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Mode-conversion of the scattering kernel for the elastic wave equation

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§ 0. Introduction.

Let Ω be an exterior domain in $\boldsymbol{R}^3$ with a smooth and compact boundary $\partial\Omega$. We consider the isotropic elastic wave equation

$$
(0.1)\qquad \begin{cases} Lu \equiv (A(\partial_x)-\partial_t^2)u = 0 & \text{in } \boldsymbol{R}\times\Omega, \\ u(t, x) = 0 & \text{on } \boldsymbol{R}\times\partial\Omega, \\ u(0, x) = f_0, \quad \partial_t u(0, x) = f_1 & \text{on } \Omega. \end{cases}
$$

Here $u={}^t(u_1, u_2, u_3)$ is the displacement vector and $A(\partial_x)$ is of the form

$$
A(\partial_x)u = \mu\Delta u+(\lambda+\mu)\operatorname{grad}(\operatorname{div} u).
$$

We assume that the Lamé constants λ and μ satisfy

$$
\lambda+\frac{2}{3}\mu > 0 \qquad \text{and} \qquad \mu > 0 .
$$

Then, as is shown in Yamamoto [**13**] and Shibata and Soga [**8**], we can develop the scattering theory for (0.1) in a similar way to that in Lax and Phillips [**6**]. Let $k_-(s, \omega)$ and $k_+(s, \omega)\in L^2(\boldsymbol{R}\times S^2)$ $(=\{L^2(\boldsymbol{R}\times S^2)\}^3)$ be the incoming and outgoing translation representations of the initial data $f=(f_0, f_1)$ respectively. The mapping $S: k_-\to k_+$ is called the scattering operator, which is a unitary operator from $L^2(\boldsymbol{R}\times S^2)$ to itself. The scattering operator is represented with a distribution kernel $S(s, \theta, \omega)$ called the scattering kernel:

$$
(Sk_-)(s, \theta) = \iint_{\boldsymbol{R}\times S^2} S(s-t, \theta, \omega)k_-(t, \omega)dtd\omega .
$$

Note that the scattering kernel $S(s, \theta, \omega)$ is a 3×3-matrix whose components are smooth functions in θ and ω with the value of the distribution in s. The purpose of the present paper is to study singularities of the scattering kernel $S(s, \theta, \omega)$.

The characteristic matrix $A(\xi)$ of the operator $A(\partial_x)$ has the eigenvalues $C_L^2|\xi|^2$ and $C_T^2|\xi|^2$, where

$$
C_L = (\lambda+2\mu)^{1/2}, \qquad C_T = \mu^{1/2} .
$$

Let $P_\alpha(\xi)$ be the eigen-projectors for the eigenvalues $C_\alpha|\xi|^2$ $(\alpha=L, T)$. Then $P_L(\xi)\boldsymbol{R}^3$ is the space spaned by ξ, and $P_T(\xi)\boldsymbol{R}^3$ is the orthogonal complement of $P_L(\xi)\boldsymbol{R}^3$. Associated with the eigenvalues $C_\alpha{}^2|\xi|^2$ $(\alpha=L, T)$, there are waves of two different types (modes). The one propagates with the speed C_L, and the other with C_T. Furthermore their amplitudes are longitudinal and transverse to the propagation direction repectively, and therefore these waves are called longitudinal and transverse waves respectively.

For elastic waves there is a conspicuous phenomenom called "mode-conversion", that is, when a longitudinal or transverse incident wave hits the boundary $\partial\Omega$, both a longitudinal reflected wave and a transverse reflected wave appear. In the half-space mode-conversion is well known (cf. Chapter 5 of Achenbach [**1**]). Soga [**12**] studies the same problem in general domains for anisotropic equations. He shows in terms of the asymptotic solutions that if $\partial\Omega$ is flat at points where the incident wave hits $\partial\Omega$ perpendicularly, mode-conversion does not occur on the reflection rays emanating from those points (cf. Theorem 2.1 of [**12**]).

In view of these results concerning mode-conversion, we can expect that the corresponding phenomenon occurs for the scattering kernel $S(s, \theta, \omega)$, because $P_\beta(\theta)S(C_\beta{}^{-1}\theta\cdot x-t, \theta, \omega)\, P_\alpha(\omega)$ is regarded as the C_β-mode wave scattered in the direction θ for the incident plane C_α-mode wave in the direction ω. By Soga [**11**] we obtain

$$\text{(0.2)} \quad \operatorname{supp} P_\beta(-\omega)S(\cdot, -\omega, \omega)P_\alpha(\omega) \subset (-\infty, -(r_\alpha(\omega)+r_\beta(\omega))] \quad (\alpha, \beta=L, T),$$

$$\text{(0.3)} \quad -2r_\alpha(\omega) \in \operatorname{sing\,supp} P_\alpha(-\omega)S(\cdot, -\omega, \omega)P_\alpha(\omega) \quad (\alpha=L, T),$$

where $r_\alpha(\omega)=C_\alpha{}^{-1}r(\omega)$ $(\alpha=L, T)$ and $r(\omega)=\min\{x\cdot\omega \mid x\in\boldsymbol{R}^3\setminus\Omega\}$. He treats anisotropic equations, but he does not analize $\operatorname{sing\,supp} P_\beta(-\omega)S(\cdot, -\omega, \omega)P_\alpha(\omega)$ in the case of $\alpha\neq\beta$. In this case there is a difficulty not encountered when $\alpha=\beta$. Thus our problem is

"to examine whether $P_\beta(-\omega)S(s, -\omega, \omega)P_\alpha(\omega)$ $(\alpha\neq\beta)$
is singular or not at $s=-(r_L(\omega)+r_T(\omega))$".

This means "to study mode-conversion in the sence of the scattering kernel".

In the present paper we restrict ourselves only in the case where the incident wave is longitudinal (i.e. $\alpha=L$ and $\beta=T$ in the above statement). We set

$$\text{(0.4)} \quad S_{L\to T}(s\,;\theta, \omega) = {}^t\theta P_T(-\omega)S(s, -\omega, \omega)P_L(\omega)\omega\,.$$

Roughly speaking, our conclusion is that the singularities of $S_{L\to T}(s\,;\theta, \omega)$ at $s=-(r_L(\omega)+r_T(\omega))$ depend on symmetry of $\partial\Omega$ near $\{x\in\partial\Omega \mid x\cdot\omega=r(\omega)\}$ (in detail, see § 1). Hence, in contrast with the fact that flatness of $\partial\Omega$ determines mode-conversion for the asymptotic solutions, symmetry of $\partial\Omega$ determines mode-conversion for the scattering kernel.

The proof of our results is based on the representation formula of the scattering kernel obtained by Soga [**10**]. By means of this formula, we represent $S_{L\to T}(s\,;\theta,\omega)$ in an integral form on $\partial\Omega$ with the reflected wave for the incident plane wave in the direction ω. Next, expressing the reflected wave in this integral form by the asymptotic solutions

$$u(t,\ x\,;\tau)\sim\sum_{\beta=L,T}\exp\{i\tau(t+C_\beta\varphi^\beta(x))\}\sum_{j=0}^{\infty}u_j^\beta(t,\ x)(i\tau)^{-j}, \tag{0.5}$$

we write the Fourier transform $(\alpha_\varepsilon(\cdot)S_{L\to T}(\cdot\,;\theta,\omega))\hat{}\,(-k)$ in an oscillatory integral form, and analyze its asymptotic behaviour as $|k|\to\infty$, where $\alpha_\varepsilon(s)$ is a cutoff function. In our case, however, the principal term of this oscillatory integral form is always degenerate because mode-conversion does not occur for the principal amplitude function u_0^β in (0.5), this is, the function u_0^β vanishes at points on $\partial\Omega$ where the incident wave hits $\partial\Omega$ perpendicularly. This is quite different from the case of Majda [7] and Soga [**9**, **11**], and causes the main difficulty in our proof. Hence, only by the methods in [7], [**9**] and [**11**], we cannot obtain informations of the singularities of $S_{L\to T}(s\,;\theta,\omega)$ at $s=-(r_L(\omega)+r_T(\omega))$. But, by integration by parts and other methods, we can make that integral form non-degenerate if $\partial\Omega$ does not have symmetry (cf. §1). On the other hand, if $\partial\Omega$ is symmetric, the integrand in the representation of $S_{L\to T}(s\,;\theta,\omega)$ becomes an odd function, and so $S_{L\to T}(s\,;\theta,\omega)$ itself vanishes near $s=-(r_L(\omega)+r_T(\omega))$ (cf. Theorem 1.3).

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§1. Main theorems.

We fix the progagation direction $\omega\in S^2$ of the incident wave, and set

$$M(\omega)=\{x\in\partial\Omega\mid x\cdot\omega=r(\omega)\}.$$

Let Ξ be the mean curvature of the boundary $\partial\Omega$, and denote by ∇' the gradient operator on $\partial\Omega$. The first main result is the following:

THEOREM 1.1. *Assume that*

(1.1) $\theta\cdot\nabla'\Xi\neq 0$ *on* $M(\omega)$ *and the sign of* $\theta\cdot\nabla'\Xi$ *does not change on* $M(\omega)$.

Then the distribution $S_{L\to T}(s\,;\theta,\omega)$ *is singular at* $s=-(r_L(\omega)+r_T(\omega))$.

We give a typical example which satisfies the condition (1.1) in Theorem 1.1:

EXAMPLE 1.2. Let Ω be the outside of the ellipsoid $\{x\in\boldsymbol{R}^3\mid(x_1/a_1)^2+$

$(x_2/a_2)^2+(x_3/a_3)^2>1\}$ with $a_1\neq a_2$, and $\omega={}^t(\omega_1, \omega_2, 0)\in S^2$. We set $e_1={}^t(-\omega_2, \omega_1, 0)$ and $e_2={}^t(0, 0, 1)$. Then we can easily check that the condition (1.1) is satisfied if ω and θ fulfil the following (1.2) and (1.2)′:

(1.2) $\qquad \omega\neq\pm{}^t(1, 0, 0) \quad$ or $\quad \omega\neq\pm{}^t(0, 1, 0)$,

(1.2)′ $\qquad \theta\cdot e_1\neq 0$.

In the above example if ω satisfies (1.2)′, the boundary $\partial\Omega$ near $M(\omega)$ is not symmetric with respect to the direction e_1. Thus we can understand that the condition (1.1) explains that $\partial\Omega$ near $M(\omega)$ is not symmetric.

On the other hand, if the boundary $\partial\Omega$ is symmetric near $M(\omega)$, then we can show that not only $S_{L\to T}(s; \theta, \omega)$ is smooth at $s=-(r_L(\omega)+r_T(\omega))$ but also $S_{L\to T}(s; \theta, \omega)$ vanishes near $s=-(r_L(\omega)+r_T(\omega))$. In the present paper, we say that the set W is symmetric in the direction ν, if the following condition is satisfied:

(1.3) There exist planes $\pi_1, \pi_2, \cdots, \pi_r$ normal to ν and disjoint open sets $U_1, U_2, \cdots, U_r$ in $\boldsymbol{R}^3$ covering W such that $W\cap U_j$ is symmetric with respect to π_j for each $j=1, 2, \cdots, r$.

