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Construction of irreducible unitary representations of amalgams of discrete abelian groups

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§ 0. Introduction.

Mackey's induced representation is one of the most fundamental concepts of unitary representation theory of groups and has played a significant role also in the recent development of representation theory of discrete groups. The author [**13**], [**15**] constructed new irreducible representations of the infinite symmetric group by means of induced representations and discussed a relation to indecomposable central characters or factor representations of type II. Hirai [**6**], [**7**] has made a systematic approach to representations induced from infinite-dimensional ones and obtained a fairly big family of irreducible representations of infinite wreath products and the infinite symmetric group. Although the central interest lies in the infinite symmetric group in those works mentioned above, the basic idea of constructing irreducible representations established there covers a large part of discrete groups.

The main purpose of this paper is to develop a study of induced representations of an amalgam of discrete abelian groups. We shall construct irreducible representations induced from maximal abelian subgroups and discuss irreducible decompositions of the regular representation. Our discussion contains a classical result on free groups due to Yoshizawa [**21**] and its generalization by Kawakami [**9**]. Moreover, analogous results for $SL(2, \boldsymbol{Z})$ due to Saito [**17**] are fully reproduced within our framework. While, there have been considerable interests in harmonic analysis of free groups, free products and amalgams, see [**1**], [**2**], [**3**], [**5**], [**16**], etc. Their discussions, being based on spectral theory of random walks on graphs and spherical functions on groups, seem to be independent of induced representations.

Let $G=*_Z G_i$ be an amalgam of discrete abelian groups G_i, $i\in I$, with the commom subgroup Z being amalgamated. Note that Z becomes the center of G. We assume that each free factor G_i satisfies the condition

$$\text{(A)} \qquad |G_i : Z| = \infty \quad \text{if } |G_i| = \infty .$$

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Throughout a central role will be played by the set $\mathfrak{M}_\infty(G)$ of maximal abelian subgroups H of G such that $|H|=\infty$. The following two properties of $\mathfrak{M}_\infty(G)$ are essential to our goal:

(C) $\mathfrak{M}_\infty(G)$ is closed under conjugation;

(P) If $H, K\in\mathfrak{M}_\infty(G)$ satisfy the condition that $|H: H\cap K|<\infty$ and $|K: H\cap K|<\infty$, then $H=K$.

For each maximal abelian subgroup $H\in\mathfrak{M}_\infty(G)$ we have:

(1) H is either conjugate to a free factor or a direct product of the center Z and an infinite cyclic group;

(2) $|N_G(H): H|\leqq 2$, where $N_G(H)$ stands for the normalizer of H;

(3) If H is conjugate to a free factor, then $N_G(H)=H$.

Accordingly, $\mathfrak{M}_\infty(G)$ is divided into two subclasses:

$$\mathfrak{M}_\infty(G)=\mathfrak{M}_\infty^+(G)\cup\mathfrak{M}_\infty^-(G),$$

where

$$\mathfrak{M}_\infty^+(G)=\{H\in\mathfrak{M}_\infty(G);\ N_G(H)=H\},$$

$$\mathfrak{M}_\infty^-(G)=\{H\in\mathfrak{M}_\infty(G);\ |N_G(H): H|=2\}.$$

Let $H\in\mathfrak{M}_\infty^-(G)$ and put $H=Z\times\langle h\rangle$. Then there exists an element $z_0\in Z$ such that $ghg^{-1}=z_0h^{-1}$ for all $g\in N_G(H)-H$. We then come to the main results (see also Theorems 3.6 and 3.8).

THEOREM A. *Let $G=*_ZG_i$ be an amalgam of discrete abelian groups G_i, $i\in I$, satisfying* (A). *Let χ and ϕ be one-dimensional representations of $H\in\mathfrak{M}_\infty(G)$ and $K\in\mathfrak{M}_\infty(G)$, respectively. Then:*

(1) $\mathrm{Ind}_H^G\chi$ *and* $\mathrm{Ind}_K^G\phi$ *are mutually equivalent or disjoint. They are mutually equivalent if and only if $H=K^g$ and $\chi=\phi^g$ for some $g\in G$, where ϕ^g is a representation of $K^g=g^{-1}Kg$ defined as $\phi^g(g^{-1}kg)=\phi(k)$, $k\in K$.*

(2) *If $H\in\mathfrak{M}_\infty^+(G)$, then* $\mathrm{Ind}_H^G\chi$ *is irreducible.*

(3) *Assume $H\in\mathfrak{M}_\infty^-(G)$ and put $H=Z\times\langle h\rangle$. Then* $\mathrm{Ind}_H^G\chi$ *is irreducible if and only if $\chi(h)^2\neq\chi(z_0)$. If $\chi(h)^2=\chi(z_0)$, then* $\mathrm{Ind}_H^G\chi$ *is decomposed into a direct sum of two irreducible representations which are not mutually equivalent.*

THEOREM B. *Let $G=*_ZG_i$ be an amalgam of discrete abelian groups G_i, $i\in I$, satisfying* (A). *For each $H\in\mathfrak{M}_\infty(G)$ we have*

$$(\textit{regular representation of } G)\cong\int_{\hat{H}}^{\oplus}\mathrm{Ind}_H^G\chi\,d\chi,$$

where $\mathrm{Ind}_H^G\chi$ *is irreducible for almost all $\chi\in\hat{H}$ with respect to the Haar measure $d\chi$. Moreover, if two maximal abelian subgroups in $\mathfrak{M}_\infty(G)$ are not conjugate in G, the corresponding irreducible decompositions are completely different.*

A particularly interesting example of amalgams of discrete abelian groups is the group $SL(2, \boldsymbol{Z})$ of 2×2 matrices of integers with determinant one. In fact, $SL(2, \boldsymbol{Z})\cong(\boldsymbol{Z}/4\boldsymbol{Z})*_{\boldsymbol{Z}/2\boldsymbol{Z}}(\boldsymbol{Z}/6\boldsymbol{Z})$ is a classical result (see, e.g., Serre [18]). In [17] Saito dealt with representations of $SL(2, \boldsymbol{Z})$. His construction of irreducible representations is based on induced representations from Cartan subgroups and his motivation bears some resembrance to representation theory of linear algebraic groups. However, it will turn out that Saito's results are simple consequences of our theorems. In other words, the fact that $SL(2, \boldsymbol{Z})$ is an amalgam of discrete abelian groups is essential to our goal. As a result, the Cartan subgroups of $SL(2, \boldsymbol{Z})$ are in coincidence with $\mathfrak{M}_\infty(SL(2, \boldsymbol{Z}))$.

The paper is organized as follows. In Section 1 we establish a general criterion for irreducibility and mutual equivalence of induced representations of discrete groups. We then derive some practical criteria and mention relationship between known results and ours. In Section 2 we study structure of maximal abelian subgroups of a free product of discrete groups. In Section 3 we give a complete description of maximal abelian subgroups of an amalgam of discrete abelian groups and prove the main theorems. In Section 4 the case of $SL(2, \boldsymbol{Z})$ is discussed. We determine all maximal abelian subgroups in terms of matrices and see that Saito's results are fully covered with our argument.

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§1. Induced representations of discrete groups in general.

In this section we assemble and establish general results on induced representations of discrete groups. Let G be a discrete group and simply by a representation of G we always mean a unitary representation on a Hilbert space. A representation (π, W) of a subgroup H of G is denoted by $(\pi; H)=(\pi, W; H)$. Let $\mathfrak{R}$ be a family of irreducible representations of subgroups of G. We say that H is a *underlying subgroup* of $\mathfrak{R}$ if it contains a representation of H. The main purpose of this section is to establish criteria for irreducibility and mutual equivalence of representations in $\tilde{\mathfrak{R}}=\{\mathrm{Ind}_H^G\pi; (\pi; H)\in\mathfrak{R}\}$ in terms of $\mathfrak{R}$.

