J. Math. Soc. Japan
Vol. 42, No. 4, 1990

The short time asymptotics of the traces of the heat kernels for the magnetic Schrödinger operators

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(Received Dec. 12, 1988)
(Revised Nov. 13, 1989)

§1. Introduction.

The short time asymptotics of the traces of the heat kernels for the Schrödinger operators without magnetic fields on Euclidean spaces has been considered from the old times mainly because it is directly related to the asymptotic distributions of the eigenvalues by virtue of the Tauberian theorem. See, for example, [**6**], [**8**] and the references therein. Moreover Tamura [**10**] has studied this short time asymptics for the Schrödinger operators only with magnetic fields whose magnitudes grow unboundedly at infinity.

In this paper we will consider the Schrödinger operators H_0 and H on $\boldsymbol{R}^d$, $d \geqq 2$, which are given by

$$H_0 = -\frac{1}{2}\Delta + V(x)$$

and

$$H = \frac{1}{2}(-\sqrt{-1}\nabla + A(x))^2 + V(x),$$

both of which act on $L^2(\boldsymbol{R}^d)$, and we will study the difference between the short time asymptotics of the traces of the heat kernels for $-H_0$ and $-H$.

We will assume that the scalar potential V is bounded from below and is bounded from below by some polynomial of $|x|$ at infinity. This implies that e^{-tH_0} and e^{-tH} are of the trace class. Then we will see that, if the derivatives $\partial^\alpha A(x)$, $|\alpha|=1, 2$, of the vector potential A grow more slowly than V, the leading term of the asymptotics of the trace of e^{-tH} as t tends to 0 coincides with that of e^{-tH_0}. Therefore such vector potentials or magnetic fields, which are given by $\mathrm{curl}(A(x))$, do not affect the asymptotic behavior of the trace so seriously. Moreover this result says that the leading terms of the asymptotic distributions of the eigenvalues of H_0 and H are identical.

Similar problem has been considered by Odencrantz [**5**] in the case of a uniform magnetic field. This important case will be discussed in detail also in this paper. The main methods used in [**5**] are the canonical order calculus developed by Simon [1] and the theory of pseudodifferential operators based on the weighted Sobolev spaces. In this paper we will use the probabilistic method, which is also different from that of [6]. By virtue of this approach we can show the result directly and can prove the result without any differentiability conditions on the scalar potential V.

Finally the author would like to thank the referee for useful comments.

§ 2. Main results.

We consider the operators H_0 and H on $\boldsymbol{R}^d$, $d\geqq 2$, defined by

$$H_0=-\frac{1}{2}\Delta+V(x)$$

and

$$H=\frac{1}{2}(-\sqrt{-1}\nabla+A(x))^2+V(x),$$

respectively.

First of all we state the assumptions on the scalar potential $V(x)$ and on the magnetic vector potential $A(x)=(a_1(x), a_2(x), \cdots, a_d(x))$. Throughout this paper we assume:

(V1) $V(x)$ is real valued continuous function on $\boldsymbol{R}^d$ and there exist a compact set $K\subset\boldsymbol{R}^d$ and positive constants C_1 and k such that

$$V(x)\geqq C_1|x|^{2k} \qquad \text{for } x\in\boldsymbol{R}^d\setminus K.$$

(A1) $a_j(x)$, $1\leqq j\leqq d$, are real valued C^2 functions and there exist positive constants C_2 and r with $r<2k$ such that

$$|D^\alpha a_j(x)|\leqq C_2(1+|x|^r)$$

for each j and each multi-index α with $|\alpha|=1, 2$.

Under these assumptions it is known that H_0 and H are essentially selfadjoint in $C_0^\infty(\boldsymbol{R}^d)$. We will denote the selfadjoint realizations of H_0 and H on $L^2(\boldsymbol{R}^d)$ by the same notations, respectively. Moreover the assumptions imply that $\operatorname{Tr} e^{-tH_0}$, the trace of the semigroup e^{-tH_0} generated by H_0, is finite for any $t>0$ and that, since

$$\operatorname{Tr} e^{-tH}\leqq\operatorname{Tr} e^{-tH_0},$$

e^{-tH} is also of the trace class. For details of these fundamental facts, we refer to [7] and [8].

Now we state the first result which is concerned in the rough estimate for the asymptotics of the difference between the traces of e^{-tH} and e^{-tH_0} as t tends to 0.