Note that if $\partial\Omega$ is symmetric with respect to an axis parallel to ω, for any ν orthogonal to ω, $\partial\Omega$ is symmetric in the direction ν.

THEOREM 1.3. *If the intersection Ω and a neighborhood U of $M(\omega)$ is the symmetric in the direction ν orthogonal to ω (cf.* (1.3)), *then there exists a positive constant ε such that*

$$S_{L\to T}(s; \nu, \omega)=0 \qquad \textit{for any } s>-(r_L(\omega)+r_T(\omega))-\varepsilon\,.$$

REMARK. The ε in Theorem 1.3 is a constant such that $C_L\varepsilon$-neighborhood of the set $\{x\in\partial\Omega\cap U_j \mid x\cdot\omega<r(\omega)+C_T\varepsilon\}$ is contained in the open set U_j stated in (1.3) for each $j=1, 2, \cdots, r$.

From Theorem 1.3, we have the following corollary immediately:

COROLLARY. *If $\partial\Omega$ is symmetric with respect to an axis parallel to ω, then $S_{L\to T}(s; \theta, \omega)=0$ for any $s\in\boldsymbol{R}$ and any $\theta\in S^2$ normal to ω.*

We shall prove Theorem 1.3 in §2 and Theorem 1.1 in §§3, 4 and 5.

§2. Proof of Theorem 1.3.

Before proving Theorem 1.3, we review some results obtained in Soga [**10**]. Let $v^L(t, x; \omega)$ be the solution of the equation

$$(2.1)\qquad \begin{cases} Lv^{L}(t,\ x\ ;\omega)=0 & \text{in } \boldsymbol{R}\times\Omega\,, \\ v^{L}(t,\ x\ ;\omega)=-2^{-1}(-2\pi i)^{-2}C_{L}^{-3/2}\delta(t-C_{L}^{-1}x\cdot\omega)\omega & \text{on } \boldsymbol{R}\times\partial\Omega, \\ v^{L}(t,\ x\ ;\omega)=0 & \text{if } t \text{ is small enough.} \end{cases}$$

The solution $v^{L}(t, x ; \omega)$ is a smooth function of x and ω with the value of the distribution in t. By Theorem 2 in [**10**], we have

PROPOSITION 2.1. *Let N be the conormal derivative of the operator $A(\partial_{x})$, i.e.*

$$Nu=\lambda(\operatorname{div}u)n(x)+2\mu\frac{\partial}{\partial n}u+\mu n(x)\times(\operatorname{curl}u),$$

where $n(x)={}^{t}(n_{1}(x),\ n_{2}(x),\ n_{3}(x))$ is the unit outer normal to Ω at $x\in\partial\Omega$. Then we have

$$(2.2)\qquad S_{L\to T}(s\,;\,\theta,\,\omega)=C_{T}^{-3/2}\int_{\partial\Omega}{}^{t}\theta(\partial_{t}Nv^{L})(-C_{T}^{-1}x\cdot\omega-s,\ x\,;\,\omega)dS_{x}\,.$$

In the above proposition, the integral $\int\cdot dS_{x}$ is in the sence of the Riemann integral with the value of the distribution in t.

PROOF OF THEOREM 1.3. At first we deal with the particular case where the boundary $\partial\Omega$ is symmetric with respect to a plane π normal to ν. Let us note that the isotropicity of the equation yields the following equalities:

$$(2.3)\qquad A(\partial_{y})u\,|_{y=y(x)}=TA(\partial_{x})(({}^{t}Tu)(y(x)))\qquad \text{for any } x\in\Omega,$$

$$(2.4)\qquad (N_{y}u)(y(x))=TN_{x}(({}^{t}Tu)(y(x)))\qquad \text{for any } x\in\partial\Omega.$$

Here the transformation: $x\rightarrow y(x)$ means the reflection with respect to the plane π and T is the 3×3-orthogonal matrix representing the reflection with respect to the plane parallel to π containing the origin.

The equality (2.3) and the assumption for the boundary imply that $Tv^{L}(t, x(y) ; \omega)$ is also a solution of (2.1), where $x(y)$ is the inverse transformation for $y(x)$. Hence we obtain

$$(2.5)\qquad Tv^{L}(t,\ x(y)\,;\,\omega)=v^{L}(t,\ y\,;\,\omega)\qquad \text{for any } y\in\Omega,$$

because of the uniqueness of the solutions of (2.1). From (2.4) and (2.5) it followes that

$$N_{y}v^{L}(t,\ y\,;\,\omega)|_{y=y(x)}=TN_{x}v^{L}(t,\ x\,;\,\omega)\qquad \text{for any } x\in\partial\Omega.$$

Hence we get

$$(2.6)\qquad {}^{t}\nu(\partial_{t}N_{y}v^{L})(t,\ y\,;\,\omega)|_{y=y(x)}=-{}^{t}\nu(\partial_{t}N_{x}v^{L})(t,\ x\,;\,\omega)\qquad \text{for any } x\in\partial\Omega.$$

This equality shows that the integrand in (2.2) is an odd function on the boundary since we assume that $\partial\Omega$ is symmetric. This proves the theorem in the particular case. The idea in general case is essentially the same.

Now we give the proof in general case. The transformation: $x \to y_j(x)$ means the reflection with respect to the plane π_j for each $j=1, 2, \cdots, r$. Since the propagation speed is less than C_L, by the same methods as for the equality (2.6) we see that there exist a positive constant ε_1 and open sets U'_j with $\bar{U}'_j \subset U_j$ ($j=1, 2, \cdots, r$) such that

$$(2.7) \qquad {}^t\nu(\partial_t N_y v^L)(t, y; \omega)|_{y=y_j(x)} = -{}^t\nu(\partial_t N_x v^L)(t, x; \omega)$$
$$\text{for any } (t, x) \in I \times (\Omega \cap U'_j) \qquad (j=1, 2, \cdots, r)$$

where $I=(r_L(\omega)-\varepsilon_1, r_L(\omega)+\varepsilon_1)$.

We take $\alpha(s) \in C^\infty(\boldsymbol{R})$ such that $0 \leqq \alpha \leqq 1$ on $\boldsymbol{R}$, $\alpha(s)=1$, for $|s| \leqq 1/2$ and $\alpha(s)=0$ for $|s| \geqq 1$. For any $\varepsilon>0$ we set

$$\alpha_\varepsilon(s) = \alpha((s+r_L(\omega)+r_T(\omega))/2\varepsilon).$$

Note that $\operatorname{supp} \alpha_\varepsilon(-C_T^{-1}x\cdot\omega - s) \cap I \times (\partial\Omega \cap U_j) \subset I \times (\partial\Omega \cap U'_j)$ ($j=1, 2, \cdots, r$) if $\varepsilon>0$ is small enough. The equality (2.2) yields

$$(\alpha_\varepsilon(\cdot)S_{L\to T}(\cdot, \nu, \omega))^\wedge(-k)$$
$$= C_T^{-2/3} \sum_{j=1}^{r} \int_{\partial\Omega \cap U_j} \langle {}^t\nu(\partial_t N v^L)(s, x; \omega), \alpha_\varepsilon(-C_T^{-1}x\cdot\omega - s)\exp(-ik(C_T^{-1}x\cdot\omega+s))\rangle dS_x$$

for small $\varepsilon>0$, where $\hat{f}(k)=\int \exp(-ik\cdot s)f(s)ds$ and $\langle\, ,\rangle$ means the pairing of the distribution in s and their test functions. Representing these surface integrals by the local coordinates, we see from (2.7) and the assumption in the theorem that each integrand is an odd functions in the local coordinates. Hence we obtain $\alpha_\varepsilon(s)S_{L\to T}(s; \nu, \omega)=0$, which proves Theorem 1.3.

§3. The Fourier transform of the scattering kernel.

In this section, by the same procedures as in Majda [7] and Soga [9] we derive the precise form of the Fourier transform of $\alpha_\varepsilon(s)S_{L\to T}(s; \theta, \omega)$ in s (see Proposition 3.2). For this purpose we construct an approximate solution of (2.1), and represent the Neumann operator by a classical pseudo-differential operator on $\boldsymbol{R}\times\partial\Omega$, which gives informations about the singularities of $Nv^L|_{\boldsymbol{R}\times\partial\Omega}$.

We take an orthonormal frame $\{\omega, e_1, e_2\}$ in $\boldsymbol{R}^3$, and choose the local coordinate system $x(\sigma)$ of $\partial\Omega$ near $M(\omega)$ in the following way:

$$U_0 \ni \sigma = (\sigma_1, \sigma_2) \longrightarrow x(\sigma) = \sigma_1 e_1 + \sigma_2 e_2 + g(\sigma)\omega \in \partial\Omega,$$

where U_0 is an open set in $\boldsymbol{R}^2$ and g is a smooth function on $\bar{U}_0$. We set $M_0(\omega)=\{\sigma \in U_0 \mid x(\sigma) \in M(\omega)\}$. Note that each element of $M_0(\omega)$ is a stationary point of g. We construct the approximate solution for $v^L(t, x; \omega)$ modulo smooth function near the boundary by means of the asymptotic solutions

$$u(t, x; \tau, \xi') \sim \exp\{i\tau(t+\varphi^L(x, \xi'))\} \sum_{l=0}^{\infty} u_l^L(t, x; \xi')(i\tau)^{-l} + \exp\{i\tau(t+\varphi^T(x, \xi'))\} \sum_{l=0}^{\infty} u_l^T(t, x; \xi')(i\tau)^{-l} \tag{3.1}$$

satisfying asymptotically the equation

$$\begin{cases} Lu = 0 & \text{in } \boldsymbol{R}\times(\Omega\cap V), \\ u = \exp\{i\tau(t+\xi'\cdot\sigma(x))\}I & \text{on } \boldsymbol{R}\times(\partial\Omega\cap V), \\ u \text{ is outgoing,} \end{cases} \tag{3.2}$$

where $\sigma(x)$ $(x\in\partial\Omega)$ is the inverse of $x(\sigma)$, V is an open set in $\boldsymbol{R}^3$ such that $V\cap\partial\Omega=\{x(\sigma)\in\partial\Omega \mid \sigma\in U_0\}$ and I is the 3×3-unit matrix. Karal and Keller [3] constructed asymptotic solutions of the form (3.1) for the Cauchy problem of the isotropic elastic wave equation (cf. §2 of [3]). Our construction is their modification (for the detailed procedures e.g. cf. §2 of Kawashita [4]).