To this end we need an intertwining number theorem. The conjugate of a representation $(\pi, W; H)$ by $g\in G$, denoted by $(\pi^g, W; H^g)$, is a representation of $H^g=g^{-1}Hg$ defined as $\pi^g(g^{-1}hg)=\pi(h)$, $h\in H$. For two representations $(\pi_1, W_1; H)$ and $(\pi_2, W_2; K)$ we denote by $\mathfrak{I}(\pi_1, \pi_2; H, K)$ the space of bounded operators $L\in B(W_1, W_2)$ satisfying the following three conditions:

($\mathfrak{I}$1) $L\in\mathrm{Hom}_{H\cap K}(\pi_1, \pi_2)$, i.e., $L\pi_1(h)=\pi_2(h)L$ for all $h\in H\cap K$;

($\mathfrak{I}$2) there exists a constant $M>0$ such that

$$\sum_{h \in H \cap K \backslash H} \| L\pi_1(h)w \|_{W_2}^2 \leq M \| w \|_{W_1}^2, \qquad w \in W_1;$$

($\mathfrak{J}$3) there exists a constant $M>0$ such that

$$\sum_{k \in K \cap H \backslash K} \| L^*\pi_2(k)v \|_{W_1}^2 \leq M \| v \|_{W_2}^2, \qquad v \in W_2.$$

With these notations we can state the following intertwining number theorem.

THEOREM 1.1. *For any pair of representations* $(\pi_1; H)$ *and* $(\pi_2; K)$ *it holds that*

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G \pi_1, \mathrm{Ind}_K^G \pi_2) = \sum_{g \in K \backslash G / H} \dim \mathfrak{J}(\pi_1, \pi_2^g; H, K^g).$$

This useful formula is direct from known results, see Mackey [**11**] and Hirai [**6**, §1]. As usual, the normalizer of H is defined as $N_G(H)=N(H)=\{g \in G; H=H^g\}$.

THEOREM 1.2. *Let* $\mathfrak{R}$ *be a family of irreducible representations of subgroups of a discrete group* G. *Suppose that* $\mathfrak{R}$ *satisfies the following two hypotheses:*

(H1) $\mathfrak{R}$ *is closed under conjugation, i.e., if* $(\pi; H) \in \mathfrak{R}$ *and* $g \in G$, *then* $(\pi^g; H^g) \in \mathfrak{R}$;

(H2) $\mathfrak{J}(\pi_1, \pi_2; H, K)=\{0\}$ *for any pair* $(\pi_1; H) \in \mathfrak{R}$, $(\pi_2; K) \in \mathfrak{R}$ *with* $H \neq K$.

Then:

(1) *For any pair* $(\pi_1; H) \in \mathfrak{R}$ *and* $(\pi_2; K) \in \mathfrak{R}$,

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G \pi_1, \mathrm{Ind}_K^G \pi_2) = \#\{g \in K \backslash G / H; H=K^g, \pi_1 \cong \pi_2^g\}.$$

(2) *Any two representations in* $\mathfrak{R}$ *are mutually equivalent or disjoint.*

(3) *Let* $(\pi_1; H) \in \mathfrak{R}$ *and* $(\pi_2; K) \in \mathfrak{R}$. *Then* $\mathrm{Ind}_H^G \pi_1 \cong \mathrm{Ind}_K^G \pi_2$ *if and only if* $(\pi_1; H)$ *and* $(\pi_2; K)$ *are conjugate, i.e.,* $H=K^g$ *and* $\pi_1 \cong \pi_2^g$ *for some* $g \in G$.

(4) *For any* $(\pi; H) \in \mathfrak{R}$,

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G \pi, \mathrm{Ind}_H^G \pi) = \#\{g \in N_G(H)/H; \pi \cong \pi^g\}.$$

In particular, $\mathrm{Ind}_H^G \pi$ *is irreducible if and only if* π *and* π^g *are not equivalent for any* $g \in N_G(H)-H$.

PROOF. (1) Let $g \in G$ satisfies the condition that $\dim \mathfrak{J}(\pi_1, \pi_2^g; H, K^g)>0$. Then $H=K^g$ by the hypotheses (H1) and (H2). Moreover, in view of the definition of $\mathfrak{J}(\pi_1, \pi_2^g; H, K^g)$, we have

$$\mathfrak{J}(\pi_1, \pi_2^g; H, K^g) = \mathrm{Hom}_{H \cap K^g}(\pi_1, \pi_2^g) = \mathrm{Hom}_H(\pi_1, \pi_2^g).$$

Since both π_1 and π_2^g are irreducible representations of H, we see that $\dim \mathfrak{J}(\pi_1, \pi_2^g; H, K^g)=1$ and $\pi_1 \cong \pi_2^g$. It then follows from Theorem 1.1 that

$$\begin{aligned}\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G\pi_1, \mathrm{Ind}_K^G\pi_2) &= \sum_{g\in K\backslash G/H} \dim \mathfrak{J}(\pi_1, \pi_2^g ; H, K^g) \\ &= \#\{g\in K\backslash G/H ; \dim \mathfrak{J}(\pi_1, \pi_2^g ; H, K^g)>0\} \\ &= \#\{g\in K\backslash G/H ; H=K^g, \pi_1\cong\pi_2^g\}.\end{aligned}$$

(2) Let $(\pi_1 ; H)\in\mathfrak{R}$ and $(\pi_2 ; K)\in\mathfrak{R}$, and assume that $\mathrm{Ind}_H^G\pi_1$ and $\mathrm{Ind}_K^G\pi_2$ are not disjoint. Then, $\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G\pi_1, \mathrm{Ind}_K^G\pi_2)>0$. By (1) there exists some $g\in G$ such that $H=K^g$ and $\pi_1\cong\pi_2^g$, and therefore $\mathrm{Ind}_H^G\pi_1\cong\mathrm{Ind}_K^G\pi_2$.

(3) Suppose that $\mathrm{Ind}_H^G\pi_1\cong\mathrm{Ind}_K^G\pi_2$. Then, in particular, they are not disjoint. In view of the argument of (2) we conclude that $H=K^g$ and $\pi_1\cong\pi_2^g$ for some $g\in G$. The converse assertion is obvious.

(4) Immediate from (1). Q. E. D.

REMARK ON THEOREM 1.2(4). If $\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G\pi, \mathrm{Ind}_H^G\pi)<\infty$, we have a more presice result, namely, the intertwining algebra $\mathrm{Hom}_G(\mathrm{Ind}_H^G\pi, \mathrm{Ind}_H^G\pi)$ is isomorphic to the group algebra $C[W(\pi)/H]$, where $W(\pi)=\{g\in N_G(H) ; \pi\cong\pi^g\}$.

Noticeably, Theorem 1.2 is quite useful in constructing irreducible representations of discrete groups. For example, Hirai's somewhat complex argument in the first half of [**6**] can be clarified with our result. Moreover, as is seen below, many practical criteria for irreducibility and mutual equivalence of induced representations are derived from Theorem 1.2.

For two subgroups H and K of G we write $H\sim K$ if $|H: H\cap K|<\infty$ and $|K: H\cap K|<\infty$. This becomes an equivalence relation among subgroups. For a subgroup H of G put

$$Q(H) = Q_G(H) = \{g\in G ; H\sim H^g\}.$$

Then $Q_G(H)$ becomes a subgroup of G satisfying $H\subset N_G(H)\subset Q_G(H)$. Following Corwin [**4**] we call $Q(H)$ the *quasi-normalizer* of H. For two subgroups H and K of G we put

$$(K\backslash G/H)_f = \{g\in K\backslash G/H ; H\sim K^g\}.$$

Clearly, $(H\backslash G/H)_f=H\backslash Q(H)/H$.

REMARK. In some literature (e.g., Shimura [**19**, Chap. 3]) two subgroups H and K with $H\sim K$ are called *commensurable* and $Q(H)$ the *commensurator* of H. The purpose is, however, quite different from ours.

