J. Math. Soc. Japan
Vol. 42, No. 4, 1990

Manifolds which have two projective space bundle structures from the homotopical point of view

Dedicated to Professor Shôrô Araki on his 60th birthday

By Kouyemon IRIYE

(Received July 24, 1989)
(Revised Oct. 9, 1989)

Introduction

E. Sato [2] considered the structure of varieties which have two bundle structures whose fibers are projective spaces. It is interesting to consider this problem from a different point of view. In this paper we consider it from the homotopical point of view.

Let X be a manifold which has the following two bundle structures:

$$CP^r \xrightarrow{i_1} X \xrightarrow{p_1} CP^m, \qquad CP^s \xrightarrow{i_2} X \xrightarrow{p_2} Y,$$

where $r, s, m \geqq 1$ and Y is a manifold. The purpose of this paper is to classify the cohomology ring of X and describe the cohomology ring of Y in terms of that of X. But when we consider this problem from the homotopical point of view, it is difficult to distinguish fiber bundles from fibrations. Hence what we do in this paper is to consider manifolds with two maps $p_1: X \to CP^m$ and $p_2: X \to Y$, where Y is a manifold, whose homotopy fibers are complex projective spaces.

Before we state the results of this paper we list the non-trivial examples to see that there are many examples.

EXAMPLE. (1) By $H_{m,m}$ we denote the Milnor manifold, that is,

$$H_{m,m} = \{([x_0 : \cdots : x_m], [y_0 : \cdots : y_m]) \in CP^m \times CP^m \mid x_0 y_0 + \cdots + x_m y_m = 0\}.$$

The first and second projections $CP^m \times CP^m \to CP^m$ induce the two projective bundle structures on $H_{m,m}$.

$$\begin{array}{cccc}
 & CP^n = U(n+1)/U(1)\times U(n) & & U(2)/U(1)\times U(1) = CP^1 \\
 & \downarrow & & \downarrow \\
(2) & U(n+2)/U(1)\times U(1)\times U(n) & = & U(n+2)/U(1)\times U(1)\times U(n) \\
 & \downarrow & & \downarrow \\
 & CP^{n+1} = U(n+2)/U(1)\times U(n+1) & & U(n+2)/U(2)\times U(n).
\end{array}$$

In the case $n=1$ $Y=CP^2$. For $n>1$ $H^*(Y)=\langle t_2,\ t_1^2+t_1t_2\rangle$ (see § 1).

$$\begin{tikzcd}
CP^1 = Sp(1)/S^1 \arrow[d] & SO(3)/SO(2) = CP^1 \arrow[d] \\
Sp(2)/S^1\times S^1 \arrow[d] \arrow[r, phantom, "\cong"] & SO(5)/SO(2)\times SO(2) \arrow[d] \\
CP^3 = Sp(2)/S^1\times Sp(1) & SO(5)/SO(2)\times SO(3).
\end{tikzcd} \tag{3}$$

In this example $H^*(Y)=\langle t_2,\ t_1^2+t_1t_2\rangle$.

(4) Let ξ be an $r+1$-dimensional quaternion vector bundle over CP^m and $S(\xi)$ be its unit sphere bundle. Then we have the following example:

$$\begin{tikzcd}
 & S^{4r+3}/S^1 = CP^{2r+1} \arrow[d] & \\
CP^1 = S^3/S^1 \arrow[r] & S(\xi)/S^1 \arrow[r] \arrow[d] & S(\xi)/S^3 \\
 & CP^m. &
\end{tikzcd}$$

In this example $H^*(Y)=\langle t_1,\ t_2^2\rangle$.

(5) Consider a CP^1-bundle $p: CP^{2m+1}=S^{4m+3}/S^1\to HP^m=S^{4m+3}/S^3$ and an $(r+1)$-dimensional complex vector bundle ξ' over HP^m. Let $\xi=p^*(\xi')$ be the induced bundle over CP^{2m+1}. By $P(\xi)$, $P(\xi')$ we denote the associated projective bundles. Then we have

$$\begin{tikzcd}
 & CP^r \arrow[d] & \\
CP^1 \arrow[r] & P(\xi) \arrow[r] \arrow[d] & P(\xi') \\
 & CP^{2m+1}. &
\end{tikzcd}$$

In this example $H^*(Y)=\langle t_1^2,\ t_2\rangle$.

The first result of this paper is

THEOREM A. *If Y is homotopy equivalent to a complex projective space, then Y is homotopy equivalent to CP^r or CP^m. If Y is not homotopy equivalent to a complex projective space, then the fiber of $X\to Y$ is CP^1.*

By Theorem A we see that the cohomology ring of Y, if Y is not homotopy equivalent to a complex projective space, is generated by two elements which are in dimension 2 and 4 (see Proposition 5.1). To describe these elements in the cohomology ring of X we need Theorem B.

We consider the map $p_1\circ i_2: CP^1\to CP^m$ as an element of $\pi_2(CP^m)\cong H_2(CP^m)$

and under a certain isomorphism $\pi_2(CP^m)\cong \boldsymbol{Z}$ we denote it as β. $\pm\beta$ is a characteristic number of X. The second result of this paper is

THEOREM B. *If Y is not homotopy equivalent to a complex projective space, then $\beta=0$, ± 1 or ± 2.*

Theorem B says that there might exist examples with $\beta=\pm 2$, but we could not find such examples.

This paper is arranged as follows.

In §1 we prove elementary results by using the Serre spectral sequence. In §2 we prove Theorem A except for the case $s=r<m$. In this case we use mainly the Steenrod operation to prove it. In §4 we prove Theorem A in the exceptional case. §3 is devoted to investigate the relation between the cohomology rings of X and Y for the exceptional case. In §5 we describe the cohomology ring of Y in terms of that of X and prove Theorem B.

In this paper $H^*(X)$ stands for $H^*(X; \boldsymbol{Z})$.

The author would like to thank E. Sato and A. Kono for helpful conversation.

§1. Elementary results.

In this section we prove elementary results about the cohomology rings of X and Y. The K-comomology ring of X, which will be used in §4, is also determined.

PROPOSITION 1.1. *$H^*(X)\cong \boldsymbol{Z}[t_1, t_2]/(t_1^{m+1}, f)$, where*

$$f = t_2^{r+1}-c_1t_1t_2^r-\cdots-c_{r+1}t_1^{r+1}$$

for some $c_i\in \boldsymbol{Z}$, $1\leqq i\leqq r+1$.

PROOF. Consider the following Serre spectral sequence associated to the fiber bundle $CP^r\to X\to CP^m$:

$$E_2^{*,*}\cong H^*(CP^m)\otimes H^*(CP^r)\cong \boldsymbol{Z}[t_1, t_2]/(t_1^{m+1}, t_2^{r+1})\Longrightarrow H^*(X).$$

By the dimensional reason this spectral sequence collapses, from which we will obtain the proposition easily.

REMARK. If a fiber bundle $CP^r\to X\to CP^m$ is the projective bundle associated to a vector bundle over CP^m, $\{c_1, \cdots, c_{r+1}\}$ are the Chern classes of the vector bundle up to sign under the suitable choice of a generator t_2. But in general there is no canonical way of choosing a generator t_2. In Proposition 1.1 t_2 is only chosen so that $i_1^*(t_2)$ generates $H^2(CP^r)$. Therefore $\{c_1, \cdots, c_{r+1}\}$ is not determined uniquely.

In the examples in Introduction we choose t_1, t_2 so that $H^*(Y)$ can be described as simply as possible. See §5.

PROPOSITION 1.2. *Y is simply connected. $H^*(Y)$ is torsion free and $H^{\mathrm{odd}}(Y)=0$.*

PROOF. CP^s and X are simply connected, so is Y. We have the Serre exact sequence for any field k since CP^s and Y are simply connected:

$$0 \leftarrow H^3(Y;k) \leftarrow H^2(CP^s;k)(\cong k) \leftarrow H^2(X;k)(\cong k \oplus k) \xleftarrow{p^*} H^2(Y;k) \leftarrow 0\,.$$

If $H^3(Y;k)$ is isomorphic to k, $p^*: H^2(Y;k) \to H^2(X;k)$ is isomorphic by the above exact sequence. Since $H^*(X;k)$ is generated by elements of dimension 2, $p^*: H^*(Y;k) \to H^*(X;k)$ is epimorphic. This contradicts the fact that in dimension $\dim X$ $H^*(Y;k)=0$ and $H^*(X;k) \cong k$, which implies that $H^3(Y;k)=0$. Since k is any field, we have $H^3(Y)=0$. Then it is easy to obtain the desired result.