Let $u_{l:j}^\alpha(t, x; \xi')$ be the j-th columm vector of u_l^α. We can determine the phase functions φ^α and the amplitudes $u_{l:j}^\alpha$ by the methods in [3] or [4]; φ^α and $u_{l:j}^\alpha$ $(j=1, 2, 3, \alpha=L, T, l=0, 1, 2, \cdots)$ satisfy the following equations:

$$\begin{cases} (\nabla\varphi^\alpha)^2 = 1/C_\alpha^2 & \text{in } \Omega\cap V, \\ \varphi^\alpha = \xi'\cdot\sigma(x) & \text{on } \partial\Omega\cap V, \\ \dfrac{\partial}{\partial n}\varphi^\alpha > 0 & \text{on } \partial\Omega\cap V, \quad (\alpha=L, T) \end{cases} \tag{3.3.α}$$

$$\begin{cases} H^L u_{l:j}^L + \square_L u_{l-1:j}^L = 0 & \text{in } \boldsymbol{R}\times(\Omega\cap V), \\ \nabla\varphi^L\times u_{l:j}^L + \operatorname{curl} u_{l-1:j}^L = 0 & \text{in } \boldsymbol{R}\times(\Omega\cap V), \end{cases} \tag{3.4.l}$$

$$\begin{cases} H^T u_{l:j}^T + \square_T u_{l-1:j}^T = 0 & \text{in } \boldsymbol{R}\times(\Omega\cap V), \\ \nabla\varphi^T\cdot u_{l:j}^T + \operatorname{div} u_{l-1:j}^T = 0 & \text{in } \boldsymbol{R}\times(\Omega\cap V), \end{cases} \tag{3.5.l}$$

$$\begin{cases} u_0^L = (\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}\nabla\varphi^L\otimes\nabla\varphi^T & \text{on } \boldsymbol{R}\times(\partial\Omega\cap V), \\ u_0^T = I - u_0^L & \text{on } \boldsymbol{R}\times(\partial\Omega\cap V), \end{cases} \tag{3.6.0}$$

and for $l\geqq1$

$$\begin{cases} u_{l:j}^L = (\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}(h_{l:j}, \nabla\varphi^T)\nabla\varphi^L + z_{l:j}^L & \text{on } \boldsymbol{R}\times(\partial\Omega\cap V), \\ u_{l:j}^T = -u_{l:j}^L & \text{on } \boldsymbol{R}\times(\partial\Omega\cap V), \end{cases} \tag{3.6.l}$$

where $H^\beta=2C_\beta^2\nabla\varphi^\beta\cdot\nabla-2\partial_t+C_\beta^2\Delta\varphi^\beta$, $\square_\beta=C_\beta^2\Delta-(\partial_t)^2$, $u_{-1}^\alpha=u_{-2}^\alpha=0$ and

$$\begin{cases} z_{l:j}^L = C_L^2\nabla\varphi^L\times\operatorname{curl} u_{l-1:j}^L & \text{on } \boldsymbol{R}\times(\partial\Omega\cap V), \\ h_{l:j} = -z_{l:j}^L + C_T^2(\operatorname{div} u_{l-1:j}^T)\nabla\varphi^T & \text{on } \boldsymbol{R}\times(\partial\Omega\cap V), \end{cases} \tag{3.7.l}$$

$$j=1, 2, 3, \quad l=0, 1, 2, \cdots.$$

In the above expressions we have used the notations

$$a\otimes b=\left(a_ib_j\Big|\begin{matrix} i\downarrow 1,2,3\\ j\rightarrow 1,2,3\end{matrix}\right)\qquad\text{and}\qquad (a,b)=a\cdot b=\sum_{i=1}^{3}a_ib_i$$

for $a={}^t(a_1, a_2, a_3)$, $b={}^t(b_1, b_2, b_3)\in \boldsymbol{R}^3$. Remark that the amplitudes u_l^α are independent of the variable t since the amplitude of the boundary data is independent of t. Hence hereafter we abbreviate the variable t in u_l^α.

We take cutoff functions $\chi(\sigma)\in C_0^\infty(U_0)$ and $\chi_1(\sigma, \xi')\in C_0^\infty(U_0\times\boldsymbol{R}^2)$ satisfying

(i) $\chi(\sigma)=1$ near $M_0(\omega)$,

(ii) $\operatorname{supp}\chi_1$ is contained in a small neighborhood of $M_0(\omega)\times\{0\}\subset U_0\times\boldsymbol{R}^2$,

(iii) $\chi_1(\sigma, \xi')=1$ if $\sigma\in\operatorname{supp}\chi$ and $|\xi'|$ is small enough.

Set $\phi(t, x)=-2^{-1}(-2\pi i)^{-2}C_L^{-2/3}\delta(t-C_L^{-1}x\cdot\omega)$. Then the wave front set of $\phi(t, x)$ is non-glancing in $\{(t, x)\in\boldsymbol{R}\times\partial\Omega \mid -2r_L(\omega)-\eta<-C_L^{-1}x\cdot\omega-t\}$ if $\eta>0$ is small enough (cf. Soga [9] §2). This fact yields that

$$\chi(\sigma)\phi(t, x(\sigma))$$
$$\equiv(2\pi)^{-3}\iint_{\boldsymbol{R}\times\boldsymbol{R}^2}\exp(i\tau(t+\xi'\cdot\sigma))\chi_1(\sigma, \xi')I\tau^2(\chi\phi)\hat{\ }(\tau, \tau\xi')\omega d\tau d\xi' \qquad \operatorname{mod} C^\infty.$$

Hence, using the asymptotic solution (3.1), we obtain

$$(3.8)\qquad v^L(t, x;\omega)\equiv(2\pi)^{-3}\iint_{\boldsymbol{R}\times\boldsymbol{R}^2}u(t, x;\tau, \xi')\chi_1(\sigma, \xi')\tau^2(\chi\phi)\hat{\ }(\tau, \tau\xi')\omega d\tau d\xi'$$
$$\operatorname{mod} C^\infty \text{ in } \{(t, x)\in\boldsymbol{R}\times\partial\Omega \mid -2r_L(\omega)-\varepsilon_1<-C_L^{-1}x\cdot\omega-t,\ \operatorname{dist}(x, \partial\Omega)<\varepsilon'\}$$

for some small $\varepsilon_1>0$ and $\varepsilon>0$. By this approximate solution of $v^L(t, x;\omega)$, we get a representation of the Neumann operator:

LEMMA 3.1. *There exist a first order pseudo-differential operator B on $\boldsymbol{R}\times\partial\Omega$ independent of t and posessing the following properties* (i)~(iv):

(i) $(Nv^L)(t, x;\omega)|_{\boldsymbol{R}\times\partial\Omega}\equiv B(\phi(t, x)\omega) \operatorname{mod} C^\infty$ *in* $\{(t, x)\in\boldsymbol{R}\times\partial\Omega \mid -2r_L(\omega)-\varepsilon_1<-C_L^{-1}x\cdot\omega-t\}$ *for some small constant* $\varepsilon_1>0$.

(ii) $B\equiv 0 \operatorname{mod} C^\infty$ *on* $\boldsymbol{R}\times\partial\Omega\setminus\boldsymbol{R}\times V'$, *where V' is an open set on $\partial\Omega$ satisfying $M(\omega)\subset V'\subset\subset V$.*

(iii) *The symbol $B(\sigma, \tau, \xi)$ of B in the local coordinates σ has an homogeneous asymptotic expansion $B(\sigma, \tau, \xi)\sim\sum_{l=0}^\infty B_l(\sigma, \tau, \xi)$ such that $B_l(\sigma, \tau, \xi)$ are purely imaginary-valued for even l and real-valued for odd l.*

(iv) *The symbols $B_l(\sigma, \tau, \xi)$ in* (iii) *are represented of the forms*

$$B_l(\sigma, \tau, \xi)=(i\tau)^{1-l}\chi_1(\sigma, \xi/\tau)\Psi_l(\sigma, \xi/\tau)+\tilde{N}(\chi_1(\sigma, \xi/\tau)I)\delta_{1l}\quad (l=0, 1, 2, \cdots),$$

where $\tilde{N}$ is the part of N tangential to the boundary and Ψ_l are of the forms

$$(3.9)\quad \Psi_0(\sigma, \xi')=\mu\left(\frac{\partial\varphi^T}{\partial n}\right)I+\mu(\nabla\varphi^T\otimes n)+(\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}\Big\{(\lambda/C_L^2)(n\otimes\nabla\varphi^T)$$
$$+2\mu\left(\frac{\partial\varphi^L}{\partial n}\right)(\nabla\varphi^L\otimes\nabla\varphi^T)-\mu\left(\frac{\partial\varphi^T}{\partial n}\right)(\nabla\varphi^L\otimes\nabla\varphi^T)-\mu\left(\frac{\partial\varphi^L}{\partial n}\right)(\nabla\varphi^T\otimes\nabla\varphi^T)\Big\},$$

$$\Psi_l(\sigma, \xi') = (\mu I + (\lambda+\mu) n \otimes n)\Big\{\Big(\frac{\partial \varphi^L}{\partial n} - \frac{\partial \varphi^T}{\partial n}\Big) u_l^L + \frac{\partial}{\partial n}(u_{l-1}^L + u_{l-1}^T)\Big\} \quad (l \geqq 1). \tag{3.10}$$

For a 3×3-matrix valued function $u=(u_1, u_2, u_3)$ a 3×3-matrix Nu is defind as $Nu=(Nu_1, Nu_2, Nu_3)$.

PROOF OF LEMMA 3.1. Applying N to (3.8), we have

$$(Nv^L)|_{R\times\partial\Omega} \equiv (2\pi)^{-3}\iint_{R\times R^2} \exp(i\tau(t+\xi'\cdot\sigma) B(\sigma, \tau, \tau\xi')\tau^2(\chi\phi)\hat{}(\tau, \tau\xi')\omega d\tau d\xi'$$

$$\bmod C^\infty \quad \text{in } \{(t, x) \in \boldsymbol{R}\times\partial\Omega \mid -2r_L(\omega)-\varepsilon_1 < -C_L^{-1}x\cdot\omega - t\},$$

where $B(\sigma, \tau, \xi) \sim \sum_{l=1}^{\infty} B_l(\sigma, \tau, \xi)$ and

$$B_l(\sigma, \tau, \xi) = (i\tau)^{1-l}\Big[\chi_1(\sigma, \xi')\sum_{\beta=L,T}\Big\{\lambda(\nabla\varphi^\beta, u_l^\beta)n$$

$$+\mu\Big(\frac{\partial\varphi^\beta}{\partial n}\Big)u_l^\beta + \mu(n, u_l^\beta)\nabla\varphi^\beta\Big\} + \sum_{\beta=L,T} N(\chi_1(\sigma, \xi')u_{l-1}^\beta)\Big]\Big|_{\xi'=\xi/\tau}.$$

In the above equality we have used the notation

$$(a, u_l^\beta)b = \Big((a\cdot u_{l:j}^\beta)b_i\Big|_{j\to 1,2,3}^{i\downarrow 1,2,3}\Big) \qquad (a, b \in \boldsymbol{R}^3).$$

Hence (i) and (ii) are evident. Since every component of u_l^β is real-valued, we obtain (iii). The equality (3.9) also follows since (3.6.0) yields that

$$(\nabla\varphi^L, u_0^L)n = C_L^{-2}(\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}(n\otimes\nabla\varphi^T), \qquad (\nabla\varphi^T, u_0^T)n = 0,$$

$$(n, u_0^L)\nabla\varphi^L = (\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}\Big(\frac{\partial\varphi^L}{\partial n}\Big)\nabla\varphi^L\otimes\nabla\varphi^T,$$

$$(n, u_0^T)\nabla\varphi^T = \nabla\varphi^T\otimes n - (\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}\Big(\frac{\partial\varphi^L}{\partial n}\Big)\nabla\varphi^T\otimes\nabla\varphi^T.$$

Finally let us show (3.10) for $l\geqq 1$. Since $\nabla\varphi^L - \nabla\varphi^T = (\partial\varphi^L/\partial n - \partial\varphi^T/\partial n)n$ and $(n, u_l^L)n = (n\otimes n)u_l^L$ on $\partial\Omega$, it follows from (3.6.l) that

$$B_l(\sigma, \tau, \xi) = (i\tau)^{1-l}\Big[\chi_1(\sigma, \xi')\Big(\frac{\partial\varphi^L}{\partial n} - \frac{\partial\varphi^T}{\partial n}\Big)(\mu I + (\lambda+\mu)n\otimes n)u_l^L$$

$$+N(\chi_1(\sigma, \xi')(u_{l-1}^L + u_{l-1}^T))\Big]\Big|_{\xi'=\xi/\tau}.$$

Noting that $N=(\mu I+(\lambda+\mu)n\otimes n)\partial/\partial n + \tilde{N}$ and (3.6.0) we obtain

$$N(\chi_1(\sigma, \xi')(u_{l-1}^L + u_{l-1}^T))$$

$$= \chi_1(\sigma, \xi')(\mu I + (\lambda+\mu)n\otimes n)\frac{\partial}{\partial n}(u_{l-1}^L + u_{l-1}^T) + \tilde{N}(\chi_1(\sigma, \xi')I)\delta_{1l}.$$

Hence we get (3.10), which completes the proof of Lemma 3.1.