PROPOSITION 1.3. *If two representations* $(\pi_1 ; H)$ *and* $(\pi_2 ; K)$ *satisfy the condition*

(F) $\mathfrak{J}(\pi_1, \pi_2^g ; H, K^g) = \{0\}$ *for any* $g\in G$ *such that* $H\not\sim K^g$,

then it holds that

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G\pi_1, \mathrm{Ind}_K^G\pi_2) = \sum_{g\in(K\backslash G/H)_f} \dim \mathrm{Hom}_{H\cap K^g}(\pi_1, \pi_2^g).$$

Conversely, if this identity holds and is fintie, the condition (F) *holds.*

Proof. It follows from Theorem 1.1 that

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G \pi_1, \mathrm{Ind}_K^G \pi_2) = \sum_{g \in K\backslash G/H} \dim \mathfrak{I}(\pi_1, \pi_2^g; H, K^g).$$

By assumption (F) the sum is taken over the subset of $g \in K\backslash G/H$ such that $H \sim K^g$, namely,

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G \pi_1, \mathrm{Ind}_K^G \pi_2) = \sum_{g \in (K\backslash G/H)_f} \dim \mathfrak{I}(\pi_1, \pi_2^g; H, K^g).$$

We then need only to show that $\mathfrak{I}(\pi_1, \pi_2^g; H, K^g) = \mathrm{Hom}_{H\cap K^g}(\pi_1, \pi_2^g)$ for all $g \in (K\backslash G/H)_f$. Now suppose $g \in (K\backslash G/H)_f$ and $L \in \mathrm{Hom}_{H\cap K^g}(\pi_1, \pi_2^g)$. Then,

$$\sum_{h \in H\cap K^g\backslash H} \|L\pi_1(h)w\|_{W_2}^2 \leq |H: H\cap K^g| \, \|L\|^2 \|w\|_{W_1}^2, \qquad w \in W_1.$$

This proves that L satisfies ($\mathfrak{I}$2). Similarly L satisfies ($\mathfrak{I}$3), and therefore $L \in \mathfrak{I}(\pi_1, \pi_2^g; H, K^g)$. The converse assertion is already clear. Q.E.D.

The importance of the condition (F) is illustrated by the next fact (cf. Kleppner [**10**]).

Lemma 1.4. *Any pair of finite dimensional representations* $(\pi_1, W_1; H)$ *and* $(\pi_2, W_2; K)$ *satisfies the condition* (F).

Proof. Suppose that $L \in \mathfrak{I}(\pi_1, \pi_2^g; H, K^g)$, $L \neq 0$. It suffices to show that $H \sim K^g$. By virtue of ($\mathfrak{I}$2) there exists a constant $M > 0$ such that

$$\sum_{h \in H\cap K^g\backslash H} \|L\pi_1(h)w\|_{W_2}^2 \leq M\|w\|_{W_1}^2, \qquad w \in W_1.$$

Making w run over a complete orthonormal basis for W_1, we take the sum of both sides to get

$$\sum_{h \in H\cap K^g\backslash H} \|L\pi_1(h)\|_{HS}^2 \leq M \dim W_1.$$

Hence,

$$|H: H\cap K^g| \, \|L\|_{HS}^2 \leq M \dim W_1 < \infty.$$

Similarly, using ($\mathfrak{I}$3), we get

$$|K^g: H\cap K^g| \, \|L^*\|_{HS}^2 \leq M \dim W_2 < \infty.$$

Since $L \neq 0$ we have $H \sim K^g$. Q.E.D.

Thus, irreducibility and mutual equivalence of representations induced from finite dimensional ones may be discussed fully within our framework. In this connection, see also [**4**], [**10**], [**11**], [**14**], etc.

We are now going back to a discussion about $\mathfrak{N}$. Let $\mathfrak{G}$ be a family of subgroups of G and consider the following two properties:

(C) $\mathfrak{G}$ is closed under conjugation of G;

(P) for any two subgroups H and K in $\mathfrak{G}$, $H \sim K \Leftrightarrow H=K$.

As is seen in Section 3, these are concerned with irreducible decompositions of the regular representation of G. The following two facts are easily verified.

LEMMA 1.5. *If $\mathfrak{G}$ satisfies* (C) *and* (P), *then $Q(H)=N(H)$ for all $H \in \mathfrak{G}$.*

LEMMA 1.6. *If every $H \in \mathfrak{G}$ satisfies the condition that $Q(H)=H$, then $\mathfrak{G}$ satisfies* (P).

With these preparations we can derive a useful criterion for irreducibility and mutual equivalence of representations in $\tilde{\mathfrak{R}}=\{\mathrm{Ind}_H^G \pi\,;(\pi\,;H)\in\mathfrak{R}\}$.

THEOREM 1.7. *Let $\mathfrak{R}$ be a family of irreducible representations of subgroups of G. Assume that any two representations in $\mathfrak{R}$ satisfy* (F) *and that the underlying subgroups satisfy* (C) *and* (P). *Then the same assertions as in Theorem* 1.2 (1)-(4) *hold.*

PROOF. It can be proved, with no difficulty, that $\mathfrak{R}^*=\{(\pi^g\,;H^g)\,;(\pi\,;H)\in\mathfrak{R},\ g\in G\}$ satisfies the hypotheses (H1) and (H2). Q.E.D.

REMARK. The idea of dealing with a family of representations of subgroups is due to Saito [**17**]. Recently, Hirai [**6**, §1] has modified Saito's argument to get a criterion covering a wider class of induced representations. Their results are, however, simple consequences of Theorem 1.7.

§2. Maximal abelian subgroups of free products.

Given a family of discrete groups G_i, $i\in I$, let $G=*G_i$ denote their free product. Recall that each G_i is called a *free factor* of G. We always assume that $G_i\neq\{e\}$ for all $i\in I$. Each element $g\in G$, $g\neq e$, may be written uniquely in *reduced expression*, namely, as a product $g=g_1\cdots g_r$ where $g_n\in G_{i_n}$, $g_n\neq e$, $i_n\neq i_{n+1}$. In that case r is called the *length* of g. By definition the length of the unit element e is zero. In this section we shall devote ourselves to a study of maximal abelian subgroups of $G=*G_i$ where G_i, $i\in I$, is a discrete group.

We denote by $\mathfrak{M}(A)$ the set of maximal abelian subgroups of a group A and by $\mathfrak{M}_\infty(A)$ the subset of $H\in\mathfrak{M}(A)$ with $|H|=\infty$. Obviously, $\mathfrak{M}_\infty(G)$ satisfies (C), namely, is closed under the conjugation of G. In this section we shall prove the following three results.

THEOREM 2.1. *Let H be an abelian subgroup of $G=*G_i$, $H\neq\{e\}$. Then one of the following two possibilities occurs:*

(i) *$H\cap G_i^\gamma\neq\{e\}$ for some pair $i\in I$, $\gamma\in G$. In this case $H\subset G_i^\gamma$;*

(ii) *$H\cap G_i^\gamma=\{e\}$ for all $i\in I$ and $\gamma\in G$. In this case H is an infinite cyclic* *group generated by an element which does not belong to any conjugate of any free factor.*

THEOREM 2.2. *If every* $\mathfrak{M}_\infty(G_i)$ *satisfies the condition* (P), *so does* $\mathfrak{M}_\infty(G)=\mathfrak{M}_\infty(*G_i)$.

Obviously, if A is a finite group or an abelian group, $\mathfrak{M}_\infty(A)$ satisfies (P).

THEOREM 2.3. (1) *If* $H\in\mathfrak{M}_\infty(G)$ *is in a conjugate of a free factor, say,* $H\subset G_i^\gamma$, *then* $N_G(H)=N_{G_i^\gamma}(H)$.