THEOREM 1. *Under the assumptions* (V1) *and* (A1), *there exists a positive constant* C *such that*

$$|\mathrm{Tr}(e^{-tH}-e^{-tH_0})| \leqq Ct^{2-(d/2)(1+1/k)-r/k} \tag{2.1}$$

for sufficiently small $t>0$.

The proof of Theorem 1 will be given in the next section. We note here that, if $d=2$ or 3, there is some cases when the power of t in (2.1) is positive and, in particular,

$$\mathrm{Tr}(e^{-tH}-e^{-tH_0}) = o(1) \qquad \text{as } t\downarrow 0\,,$$

which cannot be seen by the result of Odencrantz [**5**].

As a corollary of Theorem 1 we can easily prove that the leading terms of the asymptotic distributions of the eigenvalues of H and H_0 are identical. To mention the result, denote by $N_0(\lambda)$ and $N(\lambda)$ the numbers of the eigenvalues less than $\lambda>0$ of H_0 and H, respectively. Then we will show:

COROLLARY 1. *Assume* (V1), (A1) *and, moreover, assume that there exists a constant* C_3 *such that*

$$V(x) \leqq C_3|x|^{2k}$$

holds for $x\in \boldsymbol{R}^d\setminus K$. *Then it holds that*

$$N(\lambda) = N_0(\lambda) \quad (1+o(1)) \qquad \textit{as } \lambda\to\infty\,.$$

The proof will also be given in the next section.

Next we consider the asymptotically uniform magnetic field. We will see that the power of t in (2.1) is in fact attained in this case. Moreover we will express the constant which appears in the leading term of the difference explicitly. For this purpose we assume also the following:

(V2) There exist a Lipschitz continuous function v on the d-sphere $\boldsymbol{S}^{d-1}$ and positive constants δ and C_4 such that

$$|V(x)-|x|^{2k}v(x/|x|)| \leqq C_4|x|^{2k-\delta} \qquad \text{for } x\in\boldsymbol{R}^d\setminus K\,.$$

(A2) For each i, j there exists a constant a_{ij} such that

$$\lim_{|x|\to\infty}\frac{\partial}{\partial x^j}a_i(x) = a_{ij}$$

and, for each multi-index α with $|\alpha|=2$, it holds that

$$\lim_{|x|\to\infty}|D^\alpha a_i(x)| = 0\,.$$

Then we will prove:

THEOREM 2. *Under the assumptions* (V1), (V2), (A1) *and* (A2), *it holds that*

$$\lim_{t\downarrow 0} t^{-2+(d/2)(1+1/k)}\,\mathrm{Tr}\,(e^{-tH}-e^{-tH_0})$$

(2.2)
$$=\frac{1}{12(2\pi)^{d/2}}\int_{R^d}\exp(-|x|^{2k}v(x/|x|))dx\sum_{i,j=1}^{d}(a_{ij}-a_{ji})^2\,.$$

The proof of Theorem 2 will be given in Section 4. Here we give another expressions of the constant which appeared in (2.2). We will show

(2.3)
$$\frac{1}{12}\sum_{i,j=1}^{d}(a_{ij}-a_{ji})^2=E\Big[\Big(\sum_{i,j=1}^{d}a_{ij}\int_0^1 X_s^j\circ dX_s^i\Big)^2\Big],$$

where E is the expectation with respect to the d-dimensional pinned Brownian motion $\{X_s\}_{0\le s\le 1}$ such that $X_0=X_1=0$ and $\circ dX_s^j$ denotes the Stratonovich integral. For the pinned Brownian motion and the Stratonovich integral, see [**3**] and [**8**].

We will end this section with some remarks on the case of uniform magnetic fields. By the gauge invariance (cf. [**1**]), we can assume that the uniform magnetic field is defined by a skew-symmetric matrix $B=(a_{ij})$, that is, by a vector potential Bx. In this case the expectation in (2.3) is the variance of a linear combination of Lévy's stochastic area, which is defined by

$$\frac{1}{2}\int_0^1 X_s^i\circ dX_s^j-X_s^j\circ dX_s^i\,.$$

This also appears when we write the heat kernel in terms of the pinned Brownian motion by using the Feynman-Kac-Itô's formula. See the next section and see also [**2**] for more informations.

Moreover, if $B=(a_{ij})$ is skew-symmetric, we can rewrite (2.3). Denote by $\pm\sqrt{-1}b_j$, $b_j>0$ and $j=1, 2, \cdots, r$, the non-zero eigenvalues of B, where $2r$ $(<d)$ is the rank of B. Then it is easily seen that

$$\frac{1}{4}\sum_{i,j=1}^{d}(a_{ij}-a_{ji})^2=\sum_{i<j}(a_{ij})^2=\sum_{j=1}^{r}(b_j)^2\,.$$

§3. Proof of Theorem 1.