Now we take the Fourier transform of $\alpha_\varepsilon(s)S_{L\to T}(s; \theta, \omega)$. Taking the constant ε so that $0<\varepsilon<\varepsilon_1/2$, by Lemma 3.1 and Proposition 2.1 we obtain

$$(\alpha_\varepsilon(\cdot)S_{L\to T}(\cdot\,;\theta,\omega))^\wedge=-2^{-1}(-2\pi i)^{-2}(C_LC_T)^{-3/2}\sum_{j=0}^{1}(ik)^{1-j}$$

$$\int_{\partial\Omega}\left\langle B[\delta(s-C_L^{-1}x\cdot\omega)\chi(\sigma)\omega],\left(\left(\frac{d}{ds}\right)^j\alpha_\varepsilon\right)(-C_T^{-1}x\cdot\omega-s)\exp(-ik(C_T^{-1}x\cdot\omega+s))\theta\right\rangle dS_x$$

$$=-2^{-1}(-2\pi i)^{-2}(C_LC_T)^{-3/2}\sum_{j=0}^{1}(ik)^{1-j}\int_{R^2}(\chi(\sigma)\omega,\,{}^tB\Big[\left(\left(\frac{d}{ds}\right)^j\alpha_\varepsilon\right)(-C_T^{-1}g(\sigma)-s)$$

$$\cdot\exp\{-ik(C_T^{-1}g(\sigma)+s)\}\rho(\sigma)\chi_2(\sigma)\theta\Big])\Big|_{s=C_L^{-1}g(\sigma)}d\sigma\,.$$

where $\rho(\sigma)=(1+|\nabla_\sigma g|^2)^{-1/2}$, $\chi_2\in C_0^\infty(U_0)$ satisfying $\chi_2=1$ on $(\operatorname{supp}\chi)\cup(\cup\{\operatorname{supp}\chi_1(\cdot,\xi')\,|\,\xi'\in\boldsymbol{R}^2\})$ and ${}^tB$ is the transposed operator of B. The symbol of ${}^tB$ has a homogeneous asymptotic expansion $\sum_{l=1}^\infty\tilde{B}_l(\sigma,\tau,\xi)$ such that each component of $\tilde{B}_l(\sigma,\tau,\xi)$ are purely imaginary-valued for even l and real-valued for odd l. Expanding

$${}^tB\Big[\left(\left(\frac{d}{ds}\right)^j\alpha_\varepsilon\right)(-C_T^{-1}g(\sigma)-s)\rho(\sigma)\chi_2(\sigma)\theta\exp(-ik(C_T^{-1}g(\sigma)+s))\Big]\quad\text{as } |k|\to\infty$$

(see Kumano-go [5]), we obtain

PROPOSITION 3.2. *For any positive integer $m\geqq 0$ we have*

$$(3.11)\quad(\alpha_\varepsilon(\cdot)S_{L\to T}(\cdot\,;\theta,\omega))^\wedge(-k)=-2^{-1}(-2\pi i)^{-2}(C_LC_T)^{-3/2}\sum_{j=0}^{m}(ik)^{2-j}$$

$$\int_{R^2}\exp(-ik(C_L^{-1}+C_T^{-1})g(\sigma))\,\beta_j(\sigma)d\sigma+O(|k|^{-m+1}),$$

where $\beta_j\in C_0^\infty(U_0)$ are real-valued functions and $\beta_j(\sigma)$ $(j=0,1,2)$ are of the following forms:

$$\beta_0(\sigma)=i^{-1}\alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\chi(\sigma)\rho(\sigma)(\omega,\tilde{B}_0(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))\theta),$$

$$\beta_1(\sigma)=\alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\chi(\sigma)\rho(\sigma)(\omega,\tilde{B}_1(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))\theta)$$

$$+i^{-1}\chi(\sigma)\Big\{\sum_{|\alpha|=1}(\omega,(\partial^\alpha_{(\tau,\xi)}\tilde{B}_0)(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))$$

$$\cdot\partial^\alpha_{(s,\sigma)}(\alpha_\varepsilon(-C_T^{-1}g(\sigma)-s)\rho(\sigma)\chi_2(\sigma))\theta)$$

$$+\sum_{|\alpha|=2}\frac{1}{\alpha!}(\omega,(\partial^\alpha_{(\tau,\xi)}\tilde{B}_0)(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))(\partial^\alpha_{(s,\sigma)}(-C_T^{-1}g(\sigma)-s))$$

$$\cdot\alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\rho(\sigma)\chi_2(\sigma)\theta)\Big\}\Big|_{s=C_L^{-1}g(\sigma)},$$

$$\beta_2(\sigma)=i\alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\chi(\sigma)\rho(\sigma)(\omega,\tilde{B}_2(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))\theta)$$

$$+\chi(\sigma)\Big\{\sum_{|\alpha|=1}(\omega,(\partial^\alpha_{(\tau,\xi)}\tilde{B}_1)(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))$$

$$\cdot\partial^\alpha_{(s,\sigma)}(\alpha_\varepsilon(-C_T^{-1}g(\sigma)-s)\rho(\sigma)\chi_2(\sigma))\theta)$$

$$+\sum_{|\alpha|=2}\frac{1}{\alpha!}(\omega,(\partial_{(\tau,\xi)}^{\alpha}\tilde{B}_1)(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))(\partial_{(s,\sigma)}^{\alpha}(-C_T^{-1}g(\sigma)-s))$$

$$\cdot\alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\rho(\sigma)\chi_2(\sigma)\theta)\Big\}\Big|_{s=C_L^{-1}g(\sigma)}$$

$$+i^{-1}\chi(\sigma)\Big\{\sum_{|\alpha|=2}\frac{1}{\alpha!}(\omega,(\partial_{(\tau,\xi)}^{\alpha}\tilde{B}_0)(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))$$

$$\cdot\partial_{(s,\sigma)}^{\alpha}(\alpha_\varepsilon(-C_T^{-1}g(\sigma)-s)\rho(\sigma)\chi_2(\sigma))\theta)$$

$$+\sum_{|\alpha|=3}\frac{1}{\alpha!}(\omega,(\partial_{(\tau,\xi)}^{\alpha}\tilde{B}_0)(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))$$

$$\cdot\sum_{\substack{0\leq\beta\leq\alpha\\ 2\leq|\beta|}}\binom{\alpha}{\beta}(\partial_{(s,\sigma)}^{\beta}(-C_T^{-1}g(\sigma)-s))\partial_{(s,\sigma)}^{\alpha-\beta}(\alpha_\varepsilon(-C_T^{-1}g(\sigma)-s)\rho(\sigma)\chi_2(\sigma))\theta)$$

$$+2^{-1}\sum_{|\alpha|=4}\frac{1}{\alpha!}(\omega,(\partial_{(\tau,\xi)}^{\alpha}\tilde{B}_0)(\sigma,-1,-C_T^{-1}(\nabla_\sigma g)(\sigma))$$

$$\cdot\sum_{\substack{0\leq\beta\leq\alpha\\ |\beta|=2}}(\partial_{(s,\sigma)}^{\beta}(-C_T^{-1}g(\sigma)-s))(\partial_{(s,\sigma)}^{\alpha-\beta}(-C_T^{-1}g(\sigma)-s))$$

$$\cdot\alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\rho(\sigma)\chi_2(\sigma)\theta)\Big\}\Big|_{s=C_L^{-1}g(\sigma)}.$$

We note that

$$\tilde{B}_0(\sigma,\tau,\xi)={}^tB_0(\sigma,-\tau,-\xi),$$

$$\tilde{B}_1(\sigma,\tau,\xi)=\sum_{|\alpha|=1}{}^t(D_\sigma^\alpha\partial_\xi^\alpha)B_0(\sigma,-\tau,-\xi)+{}^tB_1(\sigma,-\tau,-\xi),$$

$$\tilde{B}_2(\sigma,\tau,\xi)=\sum_{|\alpha|=2}\frac{1}{\alpha!}{}^t(D_\sigma^\alpha\partial_\xi^\alpha)B_0(\sigma,-\tau,-\xi)$$

$$+\sum_{|\alpha|=1}{}^t(D_\sigma^\alpha\partial_\xi^\alpha)B_1(\sigma,-\tau,-\xi)+{}^tB_2(\sigma,-\tau,-\xi),$$

where $D_\sigma^\alpha=((1/i)\partial_\sigma)^\alpha$.

It cannot be derived immediately from Proposition 3.2 that $S_{L\to T}(s;\theta,\omega)$ is singular at $s=-(r_L(\omega)+r_T(\omega))$, because in our case $\beta_0(\sigma)=0$ at the stationary points of g. This is different from the scalor-valued wave equation. But, as we shall discuss in the next section, by integration by parts we can represent $(\alpha_\varepsilon(\cdot)S_{L\to T}(\cdot;\theta,\omega))\hat{}(-k)$ by an oscillatory integral with a non-degenerate amplitude function if the assumption (1.1) is satisfied. To realize this we need to calculate precisely not only β_0 but also β_1 and β_2 in Proposition 3.2.

§ 4. Non-degenerate representation of the Fourier transform of the scattering kernel.

The goal of this section is to show the following theorem, which proves Theorem 1.1.

THEOREM 4.1. *For any positive integer $m \geqq 0$ we have*

$$(4.1) \qquad (\alpha_\varepsilon(\cdot)S_{L\to T}(\cdot,\ \theta,\ \omega))\hat{}(-k) = -2^{-1}(-2\pi i)^{-2}\sum_{l=0}^{m}(ik)^{-l} \cdot\int_{R^2}\exp(-ik(C_L^{-1}+C_T^{-1}))g(\sigma)\gamma_l(\sigma)d\sigma + O(|k|^{-m-1}) \qquad as\ |k|\to\infty\,.$$

Here $\gamma_l \in C_0^\infty(U_0)$ are real-valued functions, and

$$\gamma_0(\sigma) = a(C_L/C_T)\alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\chi(\sigma)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}\Delta_\sigma g)(\sigma),$$

where $a(x) = x^{-2/3}(1+x)^{-2}(2x^5-5x^4+x^3+5x^2+3x-2)$, $\theta_l = \theta\cdot e_l$ *and* $\Delta_\sigma = \sum_{j=1}^{2}(\partial/\partial\sigma_j)^2$.