(2) *If* $H\in\mathfrak{M}_\infty(G)$ *is not in any conjugate of any free factor, then* $g^2=e$ *for all* $g\in N_G(H)-H$ *and* $|N_G(H):H|\leqq 2$.

We first recall the following

LEMMA 2.4 ([**12**, Corollary 4.1.6]). *Let* $g, h\in G=*G_i$ *satisfy* $gh=hg$. *Then either both* g *and* h *are in the same conjugate of a free factor, or* g *and* h *are both powers of the same element of* G.

An element $g\in G$ is called *cyclically reduced* if its reduced expression is given as $g=g_1\cdots g_r$, $r\geqq 2$, where g_1 and g_r are from different free factors. The next fact is useful.

LEMMA 2.5 ([**12**, Theorem 4.2]). *Every element* $g\in G=*G_i$ *is conjugate to an element of a free factor or a cyclically reduced element.*

LEMMA 2.6. *Let* g *and* h *be cyclically reduced elements of* $G=*G_i$ *with reduced expressions* $g=g_1\cdots g_r$, $h=h_1\cdots h_s$, $r, s\geqq 2$. *Suppose that* $g^p=h^q$ *with* $p, q\geqq 1$. *Let* d *be the greatest common divisor of* r *and* s. *Then,* $pr=qs$, $d\geqq 2$ *and* $g_1\cdots g_d=h_1\cdots h_d$. *Moreover,* $g=(g_1\cdots g_d)^{r/d}$ *and* $h=(h_1\cdots h_d)^{s/d}$.

PROOF. By assumption, both sides of

$$\overbrace{(g_1\cdots g_r)\cdots(g_1\cdots g_r)}^{p\ \text{times}}=\overbrace{(h_1\cdots h_s)\cdots(h_1\cdots h_s)}^{q\ \text{times}}$$

are reduced expressions, and hence $pr=qs$. Without loss of generality we may assume that $r\leqq s$. Suppose $1\leqq i, j\leqq r$. Then, $g_i=g_j$ if there is an integer m such that $i\equiv mr+j \pmod s$. For, in that case $g_i=h_i=g_j$. On the other hand, if $i-j\equiv 0 \pmod d$, we may find two integers m_1, m_2 such that $m_1r+m_2s=i-j$. Then $i\equiv m_1r+j \pmod s$ and we get $g_i=g_j$. This proves that $g=(g_1\cdots g_d)^{r/d}$. Therefore $g_1\cdots g_d=h_1\cdots h_d$ and $h=(h_1\cdots h_d)^{s/d}$. Q.E.D.

The centralizer of $g\in G$ is denoted by $Z(g)=\{h\in G; gh=hg\}$.

LEMMA 2.7. *Let* $g\in G=*G_i$ *and* $g\neq e$. *Then:*

(1) *If* $g \in G_i$, *then* $Z(g) \subset G_i$.

(2) *If* g *is a cyclically reduced element, then* $Z(g) = \langle w \rangle$, *where* w *is a cyclically reduced element of the smallest length such that* $g = w^p$ *for some* $p \geqq 1$. *This* w *is uniquely determined.*

(3) $Z(g)$ *is a subgroup of a conjugate of a free factor or is an infinite cyclic group.*

PROOF. (1) is immediate by reduced expression. For (2) it suffices to show the inclusion $Z(g) \subset \langle w \rangle$. Let $w = w_1 \cdots w_r$, $r \geqq 2$, be the reduced expression of w and assume that $g = w^p$ with $p \geqq 1$. Take $h \in Z(g)$, $h \neq e$. In view of Lemma 2.4, g and h are powers of the same element, say, $g = v^q$, $h = v^{q'}$. We may assume that $q \geqq 1$. Let $v = v_1 \cdots v_s$ be the reduced expression. Since g is a cyclically reduced element, so is v. We now note that $w^p = g = v^q$. With the help of Lemma 2.6, taking the greatest common divisor d of r and s, we see that $w = (w_1 \cdots w_d)^{r/d}$, $v = (v_1 \cdots v_d)^{s/d}$ and $w_1 \cdots w_d = v_1 \cdots v_d$. But it follows from the assumption on w that $d = r$. Hence, $v \in \langle w \rangle$ and $h = v^{q'} \in \langle w \rangle$.

(3) follows directly from Lemma 2.5 and the above results. Q.E.D.

LEMMA 2.8. *Let* $\gamma \in G = *G_i$. *Then* $G_i \cap G_j^\gamma = \{e\}$ *unless* $i = j$ *and* $\gamma \in G_j$.

PROOF. Immediate by reduced expression. Q.E.D.

LEMMA 2.9. *Let* $g \in G = *G_i$. *If* $g^n \neq e$ *and* $g^n \in G_i$ *for some* n, *then* $g \in G_i$.

PROOF. It follows from Lemma 2.7 that $Z(g^n) \subset G_i$. On the other hand, $g \in Z(g^n)$ is obvious. Hence $g \in G_i$. Q.E.D.

PROPOSITION 2.10. *Let* $g \in G = *G_i$ *and* $g \neq e$. *Then one of the following two possibilities occurs:*

(i) $Z(g) \cap G_i^\gamma \neq \{e\}$ *for some pair* $i \in I$, $\gamma \in G$. *In this case* $Z(g) \subset G_i^\gamma$.

(ii) $Z(g) \cap G_i^\gamma = \{e\}$ *for all* $i \in I$ *and* $\gamma \in G$. *In this case* $Z(g)$ *is an infinite cyclic group generated by an element which does not belong to any conjugate of any free factor.*

PROOF. Suppose that $Z(g)$ is not a subgroup of any conjugate of any free factor. It then follows from Lemma 2.7 that $Z(g)$ is an infinite cyclic group, say, $Z(g) = \langle c \rangle$, $c \in G$. Then by Lemma 2.9 we have $Z(g) \cap G_i^\gamma = \{e\}$ for all $i \in I$ and $\gamma \in G$. In this case, c is not in any conjugates of free factors. Q.E.D.

PROOF OF THEOREM 2.1. Take $g \in H$, $g \neq e$, and note that $H \subset Z(g)$ since H is abelian. We then need only to apply Proposition 2.10 to get the results. Q.E.D.

LEMMA 2.11. *Let* $g, h \in G = *G_i$. *Assume that* g *is not in any conjugate of any free factor. If* $g^n = h^n$ *with some* $n \neq 0$, *then* $g = h$.

PROOF. By Lemma 2.5 we may assume that g is a cyclically reduced element. It then follows from Lemma 2.7 that $Z(g)=\langle w\rangle$ and $g=w^p$ with a cyclically reduced element w. Obviously, $Z(h^n)=Z(g^n)=Z(g)=\langle w\rangle$. Hence $h\in\langle w\rangle$, say, $h=w^q$. Since $w^{pn}=g^n=h^n=w^{qn}$, we have $p=q$ and therefore $g=h$. Q.E.D.

We are now in a position to give the

PROOF OF THEOREM 2.2. We put $G=*G_i$. Suppose that $H, K\in\mathfrak{M}_\infty(G)$ satisfy the condition $H\sim K$. It is sufficient to show that $H=K$. Note that $H\cap K\neq\{e\}$, for $|H: H\cap K|<\infty$ and $|H|=\infty$. According to Theorem 2.1 we consider two cases for H.

(*Case* 1) Suppose that $H\subset G_i^\gamma$ for some $i\in I$ and $\gamma\in G$. Then $\{e\}\neq H\cap K\subset G_i^\gamma\cap K$. It follows from Theorem 2.1 that $K\subset G_i^\gamma$. Thus both H and K are maximal abelian subgroups of G_i^γ with the property $H\sim K$. By assumption $\mathfrak{M}_\infty(G_i)$ satisfies (P) and therefore $H=K$.