Let us denote by $p(t, x, y)$ and $q(t, x, y)$ the heat kernels for $-H_0$ and $-H$, respectively. Then the following probabilistic representation of these heat kernels, called the Feynman-Kac-Itô's formula, is well known ([8]);

$$p(t, x, y)=(2\pi t)^{-d/2}E_{0;x}^{t;y}\Big[\exp\Big(-\int_0^t V(X_s)ds\Big)\Big]e^{-|x-y|^2/2t}$$

$$q(t, x, y)=(2\pi t)^{-d/2}E_{0;x}^{t;y}\Big[\exp\Big(\sqrt{-1}\int_0^t A(X_s)\circ dX_s-\int_0^t V(X_s)ds\Big)\Big]e^{-|x-y|^2/2t},$$

where $E_{0;x}^{t;y}$ is the expectation with respect to the d-dimensional pinned Brownian motion $\{X_s\}_{0\le s\le t}=\{(X_s^1, X_s^2, \cdots, X_s^d)\}_{0\le s\le t}$ such that $X_0=x$ and $X_t=y$. Moreover, under the assumptions (V1) and (A1), $p(t, x, y)$ and $q(t, x, y)$ are continuous in $t>0$ and $x, y\in \boldsymbol{R}^d$. Since we are interested in the trace, we will consider the heat kernels on the diagonal set only. Then, as is also well known, setting $t=\varepsilon^2$ from the probabilistic point of view, the self-similarity of the Brownian motion implies that

$$
\begin{aligned}
(3.1)\qquad & p(\varepsilon^2, x, x) = (2\pi\varepsilon^2)^{-d/2}E\left[\exp\left(-\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right)\right] \\
& q(\varepsilon^2, x, x) = (2\pi\varepsilon^2)^{-d/2}E\left[\exp\left(\sqrt{-1}\,F^\varepsilon(x)-\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right)\right] \\
& F^\varepsilon(x) = \varepsilon\int_0^1 A(x+\varepsilon X_s)\circ dX_s\,.
\end{aligned}
$$

Here and hereafter we denote $E_{0;0}^{1;0}$ simply by E.

The argument in this paper is based on these probabilistic representations of the heat kernels.

In order to prove Theorem 1, we need the following two lemmas. The first one is related to the maximum of the pinned Brownian motion and might be a well known fact. See, e.g., [**9**]. But, since we will use it also in the proof of Theorem 2, we give it here.

Let P be the probability law of the d-dimensional pinned Brownian motion $X=\{X_s\}_{0\leqq s\leqq 1}$ such that $X_0=X_1=0$. Denote by η the maximum of X;

$$\eta = \max_{0\leqq s\leqq 1} |X_s|.$$

Then

LEMMA 1. *It holds that*

$$(3.2)\qquad P(\eta\geqq R) \leqq 2de^{-2R^2/d} \qquad \textit{for } R>0\,.$$

PROOF. Let $\{x_t\}_{t\geqq 0}$ be a 1-dimensional standard Brownian motion starting from 0 defined on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$ and $\{\xi_t\}_{t\geqq 0}$ be its maximum process,

$$\xi_t = \max_{0\leqq s\leqq t} x_s\,.$$

Then, by virture of the Lévy's work on the joint distribution of these stochastic processes, it is known that

$$P_1(x_t\in da,\ \xi_t\in db) = \left(\frac{2}{\pi t^3}\right)^{1/2}(2b-a)e^{-(2b-a)^2/2t}dadb$$

holds for $t>0$, $0\leqq b$ and $a\leqq b$. See, e.g., [**4**]. Since the probability law of each 1-dimensional pinned Brownian motion $\{X_s^i\}_{0\leqq s\leqq 1}$, $i=1, 2, \cdots, d$, coincides with the conditional probability $P_1(\cdot\,|\,x_1=0)$, it is easy to show

$$P(\max_{0\leq s\leq 1} X_s^i \geqq r) = e^{-2r^2}, \qquad i=1, 2, \cdots, d \text{ and } r>0.$$

Moreover the rotation invariance of the probability law of the Brownian motions implies that

$$P(\max_{0\leq s\leq 1} |X_s^i| \geqq r) \leqq 2e^{-2r^2}.$$

Therefore we get

$$\begin{aligned} P(\eta \geqq R) &\leqq P\Big(\bigcup_{i=1}^{d} \{\max_{0\leq s\leq 1} |X_s^i| \geqq R/\sqrt{d}\,\}\Big) \\ &\leqq \sum_{i=1}^{d} P(\max_{0\leq s\leq 1} |X_s^i| \geqq R/\sqrt{d}\,) \leqq 2de^{-2R^2/d}. \end{aligned}$$

The proof is completed.