We remark that $a(x)>0$ if $x\geqq 1$; in particular $a(C_L/C_T)>0$. Hence γ_0 in Theorem 4.1 does not vanish at the stationary points of g since $(\theta\cdot\nabla' H)(\sigma) = \sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}\Delta_\sigma g)(\sigma)$. Thus we can apply Theorem 2 in Soga [**9**] and obtain Theorem 1.1.

We set

$$I_l(k) = \int_{R^2}\exp\{-ik(C_L^{-1}+C_T^{-1})g(\sigma)\}\beta_l(\sigma)d\sigma \qquad (l=0,\ 1,\ 2),$$

$$A_\varepsilon(\sigma) = \alpha_\varepsilon(-(C_L^{-1}+C_T^{-1})g(\sigma))\chi(\sigma),$$

$$J(k) = \int_{R^2}\exp\{-ik(C_L^{-1}+C_T^{-1})g(\sigma)\}A_\varepsilon(\sigma)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}\Delta_\sigma g)(\sigma)d\sigma\,.$$

The main part of the proof of Theorem 4.1 is to reperesent $(ik)^{2-l}I_l(k)$ $(l=0, 1, 2)$ in the oscillatory integral form similar to (4.1) in Theorem 4.1. For convenience we introduce the following notations:

(i) For f, $\tilde{f}\in C_0^\infty(U_0)$, we write $f \doteqdot \tilde{f}$ if f is equal to $\tilde{f}$ in some neighborhood of $M_0(\omega)$ (which is the set defined in § 3).

(ii) For a positive integer l we denote by $[\nabla_\sigma g]_0^l$ (resp. $[\nabla_\sigma g]^l$) the sum of the form

$$\sum_{|\alpha|=l} h_\alpha(\sigma)\xi^\alpha|_{\xi=\nabla_\sigma g}\,,$$

where h_α are real-valued functions $\in C_0^\infty(U_0)$ (resp. $\in C^\infty(U_0)$).

(iii) Let $I(k)$ and $\tilde{I}(k)$ are functions bounded in k $(|k|\geqq 1)$. Then we write $I(k)\cong\tilde{I}(k)$ if there exist real-valued functions $f_j\in C_0^\infty(U_0)$ $(j=0, 1, 2, \cdots)$ such that for any integer $h\geqq 1$ we have

$$I(k)-\tilde{I}(k) = \sum_{j=1}^{h}(ik)^{-j}\int_{R^2}\exp\{-ik(C_L^{-1}+C_T^{-1})\}g(\sigma)f_j(\sigma)d\sigma + O(|k|^{-h-1}) \qquad \text{as } |k|\to\infty\,.$$

To prove Theorem 4.1, we have only to show that

$$(4.2.j) \qquad (ik)^{2-j}I_j(k) \cong \kappa_j J(k) \qquad (j=0,\ 1,\ 2)$$

where $\kappa_0=-2C_L C_T^{-2}(C_L-C_T)^2(C_L^{-1}+C_T^{-1})^{-2}$, $\kappa_1=(C_L^{-1}+C_T^{-1})\cdot\{(C_L C_T^{-1}-1)(C_L^2-C_T^2)+2C_T^2(C_L C_T^{-1}-2)\}$, $\kappa_2=-(C_L-C_T)(C_L^2-2C_L C_T+2C_T^2)$. In fact, noting (3.11) and (4.2.j) ($j=0, 1, 2$), we have (4.1) by a routine calculation. At first we prove (4.2.0). To get (4.2.0) we rewrite $\beta_0(\sigma)$:

LEMMA 4.2. *The first order amplitude function $\beta_0(\sigma)$ is of the form*

$$\beta_0(\sigma)=-C_T\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}g)A_\varepsilon(\sigma)(2f(|\nabla_\sigma g|^2)-3),$$

where

$$f(x)=(1+x)/(x+\{((C_T C_L^{-1})^2-1)x+(C_T C_L^{-1})^2\}^{1/2}).$$

We remark that $f(x)$ is not smooth for large $x\in\boldsymbol{R}$. But in our case $|\nabla_\sigma g|^2$ is so small on $\bar{U}_0$ that $f(|\nabla_\sigma g|^2)$ is smooth function on $\bar{U}_0$.

PROOF OF LEMMA 4.2. Equalities (3.3.L), (3.3.T) and (3.9) imply that

$$i^{-1}(B_0(\sigma,\tau,\tau\xi')\omega,\theta)=\tau\chi_1(\sigma,\xi')\Big[\mu(n\cdot\omega)(\theta\cdot\nabla\varphi^T)$$
$$+\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big\{\frac{\lambda}{C_L^2}n\cdot\theta+2\mu\Big(\frac{\partial\varphi^L}{\partial n}\Big)(\theta\cdot\nabla\varphi^L)-\mu\Big(\frac{\partial\varphi^T}{\partial n}\Big)(\theta\cdot\nabla\varphi^L)-\Big(\frac{\partial\varphi^L}{\partial n}\Big)(\theta\cdot\nabla\varphi^T)\Big\}\Big].$$

Noting that $\nabla\varphi^L-\nabla\varphi^T=(\partial\varphi^L/\partial n-\partial\varphi^T/\partial n)n$, and that $\nabla\varphi^L\cdot\nabla\varphi^T=C_L^{-2}+(\partial\varphi^T/\partial n-\partial\varphi^L/\partial n)(\partial\varphi^L/\partial n)=C_T^{-2}+(\partial\varphi^L/\varphi n-\partial\varphi^T/\partial n)(\partial\varphi^T/\partial n)$, we see that

$$\text{(4.3)}\qquad i^{-1}(B_0(\sigma,\tau,\tau\xi')\omega,\theta)=\tau\chi_1(\sigma,\xi')\Big[\mu(n(\sigma)\cdot\omega)(\theta\cdot\nabla\varphi^T)$$
$$+\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\cdot\Big\{\frac{\lambda}{C_L^2}n\cdot\theta+\mu\Big(\frac{\partial\varphi^L}{\partial n}\Big)\theta\cdot(\nabla\varphi^L-\nabla\varphi^T)$$
$$+\mu\Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)\Big(\theta\cdot\nabla\varphi^T+\Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)n\cdot\theta\Big)\Big\}\Big]$$
$$=\tau\chi_1(\sigma,\xi')(n\cdot\omega)(\theta\cdot\nabla\varphi^T)+\mu\Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)(\theta\cdot\nabla\varphi^T)\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}$$
$$+(\omega\cdot\nabla\varphi^T)(n\cdot\theta)(2(\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}-3\mu).$$

Furthermore, we have

$$(\nabla\varphi^T)(x(\sigma),C_T^{-1}\nabla_\sigma g)=C_T^{-1}\omega,\quad(\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}(x(\sigma),C_T^{-1}\nabla_\sigma g)=C_T^2 f(|\nabla_\sigma g|^2)$$

by means of the following equalities:

$$\text{(4.4)}\qquad\left\{\begin{aligned}&(\nabla\varphi^\beta)(\sigma,\xi')=\sum_{j,l=1}^{2}g^{jl}(\sigma)\xi_j'(\partial_{\sigma_l}x)(\sigma)+\Big(\frac{\partial\varphi^\beta}{\partial n}\Big)(\sigma,\xi')n(\sigma)\\&\Big(\frac{\partial\varphi}{\partial n}\Big)(\sigma,\xi')=\Big(C_\beta^{-2}-\sum_{j,l=1}^{2}g^{jl}(\sigma)\xi_j'\xi_l'\Big)^{1/2},\end{aligned}\right.$$

where

$$g^{jl}(\sigma)=\delta_{jl}-(1+|\nabla_\sigma g|^2)^{-1}(\partial_{\sigma_j}g)(\partial_{\sigma_l}g).$$

We can see these equalities by expressing $\nabla\varphi^{\beta}$ in the form of linear combination of n, $\partial_{\sigma_1}x$ and $\partial_{\sigma_2}x$. Hence the proof of Lemma 4.2 is complete.

Using Lemma 4.2 and integration by parts, we have

$$\begin{aligned}
I_0(k) &= -C_T\sum_{l=1}^{2}\theta_l\int_{R^2}\exp(-ik(C_L^{-1}+C_T^{-1})g(\sigma))(\partial_{\sigma_l}g)A_\varepsilon(\sigma)(2f(|\nabla_\sigma g|^2)-3)d\sigma \\
&= -4C_T(ik(C_L^{-1}+C_T^{-1}))^{-1}\sum_{l=1}^{2}\theta_l\int_{R^2}\exp(-ik(C_L^{-1}+C_T^{-1})g(\sigma)) \\
&\quad\cdot A_\varepsilon(\sigma)\partial_x f(|\nabla_\sigma g|^2)\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)d\sigma+O(|k|^{-\infty})) \\
&\cong -4C_T(ik(C_L^{-1}+C_T^{-1}))^{-2}\int_{R^2}\exp(-ik(C_L^{-1}+C_T^{-1})g(\sigma) \\
&\qquad\cdot A_\varepsilon(\sigma)\partial_x f(|\nabla_\sigma g|^2)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}\Delta_\sigma g)d\sigma\,.
\end{aligned}$$

Hence, noting that $\partial_x f(|\nabla_\sigma g|^2)=\partial_x f(0)+[\nabla_\sigma g]^1$ and $\partial_x f(0)=C_L(C_L-C_T)^2/2C_T^3$, we obtain (4.2.0).

Next, we examine $(ik)I_1(k)$. By Proposition 3.2 we get

$$\begin{aligned}
\beta_1(\sigma) \doteqdot\ & A_\varepsilon(\sigma)\rho(\sigma)(B_1(\sigma,1,C_T^{-1}(\nabla_\sigma g))\omega,\theta) \\
&+A_\varepsilon(\sigma)\rho(\sigma)i^{-1}\sum_{|\alpha|=1}((\partial_\sigma^\alpha\partial_\xi^\alpha B_0)(\sigma,1,C_T^{-1}(\nabla_\sigma g))\omega,\theta) \\
&-A_\varepsilon(\sigma)i^{-1}\sum_{|\alpha|=1}((\partial_\xi^\alpha B_0)(\sigma,1,C_T^{-1}(\nabla_\sigma g))\omega,\theta)(\partial_\sigma^\alpha\rho)(\sigma) \\
&-C_T^{-1}A_\varepsilon(\sigma)\rho(\sigma)i^{-1}\sum_{|\alpha|=2}\frac{1}{\alpha!}((\partial_\xi^\alpha B_0)(\sigma,1,C_T^{-1}(\nabla_\sigma g))\omega,\theta)(\partial_\sigma^\alpha g)(\sigma),
\end{aligned}$$

since $((d/ds)^j\alpha_\varepsilon)(-(C_L^{-1}+C_T^{-1})g(\sigma))=0$ near $M_0(\omega)$ for $j\geqq 1$.