(*Case* 2) Assume that $H\cap G_i^\gamma=\{e\}$ for all $i\in I$ and $\gamma\in G$ and put $H=\langle c\rangle$. Since $H\cap K\neq\{e\}$, K is not a subgroup of any conjugate of any free factor. Therefore, K is also an infinite cyclic group and we put $K=\langle d\rangle$. Since $H\cap K\neq\{e\}$, we may write $H\cap K=\langle c^p\rangle=\langle d^q\rangle$ with some $p\geqq 1$, $q\geqq 1$. Then $c^p=d^{\pm q}$. Moreover,

$$c^p=d^{-1}d^{\pm q}d=d^{-1}c^pd=(d^{-1}cd)^p\,.$$

Since c does not belong to any conjugate of any free factor, in view of Lemma 2.11 we have $c=d^{-1}cd$. Consequently, $\langle c, d\rangle$ is an abelian group containing both H and K. Since H and K are maximal, $H=\langle c, d\rangle=K$. We have thus proved (P) for $\mathfrak{M}_\infty(G)$. Q.E.D.

PROOF OF THEOREM 2.3. (1) For any $g\in N_G(H)$ we have $H^g=H\subset G_i^\gamma$ and $H\subset G_i^{\gamma g^{-1}}\cap G_i^\gamma$. In particular, $G_i^{\gamma g^{-1}}\cap G_i^\gamma\neq\{e\}$. It follows from Lemma 2.8 that $g\in G_i^\gamma$, namely, $N_G(H)\subset G_i^\gamma$. In other words, $N_G(H)=N_{G_i^\gamma}(H)$.

(2) It follows from Theorem 2.1 that H is an infinite cyclic group, say, $H=\langle c\rangle$. For any $g\in N_G(H)$ we have $\langle c\rangle=H=H^g=\langle g^{-1}cg\rangle$. Hence $c=g^{-1}cg$ or $c=g^{-1}c^{-1}g$. In the former case $\langle c, g\rangle$ is an abelian group containing H. Then $g\in H$, for H is a maximal abelian subgroup of G. In the latter case, observing that

$$c=g^{-1}c^{-1}g=g^{-1}(g^{-1}cg)g=g^{-2}cg^2\,,$$

we have $g^2=c^p$ for some p. Suppose first that $p\neq 0$. Then,

$$c^p=g^2=gg^2g^{-1}=gc^pg^{-1}=(gcg^{-1})^p\,.$$

Recalling that c is not in any conjugate of any free factor, we see from Lemma 2.11 that $c=gcg^{-1}$. This implies contradiction, for $c^{-1}=gcg^{-1}=c$. Hence $g^2=e$.

The action of $N_G(H)$ on H induces a homomorphism $\alpha: N_G(H)\to\mathrm{Aut}(H)$. Since H is a maximal abelian subgroup of G, we see that α induces an injective homomorphism $N_G(H)/H\to\mathrm{Aut}(H)$. On the other hand, it is evident that $|\mathrm{Aut}(H)|=2$. Consequently, $|N_G(H):H|\leqq 2$. Q.E.D.

For structure of each maximal abelian subgroup of free products we only mention the following result.

PROPOSITION 2.12. *Let $G=*G_i$ be a free product of abelian groups G_i, $i\in I$. If H is a maximal abelian subgroup of G, then $H=Z(g)$ for any $g\in H$ with $g\neq e$. In particular, the maximal abelian subgroup of G containing $g\in G$, $g\neq e$, is unique and given by $Z(g)$.*

PROOF. Suppose $H\in\mathfrak{M}(G)$ and take $g\in H$, $g\neq e$. Obviously, $H\subset Z(g)$. Then we need only to show that $Z(g)$ is abelian. But this is immediate from Proposition 2.10 and the assumption that each free factor is abelian. Q.E.D.

COROLLARY 2.13. *Let G be the same as in Proposition 2.12. For a maximal abelian subgroup H of G one of the following two possibilities occurs:*

(i) *H is conjugate to a free factor G_i;*

(ii) *$H\cap G_i^\gamma=\{e\}$ for any $i\in I$ and $\gamma\in G$. In this case H is an infinite cyclic group generated by an element which does not belong to any conjugate of any free factor.*

§3. Induced representations of amalgams of discrete abelian groups.

Given discrete groups G_i, $i\in I$, we denote by $G=*_Z G_i$ their free product with the common subgroup Z amalgamated. To avoid inessential argument we always assume that $G_i\neq Z$ for all $i\in I$. We call such a group G an *amalgam* for short. As in the case of free products, each G_i is called a *free factor* of G.

Throughout this section we assume that every free factor G_i is abelian. (Later on we add one more assumption called (A).) Then, Z becomes the center of G and there is a canonical isomorphism between G/Z and the free product $*(G_i/Z)$. Let π be the canonical projection from G onto G/Z.

We begin with the following

PROPOSITION 3.1. *For a maximal abelian subgroup H of $G=*_Z G_i$ one of the following two possibilities occurs:*

(i) *H is conjugate to a free factor;*

(ii) *$H\cap G_i^\gamma=Z$ for all $i\in I$ and $\gamma\in G$. In this case there exists an element $w\in H$ of infinite order such that $H=Z\times\langle w\rangle$.*

In any case H/Z is a maximal abelian subgroup of G/Z.

PROOF. Since Z is the center of G, $Z\subset H$ and therefore $\pi(H)=H/Z\neq\{\pi(e)\}$ is an abelian subgroup of $G/Z=*(G_i/Z)$. According to Theorem 2.1 we consider two possibilities for H/Z.

(*Case* 1) Suppose that $H/Z\subset(G_i/Z)^{\pi(\gamma)}=G_i^{\gamma}/Z$ for some $\gamma\in G$ and $i\in I$. Then, $H\subset G_i^{\gamma}$. Since H is a maximal abelian subgroup, we have $H=G_i^{\gamma}$. In this case, obviously, H/Z is a maximal abelian subgroup of $G/Z=*(G_i/Z)$.

(*Case* 2) Suppose that $H/Z\cap G_i^{\gamma}/Z=\{\pi(e)\}$ for all $i\in I$ and $\gamma\in G$, and put $H/Z=\langle\pi(w)\rangle$. Obviously, $H=\pi^{-1}(H/Z)=Z\vee\langle w\rangle$. Since H is a maximal abelian subgroup, $w\notin Z$. Moreover, $Z\cap\langle w\rangle=\{e\}$. In fact, $w^p\in Z$ implies that $\pi(w)^p=\pi(e)$ and $p=0$. Therefore H admits a direct product decomposition $H=Z\times\langle w\rangle$. We next show that $\langle\pi(w)\rangle$ is a maximal abelian subgroup of G/Z. Since $\pi(w)$ does not belong to any conjugate of any free factor of G/Z, an abelian group containing $\langle\pi(w)\rangle$ should be an infinite cyclic group, say, $\langle\pi(v)\rangle$. Then we have $Z\times\langle w\rangle\subset Z\times\langle v\rangle\subset G$. Since $Z\times\langle w\rangle$ is a maximal abelian subgroup of G, it coincides with $Z\times\langle v\rangle$ and hence $\langle\pi(v)\rangle=\langle\pi(w)\rangle$. Q.E.D.

For structure of each maximal abelian subgroup of $G=*_Z G_i$ we only mention the following fact. The proof is easy from Proposition 2.12.

PROPOSITION 3.2. *If H is a maximal abelian subgroup of $G=*_Z G_i$, then $H=Z(g)$ for any $g\in H-Z$.*

LEMMA 3.3. *$G_i\cap G_j^{\gamma}=Z$ unless $i=j$ and $\gamma\in G_j$.*

PROOF. Immediate from Lemma 2.8. Q.E.D.

From now on we impose one more assumption on the amalgam $G=*_Z G_i$ of discrete abelian groups. Let us assume

(A) $|G_i: Z|=\infty$ if $|G_i|=\infty$.

