*)=f/h_1=f_1h_2$. We will show that the restriction $\phi|f_1D: f_1D\to\{h_1H\}_*$ is an isomorphism. Let $x\in\{h_1H\}_*$. Then there is $y\in D$ such that $\phi(y)=x$. By (2) and (4.7), there exist polynomials F_1 and F_2 of Λ such that

$$F_1f_1+F_2h_2=1.$$

Hence, we see that

$$\phi(F_1f_1y)=\phi((1-F_2h_2)y)=\phi(y)-F_2h_2\phi(y)=x-F_2h_2x\,.$$

Thus, since $\Delta(\{h_1H\}_*)=h_2$, we have

$$\phi(F_1f_1y)=x\,.$$

Therefore the restriction $\phi|f_1D: f_1D\to\{h_1H\}_*$ is onto. Moreover, since $r(\operatorname{Ker}\phi|f_1D)=r(f_1D)-r(\{h_1H\}_*)=\deg\Delta(f_1D)-\deg\Delta(\{h_1H\}_*)=\deg h_2-\deg h_2=0$, it follows that $\phi|f_1D: f_1D\to\{h_1H\}_*$ is an isomorphism of f_1D onto $\{h_1H\}_*$. (Note that f_1D and $\{h_1H\}_*$ are Z-torsion free.)

By (4.9), B is Z-isomorphic to the completely decomposable Λ-module $\{H/h_1H\}_*$ with polynomial $h_1(t)=f_2(t)\cdots f_r(t)$. On the other hand, since $\Delta(\operatorname{Ker}\phi)=\Delta(A)/\Delta(H)=f_1(t)$, it follows from (4.7) and Theorem 3.6 that, for any $c\in\operatorname{Ker}\phi$ $(=C)$ and any $b\,(\neq 0)\in B$,

$$\boldsymbol{t}_A(c)\geqq\boldsymbol{t}_C(c)\geqq\boldsymbol{t}_B(b)\,.$$

Hence, By Lemma 4.9, we see that

$$A\cong_Z f_1D\oplus B\oplus C\,.$$

This completes the proof.

COROLLARY 4.10. *Let A be a Λ-module. If each irreducible factor of $\Delta(A)$ satisfies* (1.1) *and any pair of irreducible factors of $\Delta(A)$ are of comparable types, then A is completely decomposable of rank d, where $d=\deg\Delta(A)$.*

5. Completely decomposable cyclic modules.

In the previous section, we give a sufficient condition for a Λ-module to be completely decomposable. In this section, we prove that the condition is necessary in case of cyclic Λ-modules.

THEOREM 5.1. *A cyclic Λ-module A with polynomial $f(t)$ is completely decomposable of rank d if and only if $f(t)$ satisfies* (1) *and* (2) *in Theorem* 4.6, *where $d=\deg f(t)$.*

COROLLARY 5.2. *Let A be a cyclic Λ-module with $\Delta(A)=g_1(t)^{k_1}g_2(t)^{k_2}$, where $g_i(t)$ is irreducible and $k_i>0$, $i=1, 2$. Then A is completely decomposable if and only if*

(1) *$g_i(t)$, $i=1, 2$, satisfies* (1.1) *in Theorem* 1.1, *and*

(2) *$\boldsymbol{t}(g_1)$ and $\boldsymbol{t}(g_2)$ are comparable, or $(g_1(t)^{k_1}, g_2(t)^{k_2})=(1)$.*

As a corollary to Theorems 4.6 and 5.1, we obtain the following:

COROLLARY 5.3. *If a cyclic Λ-module A is completely decomposable, then so is every Λ-submodule of A.*

To prove the theorem, we prove some lemmas.

LEMMA 5.4. *If a cyclic Λ-module A is decomposed into a direct sum of two cyclic Λ-submodules A_1 and A_2 as a Λ-module, then $(\Delta(A_1), \Delta(A_2))=(1)$.*

PROOF. The lemma follows immediately from the fact that the second elementary ideal of A is (1).

LEMMA 5.5. *Let A be a cyclic Λ-module and $h(t)$ any factor of $\Delta(A)$. Then the Λ-submodule $h(t)A$ is pure in A, i.e., $\{h(t)A\}_*=h(t)A$.*

PROOF. Since A is cyclic, the quotient Λ-module $A/h(t)A$ is Λ-isomorphic to $\Lambda/(h(t))$, which is Z-torsion free. Therefore $h(t)A$ is pure.