COROLLARY 1.3. *$H^*(Y)$ is a subring of $H^*(X)$ and $0 \to H^2(Y) \to H^2(X) \to H^2(CP^s) \to 0$ is a short exact sequence.*

COROLLARY 1.4. *$(s+1)|(r+1)$ or $(s+1)|(m+1)$ and $P(Y)=(1-t^{2r+2})(1-t^{2m+2})/(1-t^2)(1-t^{2s+2})$, where $P(Y)$ is the Poincaré polynomial of Y.*

PROOF. Since the Serre spectral sequence associated to the fiber bundle $CP^s \to X \to Y$ collapses, we have $P(CP^r)P(CP^m)=P(CP^s)P(Y)$. On the other hand $P(CP^s)=(1-t^{2s+2})/(1-t^2)$ and $P(Y)$ is a polynomial. Thus we have the desired result.

PROPOSITION 1.5. (1) *If $H^*(Y)$ is generated by an element, Y is homotopy equivalent to CP^r or CP^m.*

(2) *There are two generators t and u in $H^2(X)$ such that $\langle t, v\rangle \subset H^*(Y) \subset H^*(X)$, where $v=u^{s+1}+a_s u^s t+\cdots+a_1 u t^s$ for some $a_i \in \boldsymbol{Z}$, $1 \leqq i \leqq s$ and $\langle t, v\rangle$ denotes the subring generated by the elements t, v. (The fact that $H^*(Y)=\langle t, v\rangle$ will be proved in* §5.)

PROOF. (1) The generator t of $H^2(Y)$ induces a map $t: Y \to CP^{r+m-s}$, which is a homology equivalence. Since Y and CP^{r+m-s} are simply connected, t is a homotopy equivalence. In this case by Corollary 1.4 s must be equal to r or m, which implies (1).

(2) Since the Serre spectral sequence associated to the fiber bundle $CP^s \to X \to Y$ collapses, we have

$$H^*(X) \cong H^*(Y)[u]/(u^{s+1}+a_s u^s t+\cdots+a_1 u t^s - v)\,, \qquad v \in H^{2s+2}(Y)\,,$$

where $u \in H^2(X)$ is corresponding to a generator of $H^2(CP^s)$, which implies (2).

Next we consider the K-cohomology ring of X, which will be used in § 4. t_1 and t_2 $(\in H^2(X))$ determine complex line bundles over X and we denote them by the same symbols. We set

$$\tilde{t}_i = t_i - 1 \in \tilde{K}(X) \qquad \text{for } i=1, 2\,.$$

$\{1, \tilde{t}_2, \tilde{t}_2^2, \cdots, \tilde{t}_2^r\}$ are elements in $K^*(X)$ such that $\{p_1^*1, p_1^*\tilde{t}_2, \cdots, p_1^*\tilde{t}_2^r\}$ form a basis for $K^*(CP^r)$ over $K^*(pt)$. The Leray-Hirsch theorem says that $K^*(X)$ is a free $K^*(CP^m)$-module with a basis $\{1, \tilde{t}_2, \tilde{t}_2^2, \cdots, \tilde{t}_2^r\}$. Thus there is a unique relation such that

$$\tilde{f} = \tilde{t}_2^{r+1} - \tilde{c}_1\tilde{t}_2^r - \cdots - \tilde{c}_r\tilde{t}_2 - \tilde{c}_{r+1} = 0$$

with $\tilde{c}_i \in \tilde{K}(CP^m)$, $1 \leqq i \leqq r+1$, and we obtain

PROPOSITION 1.6. $K^*(X) \cong K^*(pt)[\tilde{t}_1, \tilde{t}_2]/(\tilde{t}_1^{m+1}, \tilde{f})$.

The relation $\tilde{f}$ is given as follows. Consider the Atiyah-Hirebruch spectral sequence:

$$H^*(X; K^*(pt)) \Longrightarrow K^*(X)\,.$$

By Proposition 1.1 this spectral sequence collapses and the Chern character ch: $K(X) \to H^*(X; \boldsymbol{Q})$ is monomorphic. Let us consider an element of $K(X) \otimes \boldsymbol{Q}$:

$$\tilde{f}'' = \{\log(1+\tilde{t}_2)\}^{r+1} - \sum_{i=1}^{r+1} c_i\{\log(1+\tilde{t}_1)\}^i\{\log(1+\tilde{t}_2)\}^{r+1-i}\,.$$

Since $\mathrm{ch}\,\tilde{f}''=f$, $\tilde{f}''=0$. Put $\ln(x)=x^{-1}\log(1+x)$, then we have

$$\tilde{f}' = \tilde{t}_2^{r+1} - \sum_{i=1}^{r+1} c_i(\ln\tilde{t}_1/\ln\tilde{t}_2)^i\tilde{t}_1^i\tilde{t}_2^{r+1-i}\,. \tag{1.7}$$

From this relation (1.7) recurrently we can get the relation $\tilde{f}$ since $\tilde{t}_2^{r+1}$ does not appear in the second term of (1.7). Now the first few terms of $\tilde{c}_i$ in f, for $r>1$, is given as follows:

$$\begin{aligned}\tilde{c}_i = {} & c_i\tilde{t}_1^i + \frac{1}{2}\{(i+1)c_{i+1} - ic_i + c_1c_i\}\tilde{t}_1^{i+1} \\ & + \frac{1}{24}\{(3i+1)(i+2)c_{i+2} - (6i+4)c_1c_{i+1} + 2c_2c_i + 4c_1^2c_i \\ & - 6(i+1)c_1c_i - 6(i+1)(2r-i+1)c_{i+1} + (3i^2+5i)c_i\}\tilde{t}_1^{i+2} + \cdots,\end{aligned} \tag{1.8}$$

where $c_i=0$ for $i>r+1$.

§ 2. Proof of Theorem A (elementary case).

As for the two generators of $H^*(X)$ described in Proposition 1.5, (2) we put $t=\alpha t_1+\beta t_2$ and $u=\delta t_1+\gamma t_2$ and fix this notation from now on. Of course there are ambiguities in choosing t_2 and u.

PROPOSITION 2.1. *Y is homotopy equivalent to a projective space or $s\leqq \min\{r, m\}$.*

PROOF. If $s>\min\{r, m\}$, then $m<s\leqq r$ or $r<s\leqq m$ by Corollary 1.4.

(1) We assume that $m<s\leqq r$. Since Y is a $2(m+r-s)$ ($<2r$) dimensional manifold,

$$t^r=\sum_{i=0}^{m}\binom{r}{i}\alpha^i\beta^{r-i}t_1^it_2^{r-i}=0,$$

which implies that $\beta=0$. We may regard $t=t_1$ and $u=t_2$. If $s<r$, this implies that t_2^{s+1} is expressed as another element. Of course this is impossible. If $s=r$, Y is homotopy equivalent to CP^m.

(2) Next we assume that $r<s\leqq m$. Since Y is a $2(m+r-s)$ ($<2m$) dimensional manifold, $t^m=0$. We investigate this condition in $H^*(X; \boldsymbol{C})\cong\boldsymbol{C}[t_1, t_2]/(t_1^{m+1}, f)$. Let $\omega\in\boldsymbol{C}$ be a root of the equation $f(1, x)=0$ and put $t_2'=t_2-\omega t_1$. Then we have

$$\begin{aligned} f(t_1, t_2)&=(t_2'+\omega t_1)^{r+1}-c_1t_1(t_2'+\omega t_1)^r-\cdots-c_{r+1}t_1^{r+1}\\ &=t_2'^{r+1}-c_1't_1t_2'^r-\cdots-c_r't_1^rt_2' \end{aligned}$$

for some $c_i'\in\boldsymbol{C}$. By using this relation we obtain

$$\begin{aligned} 0=t^m&=(\alpha t_1+\beta t_2)^m=(\alpha t_1+\beta(t_2'+\omega t_1))^m\\ &=((\alpha+\beta\omega)t_1+\beta t_2')^m=(\alpha+\beta\omega)^mt_1^m+(\text{terms which contain } t_2'), \end{aligned}$$

which implies that $\alpha+\beta\omega=0$. If $\beta=0$, then $\alpha=0$, which contradicts the assumption. Thus $\beta\neq0$ and $\omega=-\alpha/\beta$. On the other hand ω is an algebraic integer and α is prime to β, so we see that ω is an integer and $\beta=\pm1$. Thus we may regard $t=t_2$, $u=t_1$. If $s<m$, this implies that t_1^{s+1} is expressed as another element. This is impossible. If $s=m$, Y is homotopy equivalent to CP^r.