For instance, an amalgam of finite abelian groups satisfies this condition. The following two results are very important to our goal.

LEMMA 3.4. *$\mathfrak{M}_\infty(G)$ satisfies* (C) *and* (P).

PROOF. (C) is obvious. We shall prove (P). Let $H\in\mathfrak{M}_\infty(G)$ and $K\in\mathfrak{M}_\infty(G)$ satisfy the condition $H\sim K$. In view of Proposition 3.1 we consider three cases:

(*Case* 1) Suppose that both H and K are conjugate to some free factors, say, $H=G_i^{\gamma}$ and $K=G_j^{\delta}$. By assumption, we have $|G_i: G_i\cap G_j^{\delta\gamma^{-1}}|<\infty$ and $|G_j: G_j\cap G_i^{\gamma\delta^{-1}}|<\infty$. Then we see from (A) that $G_i\cap G_j^{\delta\gamma^{-1}}\neq Z$. Hence by Lemma 3.3 we have $i=j$ and $\delta\gamma^{-1}\in G_j$. Therefore, $H=K$.

(*Case* 2) Suppose that H is conjugate to a free factor, say, $H=G_i^{\gamma}$, and that $K\cap G_j^{\delta}=Z$ for all $j\in I$ and $\delta\in G$. Then $H\cap K=Z$ and $|G_i: Z|=|H: H\cap K|<\infty$. But this is impossible due to the assumption (A).

(*Case* 3) Suppose that $H\cap G_i^\gamma=K\cap G_i^\gamma=Z$ for all $i\in I$ and $\gamma\in G$. In view of Proposition 3.1 we have $H/Z\sim K/Z$ in G/Z. On the other hand, note that $\mathfrak{M}_\infty(G_i/Z)$ satisfies (P) because G_i/Z is abelian. It follows from Theorem 2.2 that $\mathfrak{M}_\infty(*(G_i/Z))=\mathfrak{M}_\infty(G/Z)$ satisfies (P). Since both H/Z and K/Z are in $\mathfrak{M}_\infty(G/Z)$ by Proposition 3.1, we see that $H/Z=K/Z$, and therefore $H=K$. Q.E.D.

LEMMA 3.5. $|N_G(H):H|\leqq 2$ *for all* $H\in\mathfrak{M}_\infty(G)$.

PROOF. Suppose that $H\in\mathfrak{M}_\infty(G)$. Then, by Proposition 3.1 and assumption (A) we see that $H/Z\in\mathfrak{M}_\infty(G/Z)=\mathfrak{M}_\infty(*(G_i/Z))$. It then follows from Theorem 2.3(2) that $|N_{G/Z}(H/Z):H/Z|\leqq 2$. On the other hand, we have an obvious isomorphism

$$N_G(H)/H\cong N_{G/Z}(H/Z)/(H/Z).$$

The assertion is then immediate. Q.E.D.

According to Lemma 3.5 we put

$$\mathfrak{M}_\infty^+(G)=\{H\in\mathfrak{M}_\infty(G)\,;\,N_G(H)=H\}$$

and

$$\mathfrak{M}_\infty^-(G)=\{H\in\mathfrak{M}_\infty(G)\,;\,|N_G(H):H|=2\}.$$

Then $\mathfrak{M}_\infty(G)=\mathfrak{M}_\infty^+(G)\cup\mathfrak{M}_\infty^-(G)$. It follows from Lemma 3.3 that if $H\in\mathfrak{M}_\infty(G)$ is conjugate to a free factor, it belongs to $\mathfrak{M}_\infty^+(G)$. Suppose that $H\in\mathfrak{M}_\infty^-(G)$ and put $H=Z\times\langle h\rangle$. We see from Lemma 3.5 that there exists an element $z_0\in Z$ such that $ghg^{-1}=z_0h^{-1}$ for all $g\in N_G(H)-H$. This z_0 depends upon the choice of h in the direct product decomposition of H.

THEOREM 3.6. *Let* $G=*_ZG_i$ *be an amalgam of discrete abelian groups* G_i *satisfying* (A). *Let* $\mathfrak{N}$ *be the collection of all irreducible representations of subgroups in* $\mathfrak{M}_\infty(G)$. *Then:*

(1) *Any two representations in* $\mathfrak{R}=\{\mathrm{Ind}_H^G\chi\,;\,(\chi\,;H)\in\mathfrak{N}\}$ *are mutually equivalent or disjoint.*

(2) *Let* $(\chi\,;H)\in\mathfrak{N}$ *and* $(\psi\,;K)\in\mathfrak{N}$. *Then* $\mathrm{Ind}_H^G\chi\cong\mathrm{Ind}_K^G\psi$ *if and only if* $H=K^g$ *and* $\chi=\psi^g$ *for some* $g\in G$.

(3) *If* $H\in\mathfrak{M}_\infty^+(G)$, *then* $\mathrm{Ind}_H^G\chi$ *is irreducible.*

(4) *Assume* $H\in\mathfrak{M}_\infty^-(G)$ *and put* $H=Z\times\langle h\rangle$. *Then* $\mathrm{Ind}_H^G\chi$ *is irreducible if and only if* $\chi(h)^2\neq\chi(z_0)$. *If* $\chi(h)^2=\chi(z_0)$, *then* $\mathrm{Ind}_H^G\chi$ *is decomposed into a direct sum of two irreducible representations which are not mutually equivalent.*

PROOF. Note that only one-dimensional representations are irreducible ones of subgroups in $\mathfrak{M}_\infty(G)$. Hence by virtue of Lemma 3.4, we see that $\mathfrak{N}$ satisfies all the conditions of Theorem 1.7. Therefore (1)-(2) follow directly from Theorem 1.7 (2)-(3). Furthermore (3) follows from Theorem 1.7(4) because $Q(H)=N(H)=H$ for all $H\in\mathfrak{M}_\infty^+(G)$. We now prove (4). Suppose $H\in\mathfrak{M}_\infty^-(G)$, namely,

$|N(H):H|=2$. It follows from Theorem 1.7 (4) that $\mathrm{Ind}_H^G\chi$ is reducible if and only if $\dim \mathrm{Hom}_G(\mathrm{Ind}_H^G\chi, \mathrm{Ind}_H^G\chi)=2$. In this case, $\mathrm{Ind}_H^G\chi$ is a direct sum of two irreducible representations which are not mutually equivalent. On the other hand, $\mathrm{Ind}_H^G\chi$ is irreducible if and only if $\chi^g\neq\chi$ for all $g\in N(H)-H$. Recall that $ghg^{-1}=z_0h^{-1}$ for $g\in N(H)-H$. Then $\chi^g(h)=\chi(ghg^{-1})=\chi(z_0)\chi(h^{-1})$. Hence $\chi^g=\chi$ if and only if $\chi(h)^2=\chi(z_0)$. This completes the proof. Q.E.D.

Irreducible representations obtained in Theorem 3.6 are relevant to irreducible decompositions of the regular representation of G. We first note the following

LEMMA 3.7. *Let G be an arbitrary discrete group and H an abelian subgroup of G. Then,*

$$(\textit{regular representation of } G)\cong\int_{\hat{H}}^{\oplus}\mathrm{Ind}_H^G\chi d\chi,$$

where $\hat{H}$ denotes the dual of H and $d\chi$ the Haar measure of $\hat{H}$. If $Q(H)=H$, the above direct integral gives an irreducible decomposition. Moreover, two irreducible decompositions obtained from such abelian subgroups are completely different if and only if the abelian subgroups are not conjugate.

PROOF. The first assertion is well known and easy to see by Fourier transform on H. Note that H is a maximal abelian subgroup of G if $Q(H)=H$. Then the statements on irreducible decompositions are direct from Theorem 1.7. Q.E.D.