LEMMA 5.6. *Let A be a completely decomposable cyclic Λ-module and $\Delta(A)= f_1(t)\cdots f_m(t)$ be a decomposition of $\Delta(A)$ into non-unit factors such that each factor $f_i(t)$, $i=1, \cdots, m$, satisfies* (1.1) *in Theorem* 1.1, *and $\mathbf{t}(f_i)\neq\mathbf{t}(f_j)$, $i\neq j$. If $h_1(t)= f_1(t)\cdots f_r(t)$ and $h_2(t)=f_{r+1}(t)\cdots f_m(t)$ satisfy the condition*

$$(5.1)\qquad \mathbf{t}(f_i)\not\leqq \mathbf{t}(f_j),\qquad \mathbf{t}(f_i)\not\geqq \mathbf{t}(f_j),\qquad 0<i\leqq r<j\leqq m,$$

then $(h_1(t), h_2(t))=(1)$.

PROOF. Let $A=\bigoplus_{i=1}^m A_i$, where $A_i\cong_Z\bigoplus_{j=1}^{d_i}Z[1/\mu(f_i)]$ and $d_i=\deg f_i(t)$, $i= 1, \cdots, m$. Then, by Lemmas 4.7, 5.5 and (5.1), we have

$$h_2A = A_1\oplus\cdots\oplus A_r,\ h_1A = A_{r+1}\oplus\cdots\oplus A_m,\quad\text{and}\quad A =_Z h_2A\oplus h_1A.$$

Therefore, since h_1A and h_2A are Λ-submodules, A is the direct sum of h_2A and h_1A as a Λ-module. Thus, by Lemma 5.4, we obtain $(h_1(t), h_2(t))=(1)$.

PROOF OF THEOREM 5.1. The necessity follows from Theorem 4.6. Therefore suppose that A is completely decomposable. By Theorem 1.3, the first assertion holds. Then we will prove the second. Let $f(t)=f_1(t)\cdots f_n(t)$ be a decomposition of $f(t)$ into non-unit factors such that each factor $f_i(t)$, $i=1, \cdots, n$, satisfies (1.1) and $\mathbf{t}(f_i)\neq\mathbf{t}(f_j)$, $i\neq j$. We use induction on the number n of the factors $f_i(t)$. If $n=1$, then the assertion is trivial. Suppose that $n>1$. Let $h_1(t)$ and $h_2(t)$ be factors of $f(t)$ satisfying the assumption in (2) of Theorem 4.6. Then we may suppose that $h_1(t)$ and $h_2(t)$ are factors of $\tilde{h}_1(t)=f_1(t)\cdots f_r(t)$ and $\tilde{h}_2(t)=f_{r+1}(t)\cdots f_m(t)$ for some integers r, m $(1\leqq r<m\leqq n)$. Moreover, without loss of generality, we may assume that

(5.2) $\mathbf{t}(f_i)\not\geqq \mathbf{t}(f_j)$, $\quad\mathbf{t}(f_i)\not\leqq \mathbf{t}(f_j)$, $\quad 1\leqq i\leqq r<j\leqq m$, $\quad$ and

(5.3) there is no factor $f_k(t)$, $m<k\leqq n$, such that $\mathbf{t}(f_{i_1})\leqq\mathbf{t}(f_k)\leqq\mathbf{t}(f_{i_2})$ or $\mathbf{t}(f_{j_1})\leqq\mathbf{t}(f_k)\leqq\mathbf{t}(f_{j_2})$, $1\leqq i_1, i_2\leqq r<j_1, j_2\leqq m$.

By Remark 4.5, it suffices to show that $(\tilde{h}_1(t), \tilde{h}_2(t))=(1)$.