LEMMA 2.2. *We assume that $H^{2i}(Y; \boldsymbol{Z}/2)\cap\{u^i+x\mid x\in(t)\}=\emptyset$ for $s+1<i<2s+2$ and that $H^{2s+6}(Y; \boldsymbol{Z}/3)\cap\{u^{s+3}+x\mid x\in(t)\}=\emptyset$ if $s>1$. Then the fiber of the fiber bundle $X\to Y$ is CP^1.*

PROOF. First we investigate $H^*(X; \boldsymbol{Z}/2)$. Obviously

$$Sq^{2i}v=\binom{s+1}{i}u^{s+1+i}+(\text{terms which contain } t)\in H^*(Y; \boldsymbol{Z}/2)$$

for $0<i<s+1$. By assumption, $\binom{s+1}{i}\equiv0 \bmod 2$ for $0<i<s+1$, which implies that $s+1$ is a power of 2. So we put $s+1=2^k$, $k\geqq1$.

Next we investigate $H^*(X; \boldsymbol{Z}/3)$. By using P^1 instead of Sq^{2i} in the above argument we have

$$(s+1)u^{s+3}+(\text{terms which contain } t)$$
$$=2^k u^{2^k+2}+(\text{terms which contain } t)\in H^*(Y\,;\boldsymbol{Z}/3).$$

Thus we obtain that $k=1$, i.e., $s=1$, which completes the proof.

PROPOSITION 2.3. *If $H^*(X)\cong\boldsymbol{Z}[t_1, t_2]/(t_1^{m+1}, t_2^{r+1})$ and Y is not homotopy equivalent to a projective space, then the fiber of $X\to Y$ is CP^1 and $H^*(Y)=\langle t_1, t_2^2\rangle$, r is odd, or $H^*(Y)=\langle t_1^2, t_2\rangle$, m is odd.*

PROOF. Since Y is a $2(m+r-s)$-dimensional manifold, we have

$$\begin{aligned} t^{m+r-s+1} &= (\alpha t_1+\beta t_2)^{m+r-s+1} \\ &= \sum_{i=0}^{m+r-s+1}\binom{m+r-s+1}{i}\alpha^i\beta^{m+r-s+1-i}t_1^i t_2^{m+r-s+1-i} \\ &= \sum_{i=m-s+1}^{m}\binom{m+r-s+1}{i}\alpha^i\beta^{m+r-s+1-i}t_1^i t_2^{m+r-s+1-i}=0, \end{aligned}$$

which implies that $\alpha=0$ or $\beta=0$. So we may regard $t=t_1$ or t_2. It is sufficient to examine the case $t=t_1$.

Since $t=t_1$, we can assume that $u=t_2$. For we can choose $t_2\in H^2(X)$ so that $i_1^*(t_2)$ generates $H^2(CP^r)$. In this case we can see easily that $H^*(Y)=\langle t, v\rangle$ and that $(s+1)|(r+1)$, by using the Poincaré polynomial of Y. If $s=r$, $H^*(Y)$ is generated by t_1, which contradicts the assumption by Proposition 1.5. Thus we obtain that $s\neq r$ and $(s+1)|(r+1)$.

Now the first half of the proposition is trivial since the assumption of Lemma 2.2 is obviously satisfied.

As proved above

$$H^*(Y)=\langle t_1, v\rangle, \qquad v=t_2^2+at_1t_2, \qquad r+1=2(k+1).$$

If $a\neq 0$, then $t_1^i v^j$, $0\leqq i\leqq m$, $0\leqq j\leqq k$, and v^{k+1} are linearly independent in $H^*(Y)$, since

$$v^{k+1}=(t_2^2+at_1t_2)^{k+1}=(k+1)at_1t_2^{2k+1}+\text{lower terms}.$$

This contradicts the Poincaré polynomial of Y. Thus $v=t_2^2$, which completes the proof.

PROPOSITION 2.4. *If $s<r\leqq 2s$, $(s+1)|(m+1)$ and $s\neq m$, then the assumption of Lemma 2.2 is satisfied.*

PROOF. We see that $H^{2r+2}(Y)\cong\boldsymbol{Z}$ by the Poincaré polynomial. This implies that there is a relation between t^{r+1} and vt^{r-s}. In $(2r+2)$-dimension we have only one relation f between t and u, which should be expressed as

$$af=bt^{r+1}+cvt^{r-s}=cu^{s+1}t^{r-s}+ca_su^st^{r-s+1}+\cdots+ca_1ut^r+bt^{r+1},$$

with some $a, b, c\in\boldsymbol{Z}$, $a\neq 0$, $(b, c)\neq(0, 0)$. From this fact the assumption of

Lemma 2.2 is obviously satisfied.

The similar arguments establish the following propositions.

PROPOSITION 2.5. *If* $s<m\leqq 2s$, $(s+1)|(r+1)$ *and* $s\neq r$, *then the assumption of Lemma* 2.2 *is satisfied.*

PROPOSITION 2.6. *If* $s=m<r$, *then* Y *is homotopy equivalent to* CP^r *or* $s=1$.

The following two propositions will be easily proved.

PROPOSITION 2.7. *If* $2s<\min\{r, m\}$, *the assumption of Lemma* 2.2 *is satisfied.*

PROPOSITION 2.8. *If* $m\leqq s=r$, *then* Y *is homotopy equivalent to* CP^m. *If* $r\leqq s=m$, *then* Y *is homotopy equivalent to* CP^r.

§ 3. Preparation for exceptional case.

In this section we consider the case $r=s<m$. We assume that Y is not homotopy equivalent to a complex projective space. In this case the integral cohomology group of Y is given by

$$H^*(Y)=\boldsymbol{Z}[t, v']/(t^{s+1}-nv', (v')^{k+1}),$$

where $v'\in H^{2s+2}(Y)$ is a generator, $n\neq\pm 1$ and $m+1=(s+1)(k+1)$.

By P^i we denote the i-th Steenrod operation for an odd prime and Sq^{2i} for the prime 2.

LEMMA 3.1. *Let* Y *be a manifold whose cohomology is given by the following formula:*

$$H^*(Y)\cong\boldsymbol{Z}[x, y]/(x^{s+1}-ny, y^{k+1}),$$

where $\deg x=2$ *and* $s\geqq 1$. *Let* p *be a prime number such that* p *divides* n *but* p^2 *does not divide* n. *Then* $(s+1)|(p-1)$ *or* $k+1\leqq p$.

PROOF. Let $\tilde{Y}$ be a homotopy fiber of the map $x: Y\to CP^m\subset CP^\infty$, $m+1=(s+1)(k+1)$. Then we have

$$H^*(\tilde{Y}; \boldsymbol{Z}/p)\cong\Lambda[y']\otimes\boldsymbol{Z}/p[y]/(y^{k+1}),$$

where $\beta y'=y$ (in this proof β denotes the Bockstein coboundary operator). If we deny the result, we have

$$y^p=P^{s+1}\beta(y')=P^1\beta P^s(y')-s\beta P^{s+1}(y')=0,$$

which contradicts the assumption.

As in § 2 we put $t=\alpha t_1+\beta t_2$ and $u=\gamma t_1+\delta t_2$.

LEMMA 3.2. β *is prime to* n. *Hence, when we consider the problem in*

$H^*(\ ;R)$, $\beta^{-1}\in R$, *we can assume that* $t=t_2$ *and* $u=t_1$.