The second lemma is concerned in the moment estimate of $F^{\varepsilon}(x)$.

LEMMA 2. *Under the assumption* (A1) *there exists a positive constant* C_1 *such that*

$$E[|F^{\varepsilon}(x)|^4] \leqq C_1\varepsilon^8(1+|x|^{4r}) \tag{3.3}$$

holds for every $x\in \boldsymbol{R}^d$ *and every* ε *with* $0<\varepsilon<1$.

PROOF. We first note that the pinned Brownian motion such that $X_0=X_1=0$ can be realized as the solution of the following stochastic differential equation based on the standard d-dimensional Brownian motion $\{w_t\}_{0\leq t\leq 1}$ defined on some probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$ (see, [**3**]):

$$dX_t^i = dw_t^i - \frac{X_t^i}{1-t}dt, \qquad X_0^i=0, \qquad i=1, 2, \cdots, d. \tag{3.4}$$

Then, by the Itô's formula, $F^{\varepsilon}(x)$ is equal to

$$\begin{aligned} &\varepsilon^2 \sum \int_0^1 dw_s^i \int_0^s \partial_j a_i(x+\varepsilon X_u) dw_u^j + \varepsilon^2 \sum \int_0^1 \frac{X_s^i}{1-s} ds \int_0^s \partial_j a_i(x+\varepsilon X_u) \frac{X_u^j}{1-u} du \\ &-\varepsilon^2 \sum \int_0^1 \frac{X_s^i}{1-s} ds \int_0^s \partial_j a_i(x+\varepsilon X_u) dw_u^j + \frac{\varepsilon^3}{2} \sum \int_0^1 dw_s^i \int_0^s \partial_j^2 a_i(x+\varepsilon X_u) du \\ &-\frac{\varepsilon^3}{2} \sum \int_0^1 \frac{X_s^i}{1-s} ds \int_0^s \partial_j^2 a_i(x+\varepsilon X_u) du \\ &-\varepsilon^2 \sum \int_0^1 dw_s^i \int_0^s \partial_j a_i(x+\varepsilon X_u) \frac{X_u^j}{1-u} du + \varepsilon^2 \sum_{i=1}^{d} \int_0^1 \partial_j a_i(x+\varepsilon X_u) du, \end{aligned} \tag{3.5}$$

where we have written simply $\sum$ for $\sum_{i,j=1}^{d}$ and ∂_j for $\partial/\partial x^j$. Now, noting that X_u is a Gaussian random variable of mean 0 and variance $u(1-u)$ and, therefore, that

$$E[|X_u|^{2m}] = (2m-1)!!(u(1-u))^m \qquad \text{for } m=1, 2, \cdots. \tag{3.6}$$

(3.3) follows by the standard method using the Hölder's inequality. We will show the calculations for the first and the second terms of (3.5).

For the first term, we use the moment inequality for martingales (see, [3]). Then there exists a constant $C>0$ such that

$$\tilde{E}\left[\left|\int_0^1 dw_s^i \int_0^s \partial_j a_i(x+\varepsilon X_u)dw_u^j\right|^4\right] \leqq C\tilde{E}\left[\left(\int_0^1\left|\int_0^s \partial_j a_i(x+\varepsilon X_u)dw_u^j\right|^2 ds\right)^2\right]$$
$$\leqq C\int_0^1 ds\, \tilde{E}\left[\left|\int_0^s \partial_j a_i(x+\varepsilon X_u)dw_u^j\right|^4\right].$$

Here $\tilde{E}$ denotes the expectation with respect to $\tilde{P}$ and we have used the Schwartz's inequality at the last line. Now, using the moment inequality and the Schwartz's inequality again, we obtain

$$\tilde{E}\left[\left|\int_0^1 dw_s^i \int_0^1 \partial_j a_i(x+\varepsilon X_u)dw_u^j\right|^4\right] \leqq C^2\int_0^1 ds\, \tilde{E}\left[\left(\int_0^s \partial_j a_i(x+\varepsilon X_u)^2 du\right)^2\right]$$
$$\leqq C^2\int_0^1 ds\int_0^s \tilde{E}[\partial_j a_i(x+\varepsilon X_u)^4]du\,.$$

Therefore the assumption (A1) and (3.6) imply that the fourth moment of the first term of (3.5) is bounded by constant$\times\varepsilon^8(1+|x|^{4r})$.

