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The nonexistence of expansive homeomorphisms of Suslinian continua

Dedicated to Professor Ryōsuke Nakagawa on his 60th birthday

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1. Introduction.

All spaces under consideration are assumed to be metric. By a *continuum*, we mean a compact *connected* nondegenerate space. Let X be a compact metric space with metric d. A homeomorphism f of X is called *expansive* if there exists $c>0$ (called an *expansive constant* for f) such that if x and y are different points of X, then there is an integer n such that $d(f^n(x), f^n(y))>c$. Expansiveness does not depend on the choice of metric of X. We are interested in the following problem: What kinds of continua admit expansive homeomorphisms? Here, we consider this problem from a point of view of continuum theory.

Concerning the above problem, the following results are well known.

(i) Each compact metric space which admits an expansive homeomorphism is finite-dimensional ([**12**]).

(ii) The Cantor set, the p-adic solenoids ($p\geqq 2$) and compact orientable surfaces of positive genus admit expansive homeomorphisms ([**13**], [**14**] and [**16**]). There are solenoidal groups which admit no expansive automorphisms (see [**17**, Remark 2, p. 102] and [**18**, Theorem 3, p. 30]).

(iii) The shift homeomorphism of the inverse limit of every continuous surjection of an arc is not an expansive homeomorphism ([**3**] and [**4**]).

(iv) There are no expansive homeomorphisms on the 2-sphere ([**5**]).

(v) If X is a Peano continuum in the plane, or X is a Peano continuum which contains a 1-dimensional AR neighborhood, then X does not admit an expansive homeomorphism ([**1**], [**4**], [**6**], [**7**] and [**11**]).

(vi) There are no expansive homeomorphisms on hereditarily decomposable tree-like (or circle-like) continua ([**8**] and [**9**]).

(vii) There is a continuum in the plane which admits an expansive homeo-

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morphism. This continuum is 1-dimensional, indecomposable and separates the plane ([**15**]).

(viii) There are no expansive homeomorphisms on Suslinian, hereditary θ-continua ([**10**]).

The purpose of this paper is to prove that there are no expansive homeomorphisms on Suslinian continua. In other words, if a continuum X admits an expansive homeomorphism, then X contains an uncountable collection of mutually disjoint, nondegenerate subcontinua of X. Of course, this result is an extension of (viii). As a corollary, no rational continuum admits an expansive homeomorphism. This implies that every 1-dimensional continuum which admits an expansive homeomorphism is considerably complicated.

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2. Definitions and preliminaries.

A *continuum* is a compact metric *connected* space. A continuum is said to be *Suslinian* if each collection of mutually disjoint, nondegenerate subcontinua of it is countable. A continuum is *rational* if it has a basis of open sets whose boundaries are countable. A continuum is called *hereditarily locally connected* if each subcontinuum of it is locally connected. Then we have the following diagram:

(1-dimensional ANR) $\longrightarrow$ (hereditarily locally connected) $\longrightarrow$

(rational) $\longrightarrow$ (Suslinian) $\longrightarrow$ (1-dimensional).

Note that neither implication can be replaced by an equivalence.

From now on, we list some facts which will be needed in the sequel.

(2.1) LEMMA. *Let Y be a compact metric space. Let $\varepsilon>0$ and k be any natural number. Then there is a natural number $n=n(\varepsilon, k)\geqq k$ such that if $a_1, a_2, \cdots, a_n$ are points of Y, then there is a point a of Y such that $d(a, a_{i(j)})<\varepsilon$ for $j=1, 2, \cdots, k$, where $1\leqq i(1)<i(2)<\cdots<i(k)\leqq n$.*

The proof is trivial, hence we omit the proof.

The next lemma is well known.

(2.2) LEMMA. *Let X be a compact metric space and let U, V be open sets of X such that $Cl(V)\subset U$. If A is a subcontinuum of X such that $A\cap V\neq\emptyset$ and $A-Cl(U)\neq\emptyset$, then there is a subcontinuum B of $A\cap Cl(U)$ such that $B\cap V\neq\emptyset$ and $B\cap Bd(U)\neq\emptyset$.*

(2.3) LEMMA ([8, (2.2)]). *Let $f: X\rightarrow X$ be an expansive homeomorphism of a compact metric space X. Then there is $\delta>0$ such that for each nondegenerate* *subcontinuum A of X, there is a natural number n_0 which satisfies one of the following conditions;*

(∗) $\operatorname{diam} f^n(A) \geqq \delta$ *for each* $n \geqq n_0$, *or*

(∗∗) $\operatorname{diam} f^{-n}(A) \geqq \delta$ *for each* $n \geqq n_0$.