It is difficult to represent $\beta_1(\sigma)$ completely by the graph g of the boundary $\partial\Omega$ near $M(\omega)$ like β_0. But we can show that $(ik)I_1(k)$ is a similar form to (4.1) in Theorem 4.1. Using the Taylor expansion, we present $\beta_1(\sigma)$ in the form

$$\beta_1(\sigma)\doteqdot\sum_{l=1}^{4}\beta_{1l}(\sigma)+[\nabla_\sigma g]_0^3\,, \tag{4.5}$$

where

$$\beta_{11}(\sigma)=A_\varepsilon(\sigma)\rho(\sigma)\sum_{|\gamma|\leq 2}\frac{1}{\gamma!}(\partial_\xi^\gamma B_1(\sigma,1,0)\omega,\theta)\xi^\gamma|_{\xi=C_T^{-1}\nabla_\sigma g}\,,$$

$$\beta_{12}(\sigma)=A_\varepsilon(\sigma)\rho(\sigma)i^{-1}\sum_{|\alpha|=1}\sum_{|\gamma|\leq 2}\frac{1}{\gamma!}(\partial_\sigma^\alpha\partial_\xi^{\alpha+\gamma}B_0(\sigma,1,0)\omega,\theta)\xi^\gamma|_{\xi=C_T^{-1}\nabla_\sigma g}\,,$$

$$\beta_{13}(\sigma)=-A_\varepsilon(\sigma)i^{-1}\sum_{|\alpha|=1}\sum_{|\gamma|\leq 1}(\partial_\xi^{\alpha+\gamma}B_0(\sigma,1,0)\omega,\theta)\xi^\gamma(\partial_\sigma^\alpha\rho)(\sigma)|_{\xi=C_T^{-1}\nabla_\sigma g}\,,$$

$$\begin{aligned}
\beta_{14}(\sigma)=&-C_T^{-1}A_\varepsilon(\sigma)\rho(\sigma)i^{-1}\sum_{|\alpha|=2}\frac{1}{\alpha!}\sum_{|\gamma|\leq 2}\frac{1}{\gamma!} \\
&\cdot(\partial_\xi^{\alpha+\gamma}B_0(\sigma,1,0)\omega,\theta)\xi^\gamma(\partial_\sigma^\alpha g)(\sigma)|_{\xi=C_T^{-1}\nabla_\sigma g}\,.
\end{aligned}$$

Thus to attain our purpose we have only to calculate $\beta_{1l}(\sigma)$ ($l=1, 2, 3, 4$). This is carried out by means of the following Lemmas 4.3 and 4.4.

LEMMA 4.3. *We have the following relations:*

$$i^{-1}((\partial_{\xi_j}B_0)(\sigma, 1, 0)\omega, \theta) \doteqdot C_T^2(2-C_LC_T^{-1})\chi_1(\sigma, 0)\theta_j\rho(\sigma)^{-1} \qquad (j=1, 2),$$

$$i^{-1}((\partial_{\xi_j}\partial_{\xi_k}B_0)(\sigma, 1, 0)\omega, \theta) \doteqdot \chi_1(\sigma, 0)[(C_T-C_L)\{(C_L^2-C_LC_T-C_T^2)\delta_{jk}\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}g)$$
$$+C_T^2(\theta_j(\partial_{\sigma_k}g)+\theta_k(\partial_{\sigma_j}g))\}]+[\nabla_\sigma g]_0^3, \qquad (j, k=1, 2),$$

$$i^{-1}((\partial_{\xi_j}\partial_{\xi_k}\partial_{\xi_l}B_0)(\sigma, 1, 0)\omega, \theta) \doteqdot -C_LC_T(C_L-C_T)^2\chi_1(\sigma, 0)\rho^{-1}(\sigma)$$
$$\cdot(g^{jk}\theta_l+g^{kl}\theta_j+g^{lj}\theta_k) \qquad (j, k, l=1, 2),$$

$$i^{-1}((\partial_\xi^\alpha B_0)(\sigma, 1, 0)\omega, \theta) \doteqdot [\nabla_\sigma g]_0^1 \qquad \textit{for } |\alpha|=4.$$

LEMMA 4.4. *We have the following relations:*

$$(B_1(\sigma, 1, 0)\omega, \theta) \doteqdot (C_L-C_T)\chi_1(\sigma, 0)\rho^{-1}(\sigma)[C_T\sum_{l=1}^{2}\theta_l\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)$$
$$+2^{-1}(C_L-C_T)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}g)\Delta_\sigma g]+[\nabla_\sigma g]_0^3,$$

$$((\partial_{\xi_j}B_1)(\sigma, 1, 0)\omega, \theta) \doteqdot 2^{-1}C_T(C_L-C_T)(C_L-2C_T)\chi_1(\sigma, 0)\theta_j\Delta_\sigma g$$
$$+[\nabla_\sigma g]_0^2, \qquad (j=1, 2).$$

$$((\partial_\xi^\alpha B_1)(\sigma, 1, 0)\omega, \theta) \doteqdot [\nabla_\sigma g]_0^1 \qquad \textit{for } |\alpha|=2.$$

By (3.6.0) (4.3) and (4.4) we can easily obtain Lemma 4.3. Hence we omit its proof. The proof of Lemma 4.4 will be given in §5.

Noting that $\rho(\sigma)\doteqdot 1+[\nabla_\sigma g]^1$, by Lemma 4.4 we have

$$\beta_{11}(\sigma) \doteqdot A_\varepsilon(\sigma)(C_L-C_T)\{C_T\sum_{l=1}^{2}\theta_l\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)$$
$$+2^{-1}(2C_L-3C_T)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}g)\Delta_\sigma g\}+[\nabla_\sigma g]_0^3.$$

Noting that $\rho(\sigma)\doteqdot 1+[\nabla_\sigma g]^1$ and $(1+|\nabla_\sigma g|^2)^{-1}\doteqdot 1+[\nabla_\sigma g]^1$, by Lemma 4.3 we have

$$\beta_{12}(\sigma) \doteqdot A_\varepsilon(\sigma)\Big[C_T^2(C_LC_T^{-1}-2)\sum_{l=1}^{2}\theta_l\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)$$
$$+(1-C_LC_T^{-1})\{C_L(C_L-C_T)\sum_{l=1}^{2}\theta_l\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)$$
$$+C_T^2\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}g)\Delta_\sigma g\}\Big]+[\nabla_\sigma g]_0^3.$$

For $\beta_{13}(\sigma)$ and $\beta_{14}(\sigma)$, by the same way as $\beta_{12}(\sigma)$ we have

$$\beta_{13}(\sigma) \doteqdot C_T^2(C_LC_T^{-1}-2)A_\varepsilon(\sigma)\sum_{l=1}^{2}\theta_l\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)+[\nabla_\sigma g]_0^3,$$

$$\beta_{14}(\sigma) \doteqdot A_\varepsilon(\sigma)2^{-1}(C_L C_T^{-1}-1)\{(2C_L^2-2C_L C_T-C_T^2)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}g)\Delta_\sigma g$$
$$+(2C_L^2-2C_L C_T+2C_T^2)\sum_{l=1}^{2}\theta_l\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)\}+[\nabla_\sigma g]_0^3.$$

Hence by (4.5) we obtain

$$\beta_1(\sigma) \doteqdot A_\varepsilon(\sigma)\Big[\{2C_T^2(C_L C_T^{-1}-1)+2C_T^2(C_L C_T^{-1}-2)\}\sum_{l=1}^{2}\theta_l\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)$$
$$+(C_L^2-3C_T^2)(C_L C_T^{-1}-1)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}g)\Delta_\sigma g\Big]+[\nabla_\sigma g]_0^3,$$

which yields (4.2.1).

In the same way as the proof of (4.2.1) we get (4.2.2) if we obtain the following relation proved in the next section

$$(4.6)\qquad i(B_2(\sigma, 1, 0)\omega, \theta) \doteqdot \chi_1(\sigma, 0)2^{-1}C_T(C_L-C_T)^2\sum_{l=1}^{2}\theta_l(\partial_{\sigma_l}\Delta_\sigma g)+[\nabla_\sigma g]_0^1,$$

(for the proof of (4.6), see §5). Thus we complete the proof of Theorem 4.1.

§5. Computation of the symbols of the Neumann operators.

In this section we prove Lemma 4.4 and (4.6) stated in §4. We define the mapping $\tilde{x}(\sigma, r)$ by

$$U_0\times[-r_0, 0] \ni (\sigma, r) \longrightarrow \tilde{x}(\sigma, r) = x(\sigma)+rn(\sigma) \in \Omega\,,$$

where $r_0>0$ is so small that $\tilde{x}(\sigma, r)$ is diffeomorphic, and we denote by $\tilde{g}^{jl}(\sigma, r)$ the (j, l)-component of the inverse matrix of $\left((\partial_{\sigma_j}\tilde{x})\cdot(\partial_{\sigma_l}\tilde{x})\Big|\begin{matrix} j\to 1, 2\\ l\downarrow 1, 2\end{matrix}\right)$. Note that $\tilde{g}^{ij}(\sigma, 0)$ is equal to $g^{ij}(\sigma)$ which is introduced in the proof of Lemma 4.2. In the same way as (4.4) we see that

$$\left(\frac{\partial\varphi^\beta}{\partial n}\right)=\left(C_\beta^{-2}-\sum_{j,l=1}^{2}\tilde{g}^{jl}(\sigma, r)\left(\frac{\partial\varphi^\beta}{\partial\sigma_j}\right)(\sigma, r, \xi')\left(\frac{\partial\varphi^\beta}{\partial\sigma_l}\right)(\sigma, r, \xi')\right)^{1/2}\quad(\beta=L, T).$$

Furthermore, noting that $\varphi^\beta(\sigma, 0, \xi')=\sigma\cdot\xi'$ and

$$\left(\frac{\partial}{\partial n}\left(\frac{\partial\varphi^\beta}{\partial\sigma_k}\right)\right)(\sigma, 0, \xi')=\frac{\partial}{\partial\sigma_k}\left(\left(\frac{\partial\varphi^\beta}{\partial n}\right)(\sigma, 0, \xi')\right)$$
$$=-2^{-1}\left(\frac{\partial\varphi^\beta}{\partial n}\right)^{-1}(\sigma, 0, \xi')\sum_{j,l=1}^{2}\left(\frac{\partial}{\partial\sigma_k}g^{jl}\right)(\sigma)\xi_j'\xi_l',$$

we get

$$(5.1)\qquad \partial_\xi^\alpha\left(\frac{\partial}{\partial n}\right)^2\varphi^\beta(\sigma, 0, 0)=0\qquad(|\alpha|=0, 1).$$

At first let us check

LEMMA 5.1. *We have the following equalities and relations:*

$$C_L \Delta\varphi^L|_{\xi'=0} = C_T \Delta\varphi^T|_{\xi'=0} \qquad on\ \partial\Omega\cap V\,, \tag{5.2}$$

$$\begin{cases} \Delta\varphi^\beta|_{\xi'=0} = -C_\beta^{-1}\Delta_\sigma g + [\nabla_\sigma g]^2 & on\ \partial\Omega\cap V\,, \\ \partial_{\xi'}^\alpha \Delta\varphi^\beta|_{\xi'=0} = [\nabla_\sigma g]^1 & on\ \partial\Omega\cap V\,, \end{cases} \tag{5.3}$$

$$(|\alpha|=1,\ \beta=L,\ T)$$

$$\begin{cases} \square_L(u_0^L\omega,\ \theta)|_{\xi'=0} \doteqdot -C_L^2 \sum_{l=1}^2 \theta_l(\partial_{\sigma_l}\Delta_\sigma g) + [\nabla_\sigma g]^1 & on\ \partial\Omega\cap V, \\ \square_T(u_0^T\omega,\ \theta)|_{\xi'=0} \doteqdot C_T^2 \sum_{l=1}^2 \theta_l(\partial_{\sigma_l}\Delta_\sigma g) + [\nabla_\sigma g]^1 & on\ \partial\Omega\cap V, \end{cases} \tag{5.4}$$