We can estimate the moment of the second term of (3.5) by using the Hölder's inequality and (3.6) as follows:

$$\tilde{E}\left[\left|\int_0^1 \frac{X_s^i}{1-s}ds\int_0^s \partial_j a_i(x+\varepsilon X_u)\frac{X_u^j}{1-u}du\right|^4\right]$$
$$\leqq \prod_{k=1}^4 \int_0^1 ds_k \left\{\tilde{E}\left[\left|\int_0^{s_k} \partial_j a_i(x+\varepsilon X_u)\frac{X_u^j}{1-u}du\right|^8\right]\right\}^{1/8}\left\{\tilde{E}\left[\left(\frac{X_{s_k}^j}{1-s_k}\right)^8\right]\right\}^{1/8}$$
$$\leqq C_1\left\{\int_0^1\left\{\tilde{E}\left[\int_0^s (\partial_j a_i(x+\varepsilon X_u))^{16}du\right]\tilde{E}\left[\int_0^s\left(\frac{X_u^j}{1-u}\right)^{16}du\right]\right\}^{1/16} s^{1/2}(1-s)^{-1/2}ds\right\}^4$$
$$\leqq C_2\left\{\int_0^1 ds(1+|x|^r)s(1-s)^{-15/16}ds\right\}^4 \leqq C_3(1+|x|^{4r}),$$

where C_i's are constants independent of ε and x.

The other terms can be estimated in a similar way. The proof is completed.

Now we are ready to give the proof of Theorem 1.

PROOF OF THEOREM 1. We first note that $q(\varepsilon^2, x, x)$ is real although its probabilistic representation is of a complex form. Then we have, by (3.1),

$$|q(\varepsilon^2, x, x)-p(\varepsilon^2, x, x)| = \left|\varepsilon^{-d}E\left[\{\cos(F^\varepsilon(x))-1\}\exp\left(-\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right)\right]\right|$$
$$\leqq \frac{1}{2}\varepsilon^{-d}E\left[F^\varepsilon(x)^2\exp\left(-\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right)\right].$$

Here we have used the elementary inequality

$$0 \leqq 1-\cos\theta \leqq \frac{1}{2}\theta^2 \qquad \text{for real } \theta .$$

Therefore, by the Schwartz's inequality and (3.3), we get

$$|q(\varepsilon^2, x, x)-p(\varepsilon^2, x, x)| \leqq C_1\varepsilon^{4-d}(1+|x|^{2r})\left\{E\left[\exp\left(-2\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right]\right\}^{1/2}.$$

Here C_1 is a constant independent of ε and x. We will denote such constants by C_i's.

Next let us denote by I_1 the indicator function of the set $\{\eta \geqq |x|/2\varepsilon\}$. Note that, if $\eta<|x|/2\varepsilon$, $|x+\varepsilon X_s| \geqq |x|/2$ for all s, $0\leqq s\leqq 1$. Then we get, by the assumption (V1) and (3.2),

$$\begin{aligned}
&E\left[\exp\left(-2\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right)\right] \\
&= E\left[\exp\left(-2\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right)(1-I_1)\right] + E\left[\exp\left(-2\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\right)I_1\right] \\
&\leqq E[\exp(-C_2\varepsilon^2|x|^{2k})(1-I_1)]+\exp(C_3\varepsilon^2)P(\eta \geqq |x|/2\varepsilon) \\
&\leqq \exp(-C_2\varepsilon^2|x|^{2k})+C_4\exp(-C_5|x|^2/\varepsilon^2)
\end{aligned}$$

for every x with $x/2 \in \boldsymbol{R}^d \setminus K$. Therefore we have proved that

$$\begin{aligned}
(3.7) \qquad &|q(\varepsilon^2, x, x)-p(\varepsilon^2, x, x)| \\
&\leqq C_1\varepsilon^{4-d}(1+|x|^{2r})\{\exp(-C_6\varepsilon^2|x|^{2k})+C_7\exp(-C_8|x|^2/\varepsilon^2)\}
\end{aligned}$$

holds for every x with $x/2 \in \boldsymbol{R}^d \setminus K$.