PROOF. Let p be a prime which divides β and n. p is prime to α since α is prime to β. Then

$$t^{s+1}\equiv\alpha^{s+1}t_1^{s+1}\equiv 0 \mod p\,,$$

which contradicts the above. Hence β is prime to n.

Put $t_2'=t$ and $u'=\pm(\beta u-\delta t)=t_1$. Since β is a unit in R, $i_1^*(t_2')$ generates $H^2(CP^r;R)$ and $i_2^*(u')$ generates $H^2(CP^s;R)$. Thus we can choose t_2' and u' as the generators instead of t_2 and u.

COROLLARY. *If* $n=0$, *then* $s=1$.

PROOF. By Lemma 3.2 $\beta=\pm1$. We can assume that $t=t_2$ and $u=t_1$ in $H^*(X)$. Thus the corollary follows Proposition 2.3.

NOTICE. For the following arguments (in the proofs of Propositions 3.3, 3.5 and 4.4) we make a convention. The statement such as $x\equiv0 \bmod' (t^g, u^h)$ will be used as follows. That is, if we express x as a linear combination of the basis $\{u^it^j: 0\leqq i\leqq m,\ 0\leqq j\leqq r\}$, then $x\equiv 0 \bmod' (t^g, u^h)$ means that the coefficients of u^it^j, $i<h$, $j<g$ are zero. Such a convention will be also used when we argue in K-theory.

From now on until the end of this section we assume that p is a prime number which divides n and that we discuss the problem in $H^*(\ ;\boldsymbol{Z}_{(p)})$ or in $H^*(\ ;\boldsymbol{Z}/p)$. By Lemma 3.2 $t=t_2$ and $u=t_1$. Then we see that

$$H^*(X;\boldsymbol{Z}_{(p)})=\boldsymbol{Z}_{(p)}[u,t]/(u^{m+1},f'),\qquad H^*(Y;\boldsymbol{Z}_{(p)})=\langle t,v\rangle\,,$$

where

$$\begin{aligned}f'&=t^{s+1}-nv=t^{s+1}-n(u^{s+1}+a_su^st+\cdots+a_1ut^s)\\&=t^{s+1}-\sum_{i=1}^{s+1}c_iu^it^{s+1-i}\,,\end{aligned}$$

n, $c_i\in\boldsymbol{Z}$ and $c_i=na_i$ for $1\leqq i\leqq s+1$, $a_{s+1}=1$. Of course a_i, c_i for $1\leqq i\leqq s+1$ are different from those in Propositions 1.1 and 1.5.

PROPOSITION 3.3. *If* p^2 *divides* n *or* $k+1$ *is prime to* p, *then* $v=u^{s+1}$ *in* $H^*(X;\boldsymbol{Z}_{(p)})$.

PROOF. Let $k+1=p^lh$, $(p,h)=1$. We can put

$$v^{p^l}=\sum_{i=0}^{s}e_iu^{(s+1)p^l-i}t^i\,,\qquad e_0^{-1}\in\boldsymbol{Z}_{(p)}\,.$$

Now we assume that there is an integer $d\geqq1$ such that p^{d-1} divides e_i, $1\leqq i\leqq s$, and that p^d divides e_i, $1\leqq i\leqq g-1$, but does not divide e_g for some g, $1\leqq g\leqq s$. Then

$$
\begin{aligned}
v^{k+1} &= e_0^h+\sum_{j=1}^{h}\binom{h}{j}\Big\{\sum_{i=1}^{s} e_i u^{(s+1)p^l-i}t^i\Big\}^j e_0^{h-j}u^{(s+1)p^l j}\\
&\equiv \sum_{j=1}^{h}\binom{h}{j}\Big\{\sum_{i=g}^{r} e_i u^{(r+1)p^l-i}t^i\Big\}^j e_0^{h-j}u^{(r+1)p^l j} \quad \mathrm{mod}\, p^d\\
&\equiv he_0^{h-1}e_g u^{m+1-g}t^g \quad \mathrm{mod}'(p^d,\, t^{g+1})\\
&= 0\,,
\end{aligned}
$$

which shows that p^d divides e_g. This contradicts the assumption. Therefore the proposition is proved when $k+1$ is prime to p. So we assume that p^2 divides n. When we describe

$$
v^{p^j}=\sum_{i=0}^{s} e_i^{(j)}u^{(s+1)p^j-i}t^i\,, \qquad e_0^{(j)^{-1}}\in \boldsymbol{Z}_{(p)},
$$

we will prove $e_i^{(j)}=0$, $1\leqq i\leqq s$, inductively. In the case $j=l$ this is proved as above. We argue as in above, so we take d and g.

$$
\begin{aligned}
v^{p^{j+1}} &= e_0^{(j+1)}u^{(s+1)p^{j+1}}=(v^{p^j})^p=\Big\{\sum_{i=0}^{s} e_i^{(j)}u^{(s+1)p^j-i}t^i\Big\}^p\\
&=(e_0^{(j)})^p u^{(s+1)p^{j+1}}+\Big\{\sum_{i=1}^{s} e_i^{(j)}u^{(s+1)p^j-i}t^i\Big\}^p\\
&\quad+\sum_{f=1}^{p-1}\binom{p}{f}(e_0^{(j)})^{p-f}\Big\{\sum_{i=1}^{s} e_i^{(j)}u^{(s+1)p^j-i}t^i\Big\}^f u^{(s+1)p^j f}\\
&\equiv \sum_{f=1}^{p-1}\binom{p}{f}(e_0^{(j)})^{p-f}\Big\{\sum_{i=g}^{s} e_i^{(j)}u^{(s+1)p^j-i}t^i\Big\}^f u^{(s+1)p^j f}\\
&\quad+\Big\{\sum_{i=1}^{s} e_i^{(j)}u^{(s+1)p^j-i}t^i\Big\}^p \quad \mathrm{mod}'(p^{d+1},\, u^{(s+1)p^{j+1}})\\
&\equiv p(e_0^{(j)})^{p-1}e_g^{(j)}u^{(s+1)p^{j+1}-g}t^g\\
&\quad+\Big\{\sum_{i=g}^{s} e_i^{(j)}u^{(s+1)p^j-i}t^i\Big\}^p \quad \mathrm{mod}'(p^{d+1},\, t^{g+1},\, u^{(s+1)p^{j+1}})\\
&\equiv p(e_0^{(j)})^{p-1}e_g^{(j)}u^{(s+1)p^{j+1}-g}t^g \quad \mathrm{mod}'(p^{d+1},\, t^{g+1},\, u^{(s+1)p^{j+1}}),
\end{aligned}
$$

which shows that p^d divides $e_g^{(j)}$. This contradicts the assumption. Thus the induction step is completed and the proposition is proved.

REMARK. In the above argument when we happen to know that $e_i^{(j)}\equiv 0$ mod p for $p(i+1)<2(s+1)$ and $1\leqq j\leqq l$, the result is true even if p^2 does not divide n.

PROPOSITION 3.4. *If* $s+1=bp^a$, $(b, p)=1$, *then* $b|(p-1)$ *or* $kb<p-1$.

PROOF. In $H^*(X;\boldsymbol{Z}/p)\cong \boldsymbol{Z}/p[t, u]/(t^{s+1}, u^{m+1})$ we consider the condition

$$P^i(v) \in \binom{s+1}{i} u^{s+1+i(p-1)} + (t) \subset H^*(Y\,;\,\boldsymbol{Z}/p) = \langle t,\, v\rangle\,.$$

First we consider the case when there appear two non-zero terms in the p-adic expansion of $s+1$. In this case, since $\binom{s+1}{p^a} \neq 0$ and $s+1<s+1+p^a(p-1) <2(s+1)$, the above inclusion leads to the contradiction. So we have $0<b<p$. Then

$$P^{p^a}(v) = \binom{s+1}{p^a} v^i$$

for some i or $s+1+p^a(p-1) \geqq m+1$, which induces the result.