PROOF. Since it is seen that $C_L\varphi^L|_{\xi'=0}=C_T\varphi^T|_{\xi'=0}$ in $\Omega\cap V$, (5.2) is satisfied. Expressing Δ in the variables (σ, r) and noting that

$$\sum_{j,l=1}^2 g^{jl}(\sigma)\frac{\partial^2 x}{\partial\sigma_j\partial\sigma_l}(\sigma)\cdot n(\sigma) = \rho^{-1}(\sigma)\sum_{j,l=1}^2 g^{jl}(\sigma)(\partial_{\sigma_j}\partial_{\sigma_l}g) \doteqdot \Delta_\sigma g + [\nabla_\sigma g]^2,$$

we obtain (5.3) from (4.4) and (5.1). The transport equations (3.4.0) and (3.5.0) yield that

$$\frac{\partial}{\partial n}(u_0^\beta\omega,\ \theta) = -\Big(\frac{\partial\varphi^\beta}{\partial n}\Big)^{-1}\Big\{\sum_{j,l=1}^2 \tilde{g}^{jl}\frac{\partial\varphi^\beta}{\partial\sigma_j}\frac{\partial}{\partial\sigma_l}(u_0^\beta\omega,\ \theta) + 2^{-1}(\Delta\varphi^\beta)(u_0^\beta\omega,\ \theta)\Big\}.$$

From (3.6.0) it follows that

$$\Big(\frac{\partial}{\partial n}\Big)^j(u_0^\beta\omega,\ \theta)|_{\xi'=0} = [\nabla_\sigma g]^1 \qquad \text{on } \partial\Omega\cap V \qquad \begin{matrix}(j=0,\ 1,\ 2)\\(\beta=L,\ T)\end{matrix}.$$

Thus we have

$$\Delta(u_0^\beta\omega,\ \theta)|_{\xi'=0} = \sum_{j,l=1}^2 g^{jl}(\sigma)\frac{\partial^2}{\partial\sigma_j\partial\sigma_l}(u_0^\beta\omega,\ \theta)\Big|_{\xi'=0} + [\nabla_\sigma g]^1.$$

Hence, noting that $\square_\beta = C_\beta^2\Delta - \partial_t^2$ $(\beta=L,\ T)$ and

$$(u_0^\beta\omega,\ \theta)|_{\xi'=0} = \begin{cases} -(1+|\nabla_\sigma g|^2)^{-1}\sum_{l=1}^2 \theta_l\partial_{\sigma_l}g & \text{on } \partial\Omega\cap V \text{ if } \beta=L, \\ (1+|\nabla_\sigma g|^2)^{-1}\sum_{l=1}^2 \theta_l\partial_{\sigma_l}g & \text{on } \partial\Omega\cap V \text{ if } \beta=T, \end{cases}$$

we can get (5.4).

PROOF OF LEMMA 4.4. By (3.10) and the equality $n(\sigma)=\rho^{-1}(\sigma)(-\sum_{j=1}^2 e_j(\partial_{\sigma_j}g)+\omega)$ $(l\geqq 1)$, we have

$$(B_l(\sigma,\ \tau,\ \tau\xi')\omega,\ \theta) = (ik)^{1-l}\{\chi_1(\sigma,\ \xi')\Phi_l(\sigma,\ \xi') + \tilde{N}(\chi_1(\sigma,\ \xi')I)\delta_{1l}\}\,, \tag{5.5}$$

where

$$
\begin{aligned}
(5.6)\quad \Phi_l(\sigma,\xi') = {} & \mu\Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)(u_l^L\omega,\ \theta)\\
& -(\lambda+\mu)\rho^{-1}(\sigma)\Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)\theta\cdot n\sum_{j=1}^{2}(u_l^L\omega,\ e_j)(\partial_{\sigma_j}g)(\sigma)\\
& +(\lambda+\mu)\rho^{-1}(\sigma)\Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)\theta\cdot n(u_l^L\omega,\ \omega)+\mu\Big\{\Big(\frac{\partial}{\partial n}u_{l-1}^L\omega,\ \theta\Big)+\Big(\frac{\partial}{\partial n}u_{l-1}^T\omega,\ \theta\Big)\Big\}\\
& -(\lambda+\mu)\rho^{-1}(\sigma)\theta\cdot n\sum_{j=1}^{2}\Big\{\Big(\frac{\partial}{\partial n}u_{l-1}^L\omega,\ e_j\Big)+\Big(\frac{\partial}{\partial n}u_{l-1}^T\omega,\ e_j\Big)\Big\}(\partial_{\sigma_j}g)(\sigma)\\
& +(\lambda+\mu)\rho^{-1}(\sigma)\theta\cdot n\Big\{\Big(\frac{\partial}{\partial n}u_{l-1}^T\omega,\ \omega\Big)+\Big(\frac{\partial}{\partial n}u_{l-1}^T\omega,\ \omega\Big)\Big\}.
\end{aligned}
$$

Combining this with (5.5) and the following Lemmas 5.2∼5.4, we can obtain Lemma 4.4.

LEMMA 5.2. *We have the following relations:*

$$
\begin{aligned}
(u_1^L\omega,\ e_l)|_{\xi'=0} &\doteqdot -\rho^{-1}(\sigma)\{C_T(\partial_{\sigma_l}g)\Delta_\sigma g\\
&\qquad +C_L\sum_{j=1}^{2}(\partial_{\sigma_j}g)(\partial_{\sigma_j}\partial_{\sigma_l}g)\}+[\nabla_\sigma g]^3 \qquad \text{on } U_0,\ (l=1,\ 2),\\
\partial_{\xi'_j}(u_1^L\omega,\ e_l)|_{\xi'=0} &\doteqdot -2^{-1}C_L(C_L-2C_T)\delta_{jl}\Delta_\sigma g+C_LC_T(\partial_{\sigma_j}\partial_{\sigma_l}g)+[\nabla_\sigma g]^2\\
&\qquad\qquad \text{on } U_0,\ (j,\ l=1,\ 2),\\
\partial_{\xi'}^{\alpha}(u_1^L\omega,\ e_l)|_{\xi'=0} &\doteqdot [\nabla_\sigma g]^1 \qquad \text{on } U_0,\ (|\alpha|=2,\ l=1,\ 2).
\end{aligned}
$$

LEMMA 5.3. *We have the following relations:*

$$
\begin{aligned}
(u_1^L\omega,\ \omega) &\doteqdot C_T\rho^{-1}(\sigma)\Delta_\sigma g+[\nabla_\sigma g]^2 \qquad \text{on } U_0,\\
\partial_{\xi'_j}(u_1^L\omega,\ \omega)|_{\xi'=0} &\doteqdot [\nabla_\sigma g]^1 \qquad \text{on } U_0,\ (j=1,\ 2).
\end{aligned}
$$

LEMMA 5.4. *Set*

$$
\begin{aligned}
f_1(\sigma,\xi') &= \Big\{\Big(\frac{\partial}{\partial n}u_0^L\omega,\ \theta\Big)+\Big(\frac{\partial}{\partial n}u_0^T\omega,\ \theta\Big)\Big\}\Big|_{\partial\Omega\cap V},\\
f_2(\sigma,\xi') &= \Big\{\Big(\frac{\partial}{\partial n}u_0^L\omega,\ \omega\Big)+\Big(\frac{\partial}{\partial n}u_0^T\omega,\ \omega\Big)\Big\}\Big|_{\partial\Omega\cap V}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& f_1(\sigma,\ 0)=0 \qquad \text{on } U_0,\\
& (\partial_{\xi'_j}f_1)(\sigma,\ 0)\doteqdot (C_L-C_T)\sum_{l=1}^{2}\theta_l(\partial_{\sigma_j}\partial_{\sigma_l}g)+[\nabla_\sigma g]^2 \qquad \text{on } U_0,\ (j=1,\ 2),\\
& (\partial_{\xi'}^{\alpha}f_1)(\sigma,\ 0)\doteqdot [\nabla_\sigma g]^1 \qquad \text{on } U_0\ (|\alpha|=2),\\
& f_2(\sigma,\ 0)\doteqdot 2^{-1}\rho^{-1}(\sigma)\Delta_\sigma g+[\nabla_\sigma g]^2 \qquad \text{on } U_0,
\end{aligned}
$$

$$\partial_{\xi'_j} f_2(\sigma, 0) \doteqdot [\nabla_\sigma g]^1 \qquad on\ U_0,\ (j=1, 2).$$

PROOF OF LEMMA 5.2. From (3.6.1), (3.7.1) and the eikonal equations (3.3.L) and (3.3.T), it follows that

$$(5.7) \qquad (u_1^L\omega, e_l) = (\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}V_1(\sigma, \xi')e_l\cdot\nabla\varphi^L+(z_1^L\omega, e_l),$$

where

$$(5.8) \qquad V_1(\sigma, \xi') = -C_L^2(\nabla\varphi^L\times(\text{curl}\ u_0^L\omega))\cdot\nabla\varphi^T+\text{div}\ u_0^T\omega,$$

Since $\nabla\varphi^T\cdot(\nabla\varphi^L\times(\text{curl}\ u)) = ((\nabla\varphi^T\cdot\nabla)u, \nabla\varphi^L)-((\nabla\varphi^L\cdot\nabla)u, \nabla\varphi^T)$, we see that $V_{11}(\sigma, \xi')=(\nabla\varphi^L\times(\text{curl}\ u_0^L\omega))\cdot\nabla\varphi^T$ is of the form

$$(5.9) \qquad V_{11}(\sigma, \xi') = ((\nabla\varphi^T\cdot\nabla)u_0^L\omega, \nabla\varphi^L)-((\nabla\varphi^L\cdot\nabla)u_0^L\omega, \nabla'\varphi^T).$$

Furthermore, using the equalities $\nabla\varphi^L=\nabla\varphi^T+(\partial\varphi^L/\partial n-\partial\varphi^T/\partial n)n$, $\nabla\varphi^L=\nabla'\varphi^L+(\partial\varphi^L/\partial n)n$ and (3.4.0), we have

$$\begin{aligned} V_{11}(\sigma, \xi') &= \Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)\Big\{((\nabla\varphi^L\cdot\nabla)u_0^L\omega, n)-\Big(\frac{\partial}{\partial n}u_0^L\omega, \nabla\varphi^L\Big)\Big\} \\ &= \Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)\Big(\frac{\partial\varphi^L}{\partial n}\Big)^{-1}\{((\nabla'\varphi^L\cdot\nabla)u_0^L\omega, \nabla\varphi^L)+2^{-1}(\Delta\varphi^L)(u_0^L\omega, \nabla'\varphi^L)\}. \end{aligned}$$