Now, since the integral of $|q(\varepsilon^2, x, x)-p(\varepsilon^2, x, x)|$ over a compact set in $\boldsymbol{R}^d$ is of $O(\varepsilon^{4-d})$, the assertion of the theorem is shown by integrating (3.7) over $\boldsymbol{R}^d$. The proof is completed.

We will end this section with the proof of Corollary 1.

PROOF OF COROLLARY 1. By the assumptions we have

$$(3.8) \qquad \operatorname{Tr} e^{-tH_0} = Ct^{-l}+o(t^{-l})$$

as $t\downarrow 0$ for some $C>0$, where $l=(d/2)(1+1/k)$. Moreover, by the Tauberian theorem, it holds that

$$N_0(\lambda) = \frac{C}{\Gamma(l+1)}\lambda^l+o(\lambda^l)$$

as $\lambda\to\infty$. Since Theorem 1 says that (3.8) holds if we replace H_0 with H, we see, by using the Tauberian theorem again, that $N(\lambda)$ has the same leading term as that of $N_0(\lambda)$. The proof is completed.

§ 4. Proof of Theorem 2.

Since $q(\varepsilon^2, x, x)$ is real, we have

$$(4.1)\qquad q(\varepsilon^2, x, x)-p(\varepsilon^2, x, x)$$
$$= (2\pi\varepsilon^2)^{-d/2}E\Big[\Big\{\cos(F^\varepsilon(x))-1+\frac{1}{2}F^\varepsilon(x)^2\Big\}\exp\Big(-\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\Big)\Big]$$
$$-\frac{1}{2}(2\pi\varepsilon^2)^{-d/2}E\Big[F^\varepsilon(x)^2\exp\Big(-\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\Big)\Big]$$
$$F^\varepsilon(x) = \varepsilon\sum_{i=1}^d \int_0^1 a_i(x+\varepsilon X_s)\circ dX_s^i .$$

By some calculations as in the proof of Lemma 2, we see that

$$E[|F^\varepsilon(x)|^8] \leqq C_1\varepsilon^{16}$$

holds for some $C_1>0$ by virtue of the boundedness of $\partial_j a^i$. Therefore, by the Schwartz's inequality and the elementary inequality

$$0 \leqq \frac{1}{2}\theta^2-1+\cos\theta \leqq \frac{1}{4!}\theta^4 ,$$

we see that the first term of the right hand side of (4.1) is bounded by

$$C_2\varepsilon^{8-d}\Big\{E\Big[\exp\Big(-2\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\Big)\Big]\Big\}^{1/2} .$$

Moreover, if $|x|$ is sufficiently large, this is bounded by

$$C_2\varepsilon^{8-d}(\exp(-C_3\varepsilon^2|x|^{2k})+C_4\exp(-C_5|x|^2/\varepsilon^2))$$

as is shown in the proof of Theorem 1. Here and hereafter C_i's denote the positive constants independent of ε and x. These show that the first term of the right hand side of (4.1) is negligible.

Now, to show Theorem 2, it is sufficient to consider the integral of the second term of the right hand side of (4.1):

$$\frac{1}{2}(2\pi\varepsilon^2)^{-d/2}\int_{R^d}E\Big[F^\varepsilon(x)^2\exp\Big(-\varepsilon^2\int_0^1 V(x+\varepsilon X_s)ds\Big)\Big]dx$$
$$= \frac{(2\pi)^{-d/2}}{2}\varepsilon^{-d(1+1/k)}\int_{R^d}E\Big[F^\varepsilon(\varepsilon_k x)^2\exp\Big(-\varepsilon^2\int_0^1 V(\varepsilon_k x+\varepsilon X_s)ds\Big)\Big]dx ,$$

where $\varepsilon_k=\varepsilon^{-1/k}$. To clarify the argument in the proof of Theorem 2, we need the following lemma. To mention it, define a random variable S by

$$S = \sum_{i,j=1}^d a_{ij}\int_0^1 X_s^j\circ dX_s^i .$$

Then we will show:

LEMMA 3. *$E[\varepsilon^{-2}F^\varepsilon(\varepsilon_k x)^2]$ converges to $E[S^2]$ as ε tends to 0 uniformly on*

$\{x; |x|\geqq r\}$ *for every* $r>0$.

The proof of this lemma is a little lengthy and so we give the proof of Theorem 2 before that of Lemma 3.