(2.4) REMARK. The converse assertion of (2.3) is not true. Let I be the unit interval $[0, 1]$ and let $f: I \to I$ be a map defined by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leqq x \leqq \frac{1}{2}, \\ 2-2x & \text{if } \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

Consider the inverse limit

$$(I, f) = \{(x_i)_{i=0}^{\infty} \mid x_i \in I,\ f(x_{i+1}) = x_i\}$$

and the shift homeomorphism $\tilde{f}: (I, f) \to (I, f)$, i.e.,

$$\tilde{f}((x_i)_i) = (f(x_i))_i .$$

Then $\tilde{f}$ satisfies the condition (∗), but $\tilde{f}$ is not expansive.

3. Main theorem.

In this section, we prove the following main theorem of this paper.

(3.1) THEOREM. *There are no expansive homeomorphisms on Suslinian continua. In other words, if a continuum X admits an expansive homeomorphism, then there is a closed subset Z of X such that each component of Z is nondegenerate, the space of components of Z is a Cantor set, and the decomposition of Z into components is continuous (i.e., upper-semi and lower-semi continuous).*

To prove (3.1), we need the following notations: Let X be a continuum and let $C(X)$ be the hyperspace of X defined by

$$C(X) = \{A : A \text{ is a nonempty subcontinuum of } X\}.$$

The hyperspace $C(X)$ is metrized as follows; for $A, B \in C(X)$, $d_H(A, B) = \inf\{\varepsilon > 0 : U_\varepsilon(A) \supset B \text{ and } U_\varepsilon(B) \supset A\}$, where $U_\varepsilon(A)$ denotes the ε-neighborhood of A in X. The metric d_H is called the *Hausdorff metric*. Note that $C(X)$ is also a continuum.

For any subset M of $C(X)$, we consider the following set M^f defined by

$$M^f = \{A \in C(X) : \text{for any } \varepsilon > 0 \text{ and any natural number } k, \text{ there are points } A_1, A_2, \cdots, A_k \text{ of } M \text{ such that each } A_i \text{ is nondegenerate, } A_i \cap A_j = \emptyset\ (i \neq j) \text{ and } d_H(A, A_i) < \varepsilon\}.$$

Note that in the definition of M^f, the intersection $A \cap A_i$ may not be empty.

Then we have

(3.2) PROPOSITION. *M^f is closed in $C(X)$.*

In fact, suppose that $\{B_n\}$ is a sequence of points of M^f such that $\lim B_n = B$. Since $C(X)$ is compact, $B \in C(X)$. Let $\varepsilon > 0$ and a natural number k be given. Since $\lim B_n = B$, $d_H(B, B_n) < \varepsilon/2$ for some n. Since $B_n \in M^f$, there are points $A_1, A_2, \cdots, A_k$ of M such that each element A_i is nondegenerate, $A_i \cap A_j = \emptyset$ $(i \neq j)$ and $d_H(B_n, A_i) < \varepsilon/2$ for $j=1, 2, \cdots, k$. Hence we have

$$d_H(B, A_i) \leqq d_H(B, B_n) + d_H(B_n, A_i) < \varepsilon .$$

This implies $B \in M^f$. Therefore M^f is closed in $C(X)$.

(3.3) PROPOSITION. $$M^f \supset (M^f)^f .$$

The proof is similar to the proof of (3.2). We omit the proof.

For a subset M of $C(X)$ and ordinal numbers, define

$$M_1 = M^f, \quad M_{\alpha+1} = (M_\alpha)^f \quad \text{and} \quad M_\lambda = \bigcap_{\alpha<\lambda} M_\alpha ,$$

where λ is a limit ordinal.

Note that if f is a homeomorphism of X and $f(M)=M$, then M_α is f-invariant (i.e., $f(M_\alpha)=M_\alpha$).

Then we have

(3.4) THEOREM. *Let X be a continuum and let $M=C(X)$. Then X is Suslinian if and only if $M_\alpha=\emptyset$ for some countable ordinal α.*