PROPOSITION 3.5. *In $H^*(X; \boldsymbol{Z}/p)$ we have*

(1) $\quad v = u^{p^a} + a_{p^{a-1}} u^{p^{a-1}} t^{p^{a-1}(p-1)}$ *if* $b=1$,

(2) $\quad v = u^{s+1}$ *if* $1<b<p-1$,

(3) $\quad v = (u+\lambda t)^{s+1}$ *for some* $\lambda \in \boldsymbol{Z}/p$ *if* $b=p-1$.

PROOF. If p^2 divides n or $(k+1, p)=1$, we have already proved that $v=u^{s+1}$ in $H^*(X)$. We assume that p^2 does not divide n and that p divides $k+1$, which implies that b divides $p-1$. Especially, in this case, $u^j \neq 0$ for $j<p(s+1)$.

(1) We assume that there is an integer d, $p^{a-1}<d<p^a$, such that $a_s = \cdots = a_{d+1}=0$, $a_d \neq 0$. Since

$$\begin{aligned} P^d(v) &= P^d(u^{s+1} + a_d u^d t^{s+1-d} + \cdots) \\ &= a_d u^{dp} t^{s+1-d} + \cdots = a_d v^i t^{s+1-d} \end{aligned}$$

for some i, we have $dp=i(s+1)=ip^a$, i.e., $d=ip^{a-1}$, $1<i<p$. Then

$$\begin{aligned} P^{p^{a-1}}(v) &= P^{p^{a-1}}(u^{s+1} + a_d u^d t^{s+1-d} + \cdots) \\ &= i a_d u^{d+p^{a-1}(p-1)} t^{s+1-d} + \cdots = i a_d v^j t^{s+1-d} \end{aligned}$$

for some j. We have $ip^{a-1}+p^{a-1}(p-1)=j(s+1)=jp^a$, i.e., $i=j=1$, which contradicts the assumption. Similarily we see that $a_j=0$ for $j<p^{a-1}$.

(2) and (3) It is easy to prove that $a_i=0$ unless p^{a-1} divides i. So we can assume that $a=1$ without loss of generality. We deal with the case $a=0$ by considering $a_i=0$ unless p divides i.

We assume that there is an integer d, $(b-1)p<d<bp$, such that $a_s = \cdots = a_{d+1}=0$, $a_d \neq 0$. Since

$$\begin{aligned} P^d(v) &= P^d(u^{bp} + a_d u^d t^{bp-d} + \cdots) \\ &= a_d u^{dp} t^{bp-d} + \cdots = a_d v^i t^{bp-d} \end{aligned}$$

for some i, we have $dp=i(s+1)=ibp$, i.e., $d=ib$, $1<i<p$. Then

$$P^1(v) = P^1(u^{bp}+a_d u^d t^{bp-d}+\cdots)$$
$$= iba_d u^{d+p-1}t^{bp-d}+\cdots = iba_d v^j t^{bp-d}$$

for some j. We have $ib+(p-1)=j(s+1)=jbp$, i.e., $p-1=b(pj-i)$. This implies that $j=1$, $ib=bp-p+1$. Thus $d=(bp-p+1)p=(b-1)p+1$. Then by the remark to Proposition 3.3 we see that $v=u^{bp}$ if $b<(p-1)$, which completes the case (2).

Now we consider the case (3) and show that the assumption $a_{(b-1)p+1}\neq 0$ leads the contradiction.

First we will prove that

$$v = u^{bp}+\sum_{i=1}^{p-1} u^{(b-i)p+i}t^{i(p-1)} \tag{3.6}$$

if we assume that $a_{(b-1)p+1}\neq 0$. We look at the condition that $P^1(v)$ is in $H^*(Y;\boldsymbol{Z}/p)$ more carefully. We express v as

$$v = u^{bp}+\sum_{i=1}^{(b-1)p+1} a_i u^i t^{bp-i}$$

with $a_{(b-1)p+1}\neq 0$. Then

$$P^1(v) = \sum_{i=1}^{(b-1)p+1} a_i\{iu^{i+p-1}t^{bp-i}-iu^i t^{bp-i+p-1}\}$$
$$= \sum_{i=p}^{(b-1)p+1}\{(i+1)a_{i-p+1}-ia_i\}u^i t^{bp-i+p-1}+\sum_{i=(b-1)p+2}^{bp}(i+1)a_{i-p+1}u^i t^{bp-i+p-1}$$
$$= a_{(b-1)p+1}vt^{p-1},$$

which implies that

$$(i+1)a_{i-p+1} = 0 \qquad \text{for } (b-1)p+2\leqq i<bp,$$
$$a_{(b-1)p+1}a_i = (i+1)a_{i-p+1}-ia_i \qquad \text{for } p\leqq i\leqq(b-1)p+1.$$

Thus we obtain the condition (3.7) about a_i's:

$$a_i = 0 \qquad \text{for } (b-j)p+j+1\leqq i<(b-j+1)p, \tag{3.7}$$
$$ia_i = (i-1+a_{(b-1)p+1})a_{i+p-1} \qquad \text{for } 1\leqq i\leqq(b-2)p+2.$$

Next we consider the condition that $P^p(v)$ is in $H^*(Y;\boldsymbol{Z}/p)$.

$$P^p(v) = bu^{2bp}+(b-1)a_{(b-1)p+1}u^{2bp-p+1}t^{p-1}$$
$$+\sum_{i=1}^{(b-1)p} a_i\sum_{j=1}^{p}\binom{i}{j}\binom{bp-i}{p-j}u^{i+j(p-1)}t^{bp-i+(p-j)(p-1)}$$
$$= -u^{2bp}-2a_{(b-1)p+1}u^{2bp-p+1}t^{p-1}$$
$$+\sum_{i=1}^{(b-1)p} a_i\sum_{j=0}^{p}\binom{i}{j}\binom{bp-i}{p-j}u^{i+j(p-1)}t^{bp-i+(p-j)(p-1)}$$
$$= -\{u^{bp}+a_{(b-1)p+1}u^{(b-1)p+1}t^{p-1}+\sum_{i=1}^{(b-1)p} a_i u^i t^{bp-i}\}^2$$

$$= -u^{2bp}-2a_{(b-1)p+1}u^{2bp-p+1}t^{p-1}-a^2_{(b-1)p+1}u^{2(b-1)p+2}t^{2p-2}$$
$$-2\{u^{bp}+a_{(b-1)p+1}u^{(b-1)p+1}t^{p-1}\}\sum_{i=1}^{(b-1)p} a_i u^i t^{bp-i}-\left\{\sum_{i=1}^{(b-1)p} a_i u^i t^{bp-i}\right\}^2,$$

which implies that

$$a_{(b-2)p+2} = a^2_{(b-1)p+1}$$
$$a_{(b-2)p+1} = 2a_{(b-1)p+1}a_{(b-1)p}\,.$$

From this equation and (3.7) we obtain that $a_{(b-1)p+1}=a_{(b-2)p+2}=1$ and that $a_{(b-2)p+1}=a_{(b-1)p}=0$. Then (3.7) induces:

$$\text{(3.8)}\qquad a_i = 0 \qquad \text{for } (b-j)p+j+1\leqq i<(b-j+1)p\,,$$
$$ia_i = ia_{i+p-1} \qquad \text{for } 1\leqq i\leqq (b-j+1)p+2\,.$$

Then inductively the assertion (3.6) follows (3.8) and the above equation of $P^p(v)$.

Now we consider the formula (3.6) in $H^*(X;\boldsymbol{Z}_{(p)})$. In this case $k+1=p$ by Lemma 3.1 implies

$$v = u^{bp}+\sum_{i=1}^{p-1} e_i u^{bp-i(p-1)}t^{(p-1)i}+p\Psi\,,$$

and $e_i\equiv 1 \bmod p$. Since $k+1=p$, we have

$$0=v^p=\left\{u^{bp}+\sum_{i=1}^{p-1} e_i u^{bp-i(p-1)}t^{(p-1)i}+p\Psi\right\}^p$$
$$\equiv \left\{u^{bp}+\sum_{i=1}^{p-1} e_i u^{bp-i(p-1)}t^{(p-1)i}\right\}^p \bmod p^2$$
$$\equiv \sum_{j=1}^{p-1}\binom{p}{j}u^{bp(p-j)}\left\{\sum_{i=1}^{p-1} e_i u^{bp-i(p-1)}t^{(p-1)i}\right\}^{p-j}+\left\{\sum_{i=1}^{p-1} e_i u^{bp-i(p-1)}t^{(p-1)i}\right\}^p \bmod p^2$$
$$\equiv (p+ne_1^p)e_1u^{bp^2-p+1}t^{p-1}$$
$$+\left\{(p+ne_1^p)e_2+\binom{p}{2}e_1^2\right\}u^{bp^2-2p+2}t^{2p-2} \quad \bmod'(p^2,\ t^{3(p-1)}),$$

which means that $(p+ne_1^p)e_1\equiv(p+ne_1^p)e_2+\binom{p}{2}e_1^2\equiv 0 \bmod p^2$. But this contradicts the fact that $e_1\equiv 1 \bmod p$. Thus we proved that $a_{(b-1)pa+pa-1}=0$ in $H^*(X;\boldsymbol{Z}/p)$.