PROOF OF THEOREM 2. At first we will show

$$\lim_{\varepsilon\downarrow 0}\int_{R^d} E[S^2 g^\varepsilon(x)]dx = 0, \tag{4.2}$$

where

$$g^\varepsilon(x) = \exp\left(-\varepsilon^2\int_0^1 V(\varepsilon_k x+\varepsilon X_s)ds\right)-\exp(-|x|^{2k}v(x/|x|)).$$

For this we devide the integrand into two parts as in the proof of Theorem 1. We set

$$f_1^\varepsilon(x) = E[S^2 g^\varepsilon(x)I_2], \qquad f_2^\varepsilon(x) = E[S^2 g^\varepsilon(x)(1-I_2)],$$

where I_2 is the indicator function of the set $\{\eta\geqq\varepsilon^{-1/2}|x|\}$, $\eta=\max_{0\leqq s\leqq 1}|X_s|$. Note that $g^\varepsilon(x)$ is a uniformly bounded random variable in x and ε, and that any moment of the random variable S exists. Then we see, by the Hölder's inequality,

$$\begin{aligned} f_1^\varepsilon(x) &\leqq \{E[S^8]\}^{1/8}\{E[g^\varepsilon(x)^4]\}^{1/4}\{E[I_2]\}^{1/4} \\ &\leqq C_1\{P(\max_{0\leqq s\leqq 1}|X_s|\geqq\varepsilon^{-1/2}|x|)\}^{1/4}. \end{aligned}$$

Therefore, by Lemma 1, we get

$$\int_{R^d} f_1^\varepsilon(x)dx \leqq C_2\int_{R^d}\exp(-|x|^2/2\varepsilon d)dx = O(\varepsilon^{d/2}).$$

To estimate the integral of $f_2^\varepsilon(x)$, let us note the elementary inequality

$$|e^a-1| \leqq |a|(1+e^a)$$

and set

$$G^\varepsilon(x) = |x|^{2k}v(x/|x|)-\varepsilon^2\int_0^1 V(\varepsilon_k x+\varepsilon X_s)ds.$$

Then, by the Schwartz's inequality, we get

$$f_2^\varepsilon(x) \leqq C_4\exp(-|x|^{2k}v(x/|x|))\{E[G^\varepsilon(x)^2(1+\exp(G^\varepsilon(x))^2(1-I_2)]\}^{1/2}.$$

Moreover it is easy to see that, if $\eta<\varepsilon^{-1/2}|x|$,

$$(\varepsilon_k-1)|x| \leqq |\varepsilon_k x+\varepsilon X_s| \leqq 2\varepsilon_k|x|$$

holds for $x\neq 0$ and for $\varepsilon<1$ and, therefore, that

$$|G^\varepsilon(x)| \leqq C_4\varepsilon^{C_5}|x|^{C_6}$$

by the assumption (V2), where $C_5>0$ and $C_6\leqq 2k$. Now we have proved

$$\int_{R^d} f_2^\varepsilon(x)dx \leqq \int_{R^d} C_7\varepsilon^{C_5}(1+|x|^{C_6})\exp(-|x|^{2k}v(x/|x|)+C_8\varepsilon^{C_5}|x|^{C_6})dx$$

and, therefore, (4.2) because v is strictly positive by the assumptions.

Combining (4.2) with Lemma 3, we see that the left hand side of (2.2) is equal to

$$(2\pi)^{-d/2}\int_{R^d} E[S^2]\exp(-|x|^{2k}v(x/|x|))dx\,.$$

Finally we show the equality (2.3). By using the Itô's formula, we see that the expectation

$$E\left[\int_0^1 X_s^j\circ dX_s^i\int_0^1 X_s^k\circ dX_s^l\right]$$

does not vanish if and only if $j=k\neq i=l$ or $j=l\neq i=k$. Moreover, by the Lévy's formula for the characteristic function of the stochastic area (cf. [3] and [11]), it is easy to show

$$E\left[\left(\int_0^1 X_s^j\circ dX_s^i\right)^2\right] = -E\left[\int_0^1 X_s^j\circ dX_s^i\int_0^1 X_s^i\circ dX_s^j\right] = \frac{1}{12},$$

which implies the assertion of the theorem.

In the rest we prove Lemma 3.