PROOF. Let X be a Suslinian continuum. Suppose, on the contrary, that $M_\alpha \neq \emptyset$ for any countable ordinal α. By (3.2), M_α is closed in $C(X)$. Also, by (3.3), $M_\alpha \supset M_\beta$ if $\alpha < \beta$. Since $C(X)$ is separable, there is a countable ordinal α such that $M_\alpha = M_\beta$ if $\alpha \leqq \beta$. In particular, $(M_\alpha)^f = M_\alpha$ and $M_\alpha \neq \emptyset$. Choose $A \in M_\alpha$. Since $A \in (M_\alpha)^f$, there are two points A_0 and A_1 of M_α such that each A_i is nondegenerate, $A_0 \cap A_1 = \emptyset$. Choose $\gamma > 0$ such that $\operatorname{diam} A_i > \gamma$ $(i=0, 1)$, and choose neighborhoods U_i $(i=0, 1)$ of A_i in X such that $ClU_0 \cap ClU_1 = \emptyset$ and $ClU_i \subset U_{1/2}(A_i)$. Since A_i $(i=0, 1)$ is contained in $M_\alpha = (M_\alpha)^f$, for each i we can choose two points A_{ij} $(j=0, 1)$ of M_α such that $\operatorname{diam} A_{ij} > \gamma$, $A_{i0} \cap A_{i1} = \emptyset$ and $A_{ij} \subset U_i$. Choose neighborhoods U_{ij} of A_{ij} in U_i such that $ClU_{i0} \cap ClU_{i1} = \emptyset$ and $ClU_{ij} \subset U_{1/2^2}(A_{ij})$. Note that $A_{ij} \in M_\alpha = (M_\alpha)^f$. By induction on n, we can choose subcontinua $A_{i_1 i_2 \cdots i_n}$ $(i_j = 0$ or $1)$ of X and neighborhoods $U_{i_1 i_2 \cdots i_n}$ of $A_{i_1 i_2 \cdots i_n}$ in $U_{i_1 i_2 \cdots i_{n-1}}$ such that

(1) $ClU_{i_1 i_2 \cdots i_{n-1} 0} \cap ClU_{i_1 i_2 \cdots i_{n-1} 1} = \emptyset$,

(2) $\operatorname{diam} A_{i_1 i_2 \cdots i_n} > \gamma$, and

(3) $ClU_{i_1 i_2 \cdots i_n} \subset U_{1/2^n}(A_{i_1 i_2 \cdots i_n})$.

For any sequence $\{(i_j)_n\}$ ($i_j=0$ or 1), consider the following set

$$A_{i_1 i_2 \cdots} = ClU_{i_1} \cap ClU_{i_1 i_2} \cap ClU_{i_1 i_2 i_3} \cap \cdots .$$

By (1), (2) and (3), we can easily see that the uncountable collection $\{A_{i_1 i_2 \cdots}: i_j=0 \text{ or } 1\}$ is a collection of mutually disjoint nondegenerate subcontinua of X, which implies that X is not Suslinian. This is a contradiction. Next, suppose that X is not Suslinian. By [**2**, (2.1)], there is a closed subset Z of X such that each component of Z is nondegenerate, the space of components of Z is a Cantor set, and the decomposition of Z into components is continuous. Clearly, each component of Z is contained in M_α for any ordinal α. Hence $M_\alpha \neq \emptyset$ for any countable ordinal α. This completes the proof.

(3.5) EXAMPLE. For each ordinal number $\alpha=1, 2, \cdots, \omega, \cdots, \omega_1$, let Y_α be the following 0-dimensional compact metric space;

$$Y_1 = \{*\},$$

$$Y_2 = \bigoplus_{n=1}^{\infty} Y_1^n \cup \{\infty\},$$

$$\vdots$$

$$Y_\lambda = \bigoplus_{n=1}^{\infty} Y_{\alpha_n} \cup \{\infty\} \quad (\lambda \text{ is a limit ordinal, } \alpha_1 < \alpha_2 < \cdots \text{ and } \lim \alpha_n = \lambda),$$

$$\vdots$$

$$Y_{\omega_1} = \text{a Cantor set},$$

where Y_α^n is a copy of Y_α, $\bigoplus_{n=1}^{\infty} Y_\alpha^n$ denotes the topological sum of Y_α^n ($n=1, 2, \cdots$) and $\bigoplus_{n=1}^{\infty} Y_\alpha^n \cup \{\infty\}$ is the one point compactification of $\bigoplus_{n=1}^{\infty} Y_\alpha^n$. Let X_α be the cone of Y_α. Suppose $M=C(X_\alpha)$. Then if $\alpha<\omega_1$, $M_\alpha \neq \emptyset$, and for $\beta>\alpha$, $M_\beta=\emptyset$. Also, in the case of X_{ω_1}, $M_\lambda \neq \emptyset$ for any ordinal λ.