Then it is easy to prove that $a_i=0$ unless p^a divides i and that $v=(u+\lambda t)^{s+1}$ for some $\lambda\in\boldsymbol{Z}/p$.

By the remark to Proposition 3.3 with Proposition 3.4 we have

COROLLARY. *Let p be an odd prime and $s+1=bp^a$ with $b<p-1$, then $v=u^{s+1}$ in $H^*(X;\boldsymbol{Z}_{(p)})$.*

§ 4. Proof of Theorem A (exceptional case).

In this section we prove Theorem A for the exceptional case. That is, we will prove that if a manifold X has two bundle structures: $CP^s \to X \to CP^m$, $CP^s \to X \to Y$ and Y is not homotopy equivalent to a projective space, then $s=1$.

PROPOSITION 4.1. *If 2 divides n, then $s=1$.*

PROOF. By Proposition 3.4 $s+1=2^a$ for some $a \geqq 1$. If 4 divides n, by Proposition 3.3 we see that $v=u^{2^a}$ in $H^*(X; \boldsymbol{Z}/2)$. Applying the secondary cohomology operation to $v \in H^*(Y; \boldsymbol{Z}/2)$ we obtain a contradiction if $a>1$ (see below).

If 4 does not divide n, Lemma 3.1 says that $s=1$ or $k=1$. So let us consider the latter case. By Proposition 3.5 v is expressed as

$$v = u^{2^a} + a' u^{2^{a-1}} t^{2^{a-1}}$$

in $H^*(X; \boldsymbol{Z}/2)$. If $a \geqq 3$, we can apply the secondary cohomology operation $\Phi_{0, a-1}$ to the above equation (see [**1**, Theorem 4.5.1]). Thus we obtain

$$\Phi_{0, a-1}(v) = u^{2^a + 2^{a-1}} + a'(u^{2^{a-1}+2^{a-2}} t^{2^{a-1}} + u^{2^{a-1}} t^{2^{a-1}+2^{a-2}}),$$

which contradicts the fact that $\Phi_{0, a-1}(v) \in H^*(Y; \boldsymbol{Z}/2) = \langle t, v \rangle$.

Now we consider the case $s=3$ and $k=1$ in $H^*(X; \boldsymbol{Z}_{(2)})$. Then we have

$$t^4 = nv = n(u^4 + a_3 u^3 t + a_2 u^2 t^2 + a_1 u t^3) = c_4 u^4 + c_3 u^3 t + c_2 u^2 t^2 + c_1 u t^3 .$$

By Proposition 3.5 we see that $a_1 \equiv a_3 \equiv 0 \bmod 2$. If $a_2 \equiv 1 \bmod 2$, a coefficient of $\tilde{c}_2$ does not belong to $\boldsymbol{Z}_{(2)}$ by a routine argument using the formula (1.8). Thus $a_1 \equiv a_2 \equiv a_3 \equiv 0 \bmod 2$, which leads the contradiction as above.

Now we restrict ourselves to the case n is divisible by an odd prime p.

PROPOSITION 4.2. *Let p be an odd prime which divides n. Then $s+1<p$.*

PROOF. Let $s+1=bp^a$ with $a \geqq 1$. By Proposition 3.3 and its corollary we see that $v=(u+\lambda t)^{bp^a}$ in $H^*(X; \boldsymbol{Z}/p)$. As in the proof of Proposition 4.1 if we apply the secondary operation to this equation (see [**3**, Theorem 4.3]), we obtain the contradiction.

LEMMA 4.3. *Let p be an odd prime which divides n. Then $v=u^{s+1}$ in $H^*(X; \boldsymbol{Z}_{(p)})$.*

PROOF. This lemma is already proved except for the case $s+1=p-1$ and $k+1=p^l h$, $(p, h)=1$, $l \geqq 1$. By the corollary to Proposition 3.5 and the proof of Proposition 3.3 we have

$$v^{p^{l-1}} = (u^{p^{l-1}} + \lambda^{p^{l-1}} t^{p^{l-1}})^{s+1} + p\Psi, \qquad v^{p^l} = u^{p^l(s+1)},$$

in $H^*(X; Z_{(p)})$. By the induction on l we can prove the lemma. But for the simplicity we prove only the case $l=1$.

$$
\begin{aligned}
0 = v^p &= \{(u+\lambda t)^{s+1}+p\Psi\}^p \equiv (u+\lambda t)^{p(s+1)} \\
&\equiv \sum_{i=0}^{p(s+1)} \lambda^i \binom{p(s+1)}{i} u^{p(s+1)-i} t^i \\
&\equiv \sum_{i=1}^{s} \lambda^i \binom{p(s+1)}{i} u^{p(s+1)-i} t^i + \sum_{i=0}^{s} \lambda^{i+s+1} n \binom{p(s+1)}{s+1+i} u^{(p-1)(s+1)-i} \{(u+\lambda t)^{s+1}+p\Psi\} t^i \\
&\equiv \sum_{i=1}^{s} \lambda^i \binom{p(s+1)}{i} u^{p(s+1)-i} t^i + \lambda^p n \binom{p(s+1)}{p} u^{(p-1)(s+1)-1} \sum_{i=0}^{s+1} \lambda^i \binom{s+1}{i} u^{(s+1)-i} t^{i+1} \\
&\equiv \sum_{i=1}^{s} \left\{ \lambda^i \binom{p(s+1)}{i} + n\lambda^{p+i-1} \binom{p(s+1)}{p} \binom{s+1}{i-1} \right\} u^{p(s+1)-i} t^i \mod p^2 ,
\end{aligned}
$$

which implies that

$$\lambda p(s+1) + n\lambda^p \binom{p(s+1)}{p} \equiv 0 \mod p^2$$

and

$$\lambda^2 \binom{p(s+1)}{2} + n\lambda^{p+1} \binom{p(s+1)}{p} (s+1) \equiv 0 \mod p^2 .$$

From this equation we obtain $\lambda \equiv 0 \mod p$, which implies that $v = u^{s+1}$ in $H^*(X; \boldsymbol{Z}_{(p)})$ by the remark to Proposition 3.3.

Now we will complete the proof of Theorem A.

PROPOSITION 4.4. *Let n be an odd integer, then $s=1$.*

PROOF. Let p be the smallest prime which divides n. Then $s+1<p$ by Proposition 4.2. What we have proved up to now is that

$$(\alpha t_1 + \beta t_2)^{s+1} = n t_1^{s+1} , \qquad n \in \boldsymbol{Z} ,$$

which induces a relation in $H^*(X)$:

$$t_2^{s+1} + \sum_{i=1}^{s} \binom{s+1}{i} \left(\frac{\alpha}{\beta}\right)^i t_1^i t_2^{r+1-i} + \frac{\alpha^{s+1}-n}{\beta^{s+1}} = 0 .$$

Therefore these coefficient must be integers, i.e., $\beta = \pm 1$ or $s=1$, $\beta = \pm 2$. We consider the case $\beta = \pm 1$. By changing the generator t_2 we may assume that

$$t_2^{s+1} = n t_1^{s+1} \qquad \text{in } H^*(X).$$

Thus by (1.7) we have

$$\tilde{t}_2^{s+1} = n(\ln \tilde{t}_1 / \ln \tilde{t}_2)^{s+1} \tilde{t}_1^{s+1} \equiv n(\ln \tilde{t}_1)^{s+1} \tilde{t}_1^{s+1} \qquad \mathrm{mod}' (\tilde{t}_1^{2s+2}, \tilde{t}_2).$$

Since $s+1 < p \leqq n$,

$$n(\ln \tilde{t}_1)^{s+1} \in \boldsymbol{Z}[\tilde{t}_1]/(\tilde{t}_1^{s+1})$$

if and only if

$$(\ln \tilde{t}_1)^{s+1} \in \boldsymbol{Z}[\tilde{t}_1]/(\tilde{t}_1^{s+1}).$$

Let q be a prime which divides s. Then the coefficient of $\tilde{t}_1^{q-1}$ in the expansion of $(\ln \tilde{t}_1)^{s+1}$ is

$$(-1)^{q-1}\frac{s+1}{q}+\frac{b}{a}, \qquad (q, a)=1,$$

which is not an integer since q is prime to $s+1$. Therefore s must be equal to 1.