PROOF OF LEMMA 3. Define the function $\gamma_{ij}(x)$ and the random variable $H^\varepsilon(x)$ by

$$\gamma_{ij}(x) = \frac{\partial}{\partial x^j}a_i(x)-a_{ij}$$

$$H^\varepsilon(x) = \sum_{i,j=1}^d\int_0^1\circ dX_s^i\int_0^s\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)\circ dX_u^j\,.$$

Then we have

$$E[\varepsilon^{-2}F^\varepsilon(\varepsilon_k x)^2]-E\left[\left(\sum_{i,j=1}^d a_{ij}\int_0^1 X_s^j\circ dX_s^i\right)^2\right] = 2E[S\cdot H^\varepsilon(x)]+E[H^\varepsilon(x)^2]\,.$$

In order to prove the lemma, it is sufficient to show that $E[H^\varepsilon(x)^2]$ converges to 0 uniformly on $\{x\,;\,|x|\geqq r\}$. For this we devide the expectation into two parts as before. Let I_3 be the indicator function of the set $\{\eta\geqq|x|/\varepsilon\}$. Then we have, by Lemma 1,

$$E[H^\varepsilon(x)^2 I_3] \leqq C_1\exp(-C_2|x|^2/\varepsilon^2),$$

which tends to 0 uniformly on $\{x\,;\,|x|\geqq r\}$.

Next let $\delta>0$ be given. There exists $R>0$ such that $|\gamma_{ij}(x)|<\delta$ for $|x|\geqq R$ by the assumption (A2). Moreover let us define the stopping time τ by

$$\tau = \inf\{s>0\,;\,|X_s|\geqq\varepsilon^{-1}|x|\}$$

and fix x with $|x|\geqq r$. Then we have

$$|\varepsilon_k x+\varepsilon X_s| \geqq (\varepsilon_k-1)|x| > R \tag{4.3}$$

and, therefore,

$$|\gamma_{ij}(\varepsilon_k x+\varepsilon X_s)| < \delta$$

for $s\leqq\tau$ and for sufficiently small ε. Taking (4.3) into account, we show the uniform convergence of $E[H^\varepsilon(x)^2(1-I_3)]$ to 0.

Let us remember that the pinned Brownian motion with $X_0=X_1=0$ is realized as the solution of the stochastic differential equation (3.4) and rewrite $H^\varepsilon(x)$. Then, by using the Itô's formula, we have

$$\begin{aligned} H^\varepsilon(x) &= \sum\int_0^1 dw_s^i\int_0^s \gamma_{ij}(\varepsilon_k x+\varepsilon X_u)dw_u^j-\sum\int_0^1\frac{X_s^i}{1-s}ds\int_0^s\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)dw_u^j \\ &\quad +\frac{\varepsilon}{2}\sum\int_0^1 dw_s^i\int_0^s\partial_j\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)du \\ &\quad -\frac{\varepsilon}{2}\sum\int_0^1\frac{X_s^i}{1-s}ds\int_0^s\partial_j\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)du-\sum\int_0^1 dw_s^i\int_0^s\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)\frac{X_u^j}{1-u}du \\ &\quad +\sum\int_0^1\frac{X_s^i}{1-s}ds\int_0^s\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)\frac{X_u^j}{1-u}du+\sum_{i=1}^d\int_0^1\gamma_{ii}(\varepsilon_k x+\varepsilon X_u)du \\ &= J_1+J_2+\cdots+J_7, \end{aligned}$$

where the simple $\sum$ denotes $\sum_{i,j=1}^d$. Now it is easy to see that $E[(J_l)^2(1-I_3)]$, $l=4, 6, 7$, are bounded by constant $\times\delta^2$ by virtue of (4.3) because these J_l's do not contain any stochastic integrals.

We will prove

$$E[(J_1)^2(1-I_3)] \leqq C_3\delta^2 . \tag{4.4}$$

The others can be estimated similarly. To prove (4.4), we note that

$$E[(J_1)^2(1-I_3)] \leqq E\left[\left(\sum\int_0^{1\wedge\tau}dw_s^i\int_0^s\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)dw_u^j\right)^2\right].$$

Then, by using the optional stopping theorem and the moment inequality for martingales (see, e.g., [**3**]), we obtain

$$\begin{aligned} E[(J_1)^2(1-I_3)] &\leqq C_4\sum E\left[\int_0^{1\wedge\tau}ds\left(\int_0^{s\wedge\tau}\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)dw_u^j\right)^2\right] \\ &\leqq C_5\sum\int_0^1 ds\, E\left[\left(\int_0^{s\wedge\tau}\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)dw_u^j\right)^2\right] \\ &\leqq C_6\sum\int_0^1 ds\, E\left[\int_0^{s\wedge\tau}\gamma_{ij}(\varepsilon_k x+\varepsilon X_u)^2du\right] \leqq C_7\delta^2 , \end{aligned}$$

which proves (4.4). The proof is completed.

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