Hence the eikonal equation (3.3.L) and (3.6.0) yield that

$$(5.10) \qquad \begin{aligned} V_{11}(\sigma, \xi') = \Big(\frac{\partial\varphi^L}{\partial n}-\frac{\partial\varphi^T}{\partial n}\Big)\Big(\frac{\partial\varphi^L}{\partial n}\Big)^{-1}\Big\{&C_L^{-2}\sum_{j,l=1}^{2} g^{jl}(\sigma)\xi'_j\frac{\partial}{\partial\sigma_l}\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big) \\ &+2^{-1}(\Delta\varphi^L)\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big)\sum_{j,l=1}^{2} g^{jl}(\sigma)\xi'_j\xi'_l\Big\}. \end{aligned}$$

The operator div $\cdot$ is represented in the variables (σ, r) as follows:

$$\text{div}\ u_0^T\omega = n\cdot\frac{\partial}{\partial n}(u_0^T\omega)+\sum_{j,k=1}^{2} g^{jk}(\sigma)\frac{\partial x}{\partial\sigma_j}\cdot\frac{\partial}{\partial\sigma_k}(u_0^T\omega).$$

Hence, noting that $(\partial/\partial n)(u_0^T\omega)=-(\partial\varphi^T/\partial n)^{-1}\{(\nabla'\varphi^T\cdot\nabla)u_0^T\omega+2^{-1}(\Delta\varphi^T)u_0^T\omega\}$, $\partial x/\partial\sigma_j =e_j+(\partial g/\partial\sigma_j)\omega$ and $n(\sigma)=\rho^{-1}(\sigma)\{-\sum_{k=1}^2(\partial g/\partial\sigma_k)e_k+\omega\}$, by (3.6.0) we obtain

$$(5.11) \qquad \begin{aligned} \text{div}\ u_0^T\omega = &-\Big(\frac{\partial\varphi^T}{\partial n}\Big)^{-1}\Big[\rho^{-1}(\sigma)\Big\{\sum_{j,k,l=1}^{2} g^{jk}(\sigma)\xi'_j\frac{\partial g}{\partial\sigma_l}\frac{\partial}{\partial\sigma_k}(u_0^L\omega, e_l) \\ &-\sum_{j,k=1}^{2} g^{jk}(\sigma)\xi'_j\frac{\partial}{\partial\sigma_k}(u_0^L\omega, \omega)\Big\}+2^{-1}(\Delta\varphi^T)\Big(n\cdot\omega-\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big(\frac{\partial\varphi^L}{\partial n}\Big)\Big)\Big] \\ &-\sum_{j=1}^{2}\frac{\partial}{\partial\sigma_j}(u_0^L\omega, e_j)-(1+|\nabla_\sigma g|^2)^{-1} \\ &\cdot\Big\{-\sum_{j,k=1}^{2}\frac{\partial g}{\partial\sigma_j}\cdot\frac{\partial g}{\partial\sigma_k}\frac{\partial}{\partial\sigma_j}(u_0^L\omega, e_k)+\sum_{j=1}^{2}\frac{\partial g}{\partial\sigma_j}\frac{\partial}{\partial\sigma_j}(u_0^L\omega, \omega)\Big\}. \end{aligned}$$

Combining (5.8), (5.10) and (5.11) yields that

$$(5.12)\qquad \begin{cases} V_1(\sigma, 0) = \Delta_\sigma g + [\nabla_\sigma g]^2, \\ (\partial_{\xi'_j} V_1)(\sigma, 0) = [\nabla_\sigma g]^1 \qquad (j=1, 2). \end{cases}$$

In the same way as for (5.9) we obtain

$$(z_1^L\omega, \theta) = C_L^2\{-(\nabla\varphi^L\cdot\nabla)(u_0^L\omega, \theta) + ((\theta\cdot\nabla)u_0^L\omega, \nabla\varphi^L)\}.$$

This equality, the transport equation (3.4.0) and $\theta\cdot\nabla = \sum_{j,l=1}^2 g^{jl}(\sigma)\theta_l(\partial/\partial\sigma_j) + n\cdot\theta(\partial/\partial n)$ give

$$\begin{aligned}(z_1^L\omega, \theta) &= C_L^2\Big[2^{-1}(\Delta\varphi^L)(u_0^L\omega, \theta) + \sum_{j,l=1}^2 g^{jl}(\sigma)\theta_j\frac{\partial}{\partial\sigma_l}(u_0^L\omega, \nabla\varphi^L) \\ &\quad -\theta\cdot n\Big(\frac{\partial\varphi^L}{\partial n}\Big)^{-1}\{((\nabla'\varphi^L\cdot\nabla)u_0^L\omega, \nabla\varphi^L) + 2^{-1}(\Delta\varphi^L)(u_0^L\omega, \nabla\varphi^L)\}\Big] \\ &= C_L^2\Big[2^{-1}(\Delta\varphi^L)(u_0^L\omega, \theta) + C_L^{-2}\sum_{j,k=1}^2 g^{jl}(\sigma)\theta_j\frac{\partial}{\partial\sigma_l}\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big) \\ &\quad -C_L^{-2}(\theta\cdot n)\Big(\frac{\partial\varphi^L}{\partial n}\Big)^{-1}\Big\{\sum_{j,l=1}^2 g^{jl}(\sigma)\xi'_j\frac{\partial}{\partial\sigma_l}\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big) + 2^{-1}(\Delta\varphi^L)\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big)\Big\}\Big].\end{aligned}$$

Thus it is seen that

$$(5.13)\qquad \begin{cases} (z_1^L\omega, \theta)|_{\xi'=0} = C_L\sum_{l=1}^2\theta_l\frac{\partial}{\partial\sigma_l}\rho^{-1} + [\nabla_\sigma g]^3 \qquad \text{on } \partial\Omega\cap V, \\ \partial_{\xi'_j}(z_1^L\omega, \theta)|_{\xi'=0} = -2^{-1}C_L^2\theta_j\Delta_\sigma g + C_L C_T\sum_{l=1}^2\theta_l\frac{\partial g}{\partial\sigma_l\partial\sigma_j} + [\nabla_\sigma g]^2 \\ \qquad\qquad\qquad\qquad \text{on } \partial\Omega\cap V \ (j=1, 2), \\ \partial_{\xi'}^\alpha(z_1^L\omega, \theta)|_{\xi'=0} = [\nabla_\sigma g]^1 \qquad \text{on } \partial\Omega\cap V \ (|\alpha|=2). \end{cases}$$

Lemma 5.2 is derived from (5.7), (5.12) and (5.13).

PROOF OF LEMMA 5.3. (3.6.1) yields that

$$(u_1^L\omega, \omega) = (\nabla\varphi^L\cdot\nabla\varphi^T)^{-1}V_1(\sigma, \xi')\omega\cdot\nabla\varphi^L + (z_1^L\omega, \omega).$$

By the same methods as for (5.13), we have

$$\begin{aligned}(z_1^L\omega, \omega) &= C_L^2\Big[2^{-1}(\Delta\varphi^L)(u_0^L\omega, \omega) + C_L^{-2}(1+|\nabla_\sigma g|^2)^{-1}\sum_{l=1}^2\frac{\partial g}{\partial\sigma_l}\frac{\partial}{\partial\sigma_l}\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big) \\ &\quad -C_L^{-2}(\omega\cdot n)\Big(\frac{\partial\varphi^L}{\partial n}\Big)^{-1}\Big\{\sum_{j,l=1}^2 g^{jl}(\sigma)\xi'_j\frac{\partial}{\partial\sigma_l}\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big) + 2^{-1}(\Delta\varphi^L)\Big(\frac{\omega\cdot\nabla\varphi^T}{\nabla\varphi^L\cdot\nabla\varphi^T}\Big)\Big\}\Big].\end{aligned}$$

Hence, using (5.2), (5.3) and (4.4), we obtain

$$\begin{cases} (z_1^L\omega, \omega)|_{\xi'=0} = [\nabla_\sigma g]^2 \qquad \text{on } \partial\Omega\cap V, \\ \partial_{\xi'}^\alpha(z_1^L\omega, \omega)|_{\xi'=0} = [\nabla_\sigma g]^1 \qquad \text{on } \partial\Omega\cap V. \end{cases}$$

which proves Lemma 5.3.

PROOF OF LEMMA 5.4. By the transport equations (3.4.0) and (3.5.0) we have

$$f_1(\sigma, \xi') = -\sum_{\beta=L,T}\left(\frac{\partial\varphi^\beta}{\partial n}\right)^{-1}\left\{\nabla'\varphi^\beta\cdot\nabla(u_0^\beta\omega, \theta)+2^{-1}(\Delta\varphi^\beta)(u_0^\beta\omega, \theta)\right\}$$

$$= -\left(\frac{\partial\varphi^L}{\partial n}\right)^{-1}\left[\sum_{j,l=1}^{2} g^{jl}(\sigma)\xi_j'\frac{\partial}{\partial\sigma_l}(u_0^L\omega, \theta)+2^{-1}(\Delta\varphi^L)(u_0^L\omega, \theta)\right]$$

$$-\left(\frac{\partial\varphi^T}{\partial n}\right)^{-1}\left[-\sum_{j,l=1}^{2} g^{jl}(\sigma)\xi_j'\frac{\partial}{\partial\sigma_l}(u_0^L\omega, \theta)-2^{-1}(\Delta\varphi^L)(u_0^L\omega, \theta)\right],$$

$$f_2(\sigma, \xi') = -\left(\frac{\partial\varphi^L}{\partial n}\right)^{-1}\left[\sum_{j,l=1}^{2} g^{jl}(\sigma)\xi_j'\frac{\partial}{\partial\sigma_l}(u_0^L\omega, \omega)+2^{-1}(\Delta\varphi^L)(u_0^L\omega, \omega)\right]$$

$$-\left(\frac{\partial\varphi^T}{\partial n}\right)^{-1}\left[-\sum_{j,l=1}^{2} g^{jl}(\sigma)\xi_j'\frac{\partial}{\partial\sigma_l}(u_0^L\omega,\omega)-2^{-1}(\Delta\varphi^T)(u_0^L\omega,\omega)+2^{-1}(\Delta\varphi^T)\right].$$

Therefore, by Lemma 5.1 we can obtain Lemma 5.4 directly.

PROOF OF (4.6). In the same way as for Lemmas 5.2 and 5.4, we can prove

$$(u_2^L\omega, \theta)|_{\xi'=0} = -2^{-1}C_L(C_L-2C_T)\sum_{l=1}^{2}\theta_l\partial_{\sigma_l}\Delta_\sigma g+[\nabla_\sigma g]^1 \qquad \text{on } \partial\Omega\cap V,$$

$$\left\{\frac{\partial}{\partial n}(u_0^L\omega, \theta)+\frac{\partial}{\partial n}(u_1^L\omega, \theta)\right\}\Big|_{\xi'=0} = 2^{-1}(C_L-C_T)\sum_{l=1}^{2}\theta_l\partial_{\sigma_l}\Delta_\sigma g+[\nabla_\sigma g]^1 \qquad \text{on } \partial\Omega\cap V.$$

Noting these equalities and (5.5), we can get (4.6).

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