Moreover in the case $\beta=\pm1$ we show that $m=3$. Since $s=1$ we have

$$\tilde{t}_2^2 = n(\tilde{t}_1^2-\tilde{t}_1^3)(1+\tilde{t}_2)+\frac{n(11-5n)}{12}\tilde{t}_1^4-\frac{5n(1-n)}{6}\tilde{t}_1^5$$
$$+\left\{\frac{n(11-3n)}{12}\tilde{t}_1^4+\frac{n(3n-5)}{6}\tilde{t}_1^5\right\}\tilde{t}_2+\cdots.$$

Since n in an odd integer, one of the above coefficients can not be an integer. Thus $m=2k+1\leqq3$, which implies that $m=3$.

§ 5. Cohomology ring of Y.

In this section we describe the cohomology ring of Y in terms of that of X and prove Theorem B. We assume that Y is not homotopy equivalent to a projective space throughout this section. When we change a generator u of $H^*(X)$ for $u'=u+bt$, v changes into $v'=u'^2+(a-2b)u't+(b^2-ab)t^2$, where $v=u^2+aut$. Therefore we choose the generators of $H^*(X)$ so that

$$t = \alpha t_1+\beta t_2, \qquad u = \gamma t_1+\delta t_2,$$
$$t_1 = \pm(\delta t-\beta u), \qquad t_2 = \pm(-\gamma t+\alpha u),$$
$$v = u^2 \qquad \text{or} \qquad u^2+ut,$$

where $\alpha\delta-\beta\gamma=\pm1$.

PROPOSITION 5.1. *For $m\leqq r$ we have*

$$t_1^{m+1} = g_1(t, v) = \sum_{i=0}^{[(m+1)/2]} a_i t^{m+1-2i}v^i,$$
$$f(t_1, t_2) = g_2(t, v) = t_2^{r+1}-\sum_{i=1}^{r+1} c_i t_1^i t_2^{r+1-i} = \sum_{i=0}^{[(r+1)/2]} b_i t^{r+1-2i}v^i,$$

for some $a_i, b_i\in\boldsymbol{Z}$. For $r<m$ we have

$$t_1^{m+1}+g_2(t, v)h(t, v)u = g_1(t, v) = \sum_{i=0}^{[(m+1)/2]} a_i t^{m+1-2i}v^i,$$
$$f(t_1, t_2) = g_2(t, v) = t_2^{r+1}-\sum_{i=1}^{r+1} c_i t_1^i t_2^{r+1-i} = \sum_{i=0}^{[(r+1)/2)]} b_i t^{r+1-2i}v^i,$$

for some $a_i, b_i\in\boldsymbol{Z}$ and $h(t, v)\in\boldsymbol{Z}[t, v]$. With this notaion we have

$$H^*(Y) = \langle t, v\rangle \cong \boldsymbol{Z}[t, v]/(g_1, g_2).$$

PROOF. We prove only the case $r<m$, for the proof of the other cases are

quite similar.

By the Poincaré polynomial of Y it is easy to see that $H^{2r+2}(Y)\cong \boldsymbol{Z}^{[(r+1)/2]}$. On the other hand $\langle t, v\rangle^{2r+2}=\{t^{r+1}, t^{r-1}v, \cdots, t^{r+1-2[(r+1)/2]}v^{[(r+1)/2]}\}$. Therefore there must be a relation among these elements and the relation corresponds to $f=0$.

Next in dimension $2m+2$ we have

$$\operatorname{rank} H^{2m+2}(Y) = [r/2]$$

and

$$\text{generators of } \langle t, v\rangle^{2m+2} = \{t^{m+1}, t^{m-1}v, \cdots, t^{m+1-2[(m+1)/2]}v^{[(m+1)/2]}\}.$$

In this dimension there are $[(m-r)/2]+1$ relations which are induces by the relation $\sum a_i t^{m+1-2i}v^i=0$. If there were no other relations, we would have

$$\begin{aligned}\operatorname{rank} H^{2m+2}(Y) &\geqq \operatorname{rank}\langle t, v\rangle^{2m+2}\\ &\geqq [(m+1)/2]+1-[(m-r)/2]-1 = [r/2]+1\,.\end{aligned}$$

The last equation follows by the fact that m or r is odd. Therefore there must be a new relation among these elements and the relation can be written as

$$at_1^{m+1}+f_1(t_1, t_2)h'(t_1, t_2) \tag{5.2}$$

in $H^*(X)$. To see that this relation can be written as in the proposition we use the following elementary facts, which we state as a lemma.

LEMMA 5.3. (1) *Let R be a ring. $R[t_1, t_2]=R[t, u]$ is a free module over $R[t, v]$ with a basis $\{1, u\}$.* (2) *Let R be an integral domain. Let $f\in R[t_1, t_2]$ and $g\in R[t, v]$ such that $fg\in R[t, v]$, then $f\in R[t, v]$.*

By Lemma 5.3 (1) the relation (5.2) can be choosen so that

$$at_1^{m+1}+g_2(t, v)h(t, v)u \tag{5.2$'$}$$

is a new relation among the elements $\{t^{m+1}, t^{m-1}v, \cdots, t^{m+1-2[(m+1)/2]}v^{[(m+1)/2]}\}$. Now we will see that $a=\pm 1$ in (5.2$'$). Let p be a prime dividing a. We consider the relation (5.2$'$) in $H^*(X; \boldsymbol{Z}/p)$. The relation (5.2$'$) induces a relation $g_2(t, v)h(t, v)u$ among the elements $\{t^{m+1}, t^{m-1}v, \cdots, t^{m+1-2[(m+1)/2]}v^{[(m+1)/2]}\}$ by the naturality, which implies that $h(t, v)=0$ in $H^*(X; \boldsymbol{Z}/p)$. But of course this is impossible. Hence we see that $a=\pm 1$ in (5.2$'$), which completes the proof of the first part of the proposition in the case $r<m$.

Here we remark about the statement of the proposition in the case $m<r$. In this case there are a lot of ways to choose a relation $f(t_1, t_2)$. Therefore the relation $f(t_1, t_2)$ in Proposition 5.1 should be modified.

To prove that $H^*(Y)=\langle t, v\rangle$ it is sufficient to see that $\operatorname{rank} H^*(Y)=\operatorname{rank}\langle t, v\rangle$ and that an element of $\langle t, v\rangle$ is divisible in $\langle t, v\rangle$ if and only if it is divisible in $H^*(X)$. But these facts are trivial.

By making use of Proposition 5.1 we will prove Theorem B. First we consider the case $m \leqq r$, which is divided into four cases.

Case I. m is even and r is odd. Since by Proposition 5.1 $t=\alpha t_1+\beta t_2$ divides t_1^{m+1}, $\alpha=\pm 1$ and $\beta=0$. Therefore under the suitable choice of generators of $H^*(X)$

$$t=t_1, \quad u=t_2, \qquad g_1(t, v)=t^{m+1}.$$

Case II. m is odd and r is even. Since by Proposition 5.1 $t=\alpha t_1+\beta t_2$ divides $f(t_1, t_2)$, $\beta=\pm 1$. Therefore under the suitable choice of generators of $H^*(X)$ we have $t=t_2$, $u=t_1$.