THEOREM 3.8. *Let $G=*_Z G_i$ be an amalgam of discrete abelian groups G_i satisfying* (A). *For each $H\in\mathfrak{M}_\infty(G)$ we have*

$$(\textit{regular representation of } G)\cong\int_{\hat{H}}^{\oplus}\mathrm{Ind}_H^G\chi d\chi,$$

where $\mathrm{Ind}_H^G\chi$ is irreducible for almost all $\chi\in\hat{H}$ with respect to the Haar measure $d\chi$. Moreover, if two maximal abelian subgroups in $\mathfrak{M}_\infty(G)$ are not conjugate, the corresponding irreducible decompositions are completely different.

PROOF. If $H\in\mathfrak{M}_\infty^{+}(G)$, the assertion is immediate from Theorem 3.6 (2)(3) and Lemma 3.7. Actually, in this case every $\mathrm{Ind}_H^G\chi$ is irreducible. If $H\in\mathfrak{M}_\infty^{-}(G)$, then $\mathrm{Ind}_H^G\chi$ can be reducible for some $\chi\in\hat{H}$. But such χ's form a null set of $\hat{H}$ with respect to the Haar measure $d\chi$. Q.E.D.

REMARK. In case of G being a free group the above result was shown by Kawakami [**9**], see also Kajiwara [**8**]. In that case we have $\mathfrak{M}(G)=\mathfrak{M}_\infty(G)=\mathfrak{M}_\infty^{+}(G)$. While, using particular maximal abelian subgroup of a free group, Yoshizawa [**21**] showed this kind of diversity of irreducible decompositions of the regular representation in the initial stage of unitary representation theory.

§ 4. An example $SL(2, \boldsymbol{Z})$.

One of the most interesting examples of amalgams of discrete groups is, with no doubt, the group $SL(2, \boldsymbol{Z})$ of all 2×2 matrices of integers with determinant one. We put

$$\sigma=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad e=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, $\boldsymbol{Z}/4\boldsymbol{Z}\cong\langle\sigma\rangle$, $\boldsymbol{Z}/6\boldsymbol{Z}\cong\langle\tau\rangle$ and $\boldsymbol{Z}/2\boldsymbol{Z}\cong\{\pm e\}$. With these isomorphisms we come to a classical result:

$$SL(2, \boldsymbol{Z}) \cong (\boldsymbol{Z}/4\boldsymbol{Z})*_{\boldsymbol{Z}/2\boldsymbol{Z}}(\boldsymbol{Z}/6\boldsymbol{Z}).$$

We therefore have the assertions in Theorems 3.6 and 3.8 for $G=SL(2, \boldsymbol{Z})$. Moreover, the statement of Theorem 3.6 (4) is equivalent to the following

(4′) *Assume that* $H\in\mathfrak{M}_\infty^-(G)$. *Then* $\mathrm{Ind}_H^G\chi$ *is irreducible if and only if* $\chi\neq\bar{\chi}$. *If* $\chi=\bar{\chi}$, *then* $\mathrm{Ind}_H^G\chi$ *is decomposed into a direct sum of two irreducible representations which are not mutually equivalent.*

We give a brief account of this fact. Recall that each $H\in\mathfrak{M}_\infty^-(G)$ admits a direct product decomposition $H=Z\times\langle h\rangle$ and that there exists an element $z_0\in Z$ such that $ghg^{-1}=z_0h^{-1}$ for all $g\in N_G(H)-H$. But we have $z_0=e$ due to the fact that $\mathrm{Tr}(h)=\mathrm{Tr}(h^{-1})$. The assertion (4′) is then immediate.

We shall investigate structure of a maximal abelian subgroup of $G=SL(2, \boldsymbol{Z})$ in terms of matrices. With the help of Jordan's canonical form and Proposition 3.2 one may prove the following assertion easily.

LEMMA 4.1. *Let* $g\in G$, $g\neq\pm e$. *Then a maximal abelian subgroup of* G *containing* g *is unique and given by* $Z(g)$. *Moreover,* $Z(g)=(\boldsymbol{Q}g+\boldsymbol{Q}e)\cap SL(2, \boldsymbol{Z})$.

Given integers $n_1, \cdots, n_l\in\boldsymbol{Z}$, we denote by $((n_1, \cdots, n_l))$ their greatest common divisor, namely, the integer $d\geqq 0$ such that $n_1\boldsymbol{Z}+\cdots+n_l\boldsymbol{Z}=d\boldsymbol{Z}$.

PROPOSITION 4.2. *Given* $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$, $g\neq\pm e$, *we put*

$$\alpha=\frac{a-d}{((a-d, b, c))}, \quad \beta=\frac{b}{((a-d, b, c))}, \quad \gamma=\frac{-c}{((a-d, b, c))},$$

$$D=D(\alpha, \beta, \gamma)=\alpha^2-4\beta\gamma.$$

Then

$$Z(g)=\left\{\begin{pmatrix} (p+\alpha q)/2 & \beta q \\ -\gamma q & (p-\alpha q)/2 \end{pmatrix};\ \begin{matrix} p, q\in\boldsymbol{Z} \\ p^2-Dq^2=4 \end{matrix}\right\}.$$

PROOF. Note first that

$$\boldsymbol{Q}g+\boldsymbol{Q}e=\left\{\begin{pmatrix}\lambda a+\mu & \lambda b\\ \lambda c & \lambda d+\mu\end{pmatrix};\ \lambda,\ \mu\in\boldsymbol{Q}\right\}$$

and $((a-d,\ b,\ c))\geqq 1$. Changing parameters $\lambda,\ \mu\in\boldsymbol{Q}$ with $p,\ q\in\boldsymbol{Q}$ by the formula:

$$\begin{cases} p=(a+d)\lambda+2\mu \\ q=((a-d,\ b,\ c))\lambda\,, \end{cases}$$

we obtain $\boldsymbol{Q}g+\boldsymbol{Q}e=\{h(p,\ q)\,;\ p,\ q\in\boldsymbol{Q}\}$, where

$$h(p,\ q)=\begin{pmatrix}(p+\alpha q)/2 & \beta q\\ -\gamma q & (p-\alpha q)/2\end{pmatrix}.$$

In view of Lemma 4.1 we have $Z(g)=\{h(p,\ q)\,;\ p,\ q\in\boldsymbol{Q}\}\cap SL(2,\ \boldsymbol{Z})$. First suppose that $h(p,\ q)\in SL(2,\ \boldsymbol{Z}),\ p,\ q\in\boldsymbol{Q}$. Then $(p\pm\alpha q)/2,\ \beta q,\ -\gamma q\in\boldsymbol{Z}$, and therefore $p,\ q\in\boldsymbol{Z}$ since $((\alpha,\ \beta,\ \gamma))=1$. Moreover, since

$$1=\det h(p,\ q)=\frac{p^2-Dq^2}{4}\,,$$

$(p,\ q)$ is an integral solution of the equation $p^2-Dq^2=4$.

Conversely, suppose $(p,\ q)$ to be an integral solution of the equation $p^2-Dq^2=4$. Then, we have

$$(p+\alpha q)(p-\alpha q)=p^2-Dq^2-4\beta\gamma q^2=4(1-\beta\gamma q^2)\,.$$

Since $p+\alpha q\equiv p-\alpha q\ (\mathrm{mod}\ 2)$, we conclude that $h(p,\ q)\in SL(2,\ \boldsymbol{Z})$. Q.E.D.

COROLLARY 4.3. (1) *If* $\mathrm{Tr}(g)=0$, *then* $Z(g)$ *is conjugate to* $\langle\sigma\rangle$.
(2) *If* $|\mathrm{Tr}(g)|=1$, *then* $Z(g)$ *is conjugate to* $\langle\tau\rangle$.