If there is a factor $f_k(t)$, $m<k\leqq n$, such that $\mathbf{t}(f_k)$ is maximal in $\{\mathbf{t}(f_1), \cdots, \mathbf{t}(f_n)\}$, then, by Lemmas 4.7 and 5.5, the Λ-submodule $(f/f_k)A$ is a

maximal direct summand of type $\boldsymbol{t}(f_k)$ of A. Therefore the quotient Λ-module $A/(f/f_k)A$ $(\cong_\Lambda \Lambda/(f/f_k))$ is also completely decomposable. Since $\Delta(A/(f/f_k)A)=f/f_k=f_1(t)\cdots f_{k-1}(t)f_{k+1}(t)\cdots f_n(t)$, it follows from the inductive hypothesis that $(\tilde{h}_1(t), \tilde{h}_2(t))=(1)$.

Next suppose that $\boldsymbol{t}(f_k(t))$, $m<k\leqq n$, are not maximal. We will show that

$$\boldsymbol{t}(f_i)\not\leqq \boldsymbol{t}(f_k), \qquad 1\leqq i\leqq m<k\leqq n. \tag{5.4}$$

Suppose that $\boldsymbol{t}(f_i)\leqq\boldsymbol{t}(f_l)$ for some integers i and l, $1\leqq i\leqq m<l\leqq n$. Then, since $\boldsymbol{t}(f_k)$, $m<k\leqq n$, are not maximal, there is a factor $f_j(t)$ $(1\leqq j\leqq m)$ such that $\boldsymbol{t}(f_i)\leqq\boldsymbol{t}(f_l)\leqq\boldsymbol{t}(f_j)$. Hence, by (5.2), we see that

$$\text{either} \quad 1\leqq i, j\leqq r \quad \text{or} \quad r<i, j\leqq m. \tag{5.5}$$

However this contradicts (5.3). Thus we obtain (5.4). Therefore, by Lemmas 4.7 and 5.5, the Λ-submodule $(f_{m+1}\cdots f_n)A$ is a completely decomposable direct summand of A as an abelian group. Since $(f_{m+1}\cdots f_n)A\cong_\Lambda \Lambda/(f_1(t)\cdots f_m(t))=\Lambda/(\tilde{h}_1(t)\tilde{h}_2(t))$, it follows from (5.2) and Lemma 5.6 that $(\tilde{h}_1(t), \tilde{h}_2(t))=(1)$. Hence (2) holds. The proof is completed.

Using Corollary 5.2, we can prove Theorem 1.5:

PROOF OF THEOREM 1.5. To prove the theorem, we will give the following examples:

Let $g_1(t)=2t-3$, $g_2(t)=8t-7\in\Lambda$ and B_n, C_n be Λ-modules with $\Delta(B_n)=g_1(t)^n$, $\Delta(C_n)=g_2(t)^n$, where $n>0$. Then, since $g_1(t)^n$ and $g_2(t)^n$ satisfy (1.1), it follows that

$$B_n\cong_Z \bigoplus_{i=1}^{n} Z[1/\mu(g_1)]=\bigoplus_{i=1}^{n} Z[1/6], \qquad C_n\cong_Z \bigoplus_{i=1}^{n} Z[1/\mu(g_2)]=\bigoplus_{i=1}^{n} Z[1/14].$$

Therefore, a Λ-module $A_n=B_n\oplus C_n$ is Z-isomorphic to

$$\bigoplus_{i=1}^{n}(Z[1/6]\oplus Z[1/14])$$

and A_n is completely decomposable. On the other hand, the model M_n of A_n is given by

$$M_n=\Lambda/(g_1(t)^n g_2(t)^n).$$

To prove that A_n is not Z-isomorphic to M_n, it suffices to show that M_n is not completely decomposable. It is obvious that

$$\boldsymbol{t}(g_1)\not\geqq \boldsymbol{t}(g_2) \qquad \text{and} \qquad \boldsymbol{t}(g_1)\not\leqq \boldsymbol{t}(g_2).$$

Moreover, since $g_1(-1)^n=(-5)^n$ and $g_2(-1)^n=(-15)^n$, we see that

$$(g_1(t)^n, g_2(t)^n)\neq(1).$$

Therefore, from Corollary 5.2, the cyclic Λ-module M_n is not completely decomposable. This completes the proof.

Finally we raise the following more general question:

QUESTION 5.7. *For every Λ-module A, does there exist a Λ-module*

$$\Lambda/(f_1(t)) \oplus \cdots \oplus \Lambda/(f_n(t)),$$

where $f_1(t) \cdots f_n(t) = \Delta(A)$, that is Z-isomorphic to A?

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