PROOF OF (3.1). Let X be a Suslinian continuum. Suppose, on the contrary, that there is an expansive homeomorphism f on X. Set $M=C(X)$. Let $\delta>0$ be as in (2.4). Choose a sequence $\varepsilon_1>\varepsilon_2>\varepsilon_3>, \cdots,$ of positive numbers such that $\lim \varepsilon_i=0$. For each ε_k and k, choose a natural number $n_k=n(\varepsilon_k, k)$ as in (2.1), where we assume that $Y=C(X)$ in (2.1). Let A be any nondegenerate subcontinuum of X. By (2.2), we can choose nondegenerate subcontinua $B_1, B_2, \cdots, B_{2n_k}$ of A such that $B_i \cap B_j=\emptyset$ $(i \neq j)$. By (2.3), we may assume that for some integer n,

(1) $\operatorname{diam} f^n(B_i) \geqq \delta$, where $i=1, 2, \cdots, n_k$.

By the choice of n_k, there is a point B^k of $C(X)$ such that

(2) $d_H(B^k, f^n(B_{i_j})) < \varepsilon_k$ for $j=1, 2, \cdots, k$ and $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n_k$.

Since $C(X)$ is compact, we may assume that $\{B^k\}$ is convergent to a point A_1 of $C(X)$. By (1) and (2), $\operatorname{diam} A_1 \geqq \delta$. Also, we can easily see that $A_1 \in M_1$ $(=M^f)$, hence M_1 contains nondegenerate element. Now, we shall show that

M_1 satisfies the following condition $(*_1)$:

If $A \in M_1$ and A is nondegenerate, for any open sets U, V of X such that $Cl\,V \subset U$, $A \cap V \neq \emptyset$ and $A - Cl\,U \neq \emptyset$, there exists $B \in M_1$ such that $B \cap Cl\,V \neq \emptyset$, $B \subset A \cap Cl\,U$ and $B \cap Bd\,U \neq \emptyset$.

We can prove this as follows. Since $A \in M_1$, for each k we choose $B_1, B_2, \cdots, B_{n_k} \in C(X)$ such that each B_i is nondegenerate, $B_i \cap B_j = \emptyset$ $(i \neq j)$ and $d_H(A, B_i) < \varepsilon_k$ for each $i = 1, 2, \cdots, n_k$. We may assume that $B_i \cap V \neq \emptyset$ and $B_i - Cl\,U \neq \emptyset$ for each i. By (2.2), for each $i = 1, 2, \cdots, n_k$ we can choose a subcontinuum C_i of B_i such that $C_i \subset Cl\,U$, $C_i \cap V \neq \emptyset$ and $C_i \cap Bd\,U \neq \emptyset$. By (2.1), there is a point C^k of $C(X)$ such that $d_H(C^k, C_{i_j}) < \varepsilon_k$ for each $j = 1, 2, \cdots, k$ and $1 \leqq i_1 < i_2 < \cdots < i_k \leqq n_k$. Also, we may assume that $\{C^k\}$ is convergent to a point B of $C(X)$. Then we can easily see that $B \subset A \cap Cl\,U$, $B \cap Cl\,V \neq \emptyset$ and $B \cap Bd\,U \neq \emptyset$. Clearly, $B \in M_1$.

For a countable ordinal λ, we may assume that for $\alpha < \lambda$ M_α contains a nondegenerate element and satisfies the condition $(*_\alpha)$. We shall prove that M_λ has the same properties. We consider the following two cases.

(I) $\lambda = \alpha + 1$. Note that M_α satisfies the condition $(*_\alpha)$. By an argument similar to the above one, we can prove that M_λ contains a nondegenerate element and satisfies the condition $(*_\lambda)$.

(II) λ is a limit ordinal. In this case, take a sequence $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$, of countable ordinals such that $\lim \alpha_i = \lambda$. Since M_α is f-invariant, by (2.3) we see that for each i, there is $A_i \in M_{\alpha_i}$ such that $\operatorname{diam} A_i \geqq \delta$. We may assume that $\{A_i\}$ is convergent to a point A_λ of $C(X)$. This implies that

$$A_\lambda \in \bigcap_{\alpha < \lambda} M_\alpha = M_\lambda .$$

Also, note that $\operatorname{diam} A_\lambda \geqq \delta$. By using (2.1), we can prove that M_λ satisfies the condition $(*_\lambda)$.

Consequently, $M_\alpha \neq \emptyset$ for any countable ordinal α. By (3.4), X is not Suslinian. This is a contradiction. This completes the proof.

As corollaries, we have

(3.6) COROLLARY. *There are no expansive homeomorphisms on rational continua.*

(3.7) COROLLARY. *There are no expansive homeomorphisms on hereditarily locally connected continua.*

By an argument similar to the proof of (3.1), we have

(3.8) COROLLARY. *If $f: X \to X$ is an expansive homeomorphism of a compact metric space X and* $\dim X > 0$, *then there is a closed subset Z of X such that each* *component of Z is nondegenerate, the space of components of Z is a Cantor set, and the decomposition of Z into components is continuous.*

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