When $v=u^2+aut$, $a \neq 0$, the formula $t_1^{m+1}=g_1(t, v)$ is written as follows:

$$u^{2m'+2}=\sum_{i=0}^{m'+1} a_i t^{2m'+2-2i}(u^2+aut)^i,$$

where $m+1=2m'+2$. But this is impossible. Thus we have

$$t=t_2 \quad u=t_1, \quad v=u^2 \qquad g_1(t, v)=v^{[(m+1)/2]}.$$

Under this choice of the generators we have $c_{\text{odd}}=0$.

Case III. m and r are odd and $v=u^2$. In this case the formula $t_1^{m+1}=g_1(t, v)$ is written as follows:

$$(\delta t-\beta u)^{m+1}=\sum_{i=0}^{(m+1)/2} a_i t^{m+1-2i} u^{2i}.$$

From this formula we have $(\beta, \delta)=(\pm 1, 0)$ or $(0, \pm 1)$.

Case IV. m and r are odd and $v=u^2+ut$. In this case the formula $t_1^{m+1}= g_1(t, v)$ is written as follows:

$$\begin{aligned}(\delta t-\beta u)^{m+1} &= \sum_{i=0}^{m+1}\binom{m+1}{i}\delta^i(-\beta)^{m+1-i}t^i u^{m+1-i} \\ &= \sum_{i=0}^{(m+1)/2} a_i t^{m+1-2i}(u^2+ut)^i=\sum_{k=0}^{m+1}\left\{\sum_{i+j=k,\, i \leqq (m+1)/2} a_i\binom{i}{j}\right\} t^{m+1-k}u^k.\end{aligned}$$

Put $m+1=2m'+2$. From the above equation we have $a_{m'+1}=\beta^{2m'+2}$ and $a_{m'+1}\binom{m'+1}{m'}=-(2m'+2)\delta\beta^{2m'+1}$, which implies that $\beta^{2m'+1}(\beta+2\delta)=0$. Since β is prime to δ, we obtain $\beta=0$ or $(\beta, \delta)=\pm(-2, 1)$.

Before we proceed to consider the case $r<m$, we prepare an elementary lemma. Let p be a prime number. We write $\nu_p(n)$ for the exponent to which the prime p occurs in the decomposition of n into prime powers.

LEMMA 5.4. *For $i<m$ if p is an odd prime, $i<m-1$ if $p=2$, we have*

$$\nu_p\left(\binom{m+1}{i}p^{m-i}\right) \geqq \nu_p(m+1)+1.$$

PROOF.

$$\nu_p\left(\binom{m+1}{i}p^{m-i}\right) \geqq \nu_p(m+1)-\nu_p(m+1-i)+m+1-i-1$$
$$\geqq \nu_p(m+1)-\nu_p(m+1-i)+\nu_p(m+1-i)+2-1$$
$$= \nu_p(m+1)+1\,.$$

Case I. r is even and m is odd. Since $t=\alpha t_1+\beta t_2$ divides $f(t_1, t_2)$, $\beta=\pm 1$.

Case II. r is odd, m is even and $v=u^2$. Put $m=2m'$.

$$(\delta t-\beta u)^{m+1} = \sum_{i=0}^{m+1}\binom{m+1}{i}\delta^i(-\beta)^{m+1-i}t^i u^{m+1-i}$$
$$= -\sum_{i=0}^{m'}\binom{m+1}{2i}(\delta^2t^2)^i(\beta^2v)^{m'-i}\beta u+\sum_{i=0}^{m'}\binom{m+1}{2i+1}(\delta t)^{2i+1}(\beta^2 v)^{m'-i}.$$

By Proposition 5.1 $f(t_1, t_2)=g_2(t, v)$ divides

$$\beta\sum_{i=0}^{m'}\binom{m+1}{2i}(\delta^2t^2)^i(\beta^2v)^{m'-i}\,.$$

We will prove that this implies that $\beta=0$ or $(\beta, \delta)=(\pm 1, \pm 1)$. We assume that $\beta\neq 0$. Let p be a prime which divides β. By Lemma 5.4 we see that

$$\nu_p\left(\binom{m+1}{2i}\beta^{2m'-2i}\right) \geqq \nu_p(m+1)+1 = \nu_p((m+1)\delta^{2m'})+1$$

for $i<m'$, which means that

$$\nu_p \text{ (the coefficient of } t^{2i}v^{m'-i}) > \nu_p \text{ (the coefficient of } t^{2m'})$$

for $i<m'$. From this fact we have

$$f(t_1, t_2) = g(t, v) = g_2'(\delta^2 t^2, pv)$$

for some g_2'. If we substitute 0 for t_1 into the above formula, we obtain

$$t_2^{r+1} = g_2'(\delta^2\beta^2t_2^2, p\delta(\delta+\beta)t_2^2)\,,$$

which implies that $\delta=\pm 1$ and that there is no prime which divides β.

Case III. r is odd, m is odd and $v=u^2$. Put $m+1=2m'+2$.

$$(\delta t-\beta u)^{m+1} = \sum_{i=0}^{m+1}\binom{m+1}{i}\delta^i(-\beta)^{m+1-i}t^i u^{m+1-i}$$
$$= \sum_{i=0}^{m'+1}\binom{m+1}{2i}(\delta^2t^2)^i(\beta^2v)^{m'+1-i}-\left\{\beta\delta t\sum_{i=0}^{m'}\binom{m+1}{2i+1}(\delta^2t^2)^i(\beta^2v)^{m'-i}\right\}u.$$

By Proposition 5.1 $f(t_1, t_2)=g_2(t, v)$ divides

$$\beta\delta t\sum_{i=0}^{m'}\binom{2m'+2}{2i+1}(\delta^2t^2)^i(\beta^2v)^{m'-i}\,,$$

which implies that $(\beta, \delta)=(0, \pm 1)$, $(\pm 1, 0)$ or $(\pm 1, \pm 1)$ as before.

Case IV. r is odd, m is even and $v=u^2+ut$. Put $u^k=x_k(t, v)+y_{k-1}(t, v)u$. Using the recurrent formula

$$x_{k+1}=y_{k-1}v\,,\qquad y_k=x_k-y_{k-1}t\,,$$

we have

$$y_k=(-1)^k\sum_{i=0}^{k}\binom{k-i}{i}v^it^{k-2i}\,.$$

Then

$$(\delta t--\beta u)^{m+1}=\sum_{i=0}^{m+1}\binom{m+1}{i}\delta^i(-\beta)^{m+1-i}t^iu^{m+1-i}$$

$$=\sum_{i=0}^{m+1}\binom{m+1}{i}(\delta t)^i\{x_{m+1-i}(-\beta t,\ \beta^2v)-y_{m-i}(-\beta t,\ \beta^2v)\beta u\}.$$

By Proposition 5.1 $f(t_1, t_2)=g_2(t, v)$ divides

$$\beta\sum_{i=0}^{m}\binom{m+1}{i}(\delta t)^iy_{m-i}(-\beta t,\ \beta^2v) \tag{5.5}$$

$$=\beta\sum_{j=0}^{m}\left\{\sum_{i=0}^{m}\binom{m+1}{i}\binom{m-i-j}{j}\delta^i\beta^{m-i}\right\}v^jt^{m-2j}\,,$$

which implies that $\beta=0$ or $\beta=\pm1$ by the similar method to the case II.

Case V. r is odd, m is odd and $v=u^2+ut$. Similar to the case IV $f(t_1, t_2)=g_2(t, v)$ divides the formula (5.4), but in this case this vanishes if $\beta=0$ or $(\beta, \delta)=\pm(-2, 1)$. Thus we have $\beta=0$, ±1 or $(\beta, \delta)=\pm(-2, 1)$.

References

[1] J. F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math., **72** (1960), 20–104.

[2] E. Sato, Varieties which have two projective space bundle structures, J. Math. Kyoto Univ., **25** (1985), 445–457.

[3] N. Shimada and T. Yamanoshita, On triviality of the mod p Hopf invariant, Japan. J. Math., **31** (1961), 1–24.

Kouyemon IRIYE
Department of Mathematics
Kyoto University
Kyoto 606
Japan