PROOF. With the same notations as in Proposition 4.2 we have

$$D=\alpha^2-4\beta\gamma=\frac{(a-d)^2+4bc}{((a-d,\ b,\ c))^2}=\frac{(a+d)^2-4}{((a-d,\ b,\ c))^2}\,.$$

Hence $|\mathrm{Tr}(g)|=|a+d|<2$ if and only if $D<0$. In that case, $Z(g)$ is a finite group by Proposition 4.2. On the other hand, it follows from Lemma 4.1 that $Z(g)$ is a maximal abelian group and from Proposition 3.1 that it is conjugate to a free factor. Consequently, $Z(g)$ is conjugate to $\langle\sigma\rangle$ or $\langle\tau\rangle$ according as $\mathrm{Tr}(g)=0$ or $|\mathrm{Tr}(g)|=1$. Q.E.D.

We now give a parametrization of $\mathfrak{M}_\infty(G)$. Let $H=\mathfrak{M}_\infty(G)$ and take $g\in H$, $g\neq\pm e$. Then, by Lemma 4.1 we have $H=Z(g)$. We keep to the same notation as in Proposition 4.2. Note that $D=D(\alpha,\ \beta,\ \gamma)=\alpha^2-4\beta\gamma\geqq 0$, otherwise H becomes a finite group. Note also that $D\equiv 0$ or $1\ (\mathrm{mod}\ 4)$. Moreover, it is not difficult to see that D can not be a positive square. Accordingly, we put

$$\varDelta=\{\delta\in\boldsymbol{Z}\,;\ \delta\geqq 0,\ \delta\equiv 0,\ 1\ (\mathrm{mod}\ 4)\}-\{1^2,\ 2^2,\ 3^2,\ \cdots\}$$

and

$$\Lambda = \left\{ (\alpha, \beta, \gamma) \in \boldsymbol{Z}^3 ; \begin{array}{c} ((\alpha, \beta, \gamma))=1 \\ \alpha^2 - 4\beta\gamma \in \Delta \end{array} \right\}.$$

For $(\alpha, \beta, \gamma) \in \Lambda$ we put

$$H(\alpha, \beta, \gamma) = \left\{ \begin{pmatrix} (p+\alpha q)/2 & \beta q \\ -\gamma q & (p-\alpha q)/2 \end{pmatrix} ; \begin{array}{l} p, q \in \boldsymbol{Z} \\ p^2 - D(\alpha, \beta, \gamma) q^2 = 4 \end{array} \right\}.$$

Then, $H=Z(g)=H(\alpha, \beta, \gamma)$ by Proposition 4.2. Conversely, note that $H(\alpha, \beta, \gamma) \in \mathfrak{M}_\infty(G)$ for all $(\alpha, \beta, \gamma) \in \Lambda$. In fact, the theory of Pell's equation guarantees the existence of a non-trivial integral solution of $p^2 - D(\alpha, \beta, \gamma) q^2 = 4$, see e.g., [**20**, § 33]. More precisely, let (p_1, q_1) be its integral solution such that $p_1 > 0$ and $q_1 > 0$ are the smallest among the solutions. Let $g_1 = h(p_1, q_1)$ be the corresponding matrix. Then, $H(\alpha, \beta, \gamma) = Z(g_1) = \{\pm e\} \times \langle g_1 \rangle$. In particular, $H(\alpha, \beta, \gamma) \in \mathfrak{M}_\infty(G)$. The following assertion is now easy to see.

PROPOSITION 4.4. *Let $H \in \mathfrak{M}(G)$. Then, one of the following three possibilities occurs:*

(i) $|H|=4$ *and H is conjugate to* $\langle \sigma \rangle$;

(ii) $|H|=6$ *and H is conjugate to* $\langle \tau \rangle$;

(iii) $|H|=\infty$ *and $H=H(\alpha, \beta, \gamma)$ for some $(\alpha, \beta, \gamma) \in \Lambda$. In this case $H=\{\pm e\} \times \langle g \rangle$ for some $g \in G$.*

Furthermore, $\mathfrak{M}_\infty(G) = \{H(\alpha, \beta, \gamma) ; (\alpha, \beta, \gamma) \in \Lambda\}$ and $H(\alpha, \beta, \gamma) = H(\alpha', \beta', \gamma')$ if and only if $(\alpha, \beta, \gamma) = \pm(\alpha', \beta', \gamma')$.

We have not obtained a brief description of conjugacy classes of $\mathfrak{M}_\infty(G)$. Here we only mention a relation between $\mathfrak{M}_\infty(G)$ and quadratic forms (or real quadratic number fields). For $(\alpha, \beta, \gamma) \in \Lambda$ we put

$$F(\alpha, \beta, \gamma) = \begin{pmatrix} \gamma & \alpha/2 \\ \alpha/2 & \beta \end{pmatrix}.$$

Then by a direct calculation we have

LEMMA 4.5. *$H(\alpha, \beta, \gamma) = \{g \in G ; {}^t g F(\alpha, \beta, \gamma) g = F(\alpha, \beta, \gamma)\}$.*

We say that $F(\alpha, \beta, \gamma)$ and $F(\alpha', \beta', \gamma')$ are *equivalent* if $F(\alpha', \beta', \gamma') = {}^t u F(\alpha, \beta, \gamma) u$ for some $u \in G$. The following result is then immediate.

PROPOSITION 4.6. *$H(\alpha, \beta, \gamma)$ and $H(\alpha', \beta', \gamma')$ are conjugate in G if and only if $F(\alpha, \beta, \gamma)$ and $F(\alpha', \beta', \gamma')$ are equivalent. In that case $D(\alpha, \beta, \gamma) = D(\alpha', \beta', \gamma')$.*

Given $\delta \in \Delta$, we put

$$\Lambda_\delta = \{(\alpha, \beta, \gamma) \in \Lambda ; D(\alpha, \beta, \gamma) = \delta\}.$$

Then, as is easily verified, $\Lambda_\delta \neq \emptyset$ and $\Lambda = \cup_{\delta \in \Delta} \Lambda_\delta$. We therefore see from Proposition 4.6 that there are countably infinite number of conjugacy classes of maximal abelian subgroups in $\mathfrak{M}_\infty(G)$. Furthermore, it is noteworthy that the number of conjugacy classes of maximal abelian subgroups $H(\alpha, \beta, \gamma)$, where (α, β, γ) runs over Λ_δ, $\delta>0$, is equal to the class number of $\boldsymbol{Q}(\sqrt{\delta})$.

For $\mathfrak{M}_\infty^\pm(G)$ we only mention the following result. The proof is easy and omitted.

PROPOSITION 4.7. *Given* $H=H(\alpha, \beta, \gamma)\in\mathfrak{M}_\infty(G)$, *consider*

$$\text{(E)} \qquad \begin{cases} \alpha x-\gamma y+\beta z=0 \\ x^2+yz=-1\,. \end{cases}$$

Then:

(1) (E) *admits no integral solution if and only if* $H\in\mathfrak{M}_\infty^+(G)$.

(2) (E) *admits an integral solution* $(x, y, z)\in\boldsymbol{Z}^3$ *if and only if* $H\in\mathfrak{M}_\infty^-(G)$. *In this case* $N_G(H)=H\cup\begin{pmatrix} x & y \\ z & -x\end{pmatrix}H$.

(3) *If* $(\alpha, \beta, \gamma)\in\Lambda_0$, *i.e.*, $\alpha^2-4\beta\gamma=0$, *then* $H(\alpha, \beta, \gamma)\in\mathfrak{M}_\infty^+(G)$.

REMARK. In [**17**] Saito determined all Cartan subgroups of $SL(2, \boldsymbol{Z})$ with a direct calculation of matrices. Comparing his discussion with Proposition 4.4, we see that the Cartan subgroups of $SL(2, \boldsymbol{Z})$ are in coincidence with $\mathfrak{M}_\infty(SL(2, \boldsymbol{Z}))$. It therefore turns out that the results of Theorems 3.6 and 3.8 for $G=SL(2, \boldsymbol{Z})$ are equivalent to Saito's ones [**17**, Theorems 4-7].

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