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Periods of cusp forms associated to loxodromic elements of b-groups

Dedicated to Michio Kuga* on his sixtieth birthday

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The purpose of this note is to explore the periods of cusp forms associated to loxodromic elements of b-groups (function groups with simply connected invariant components). Let α and β be two distinct points in $\boldsymbol{C}\cup\{\infty\}$. Let

$$g_{\alpha,\beta}(z)=\frac{\alpha-\beta}{(z-\alpha)(z-\beta)}, \qquad z\in\boldsymbol{C}\cup\{\infty\}. \tag{0.1}$$

Let Γ be a finitely generated non-elementary Kleinian group with region of discontinuity $\Omega=\Omega(\Gamma)$ and limit set $\Lambda=\Lambda(\Gamma)$. Fix an integer $q\geqq 2$ and let $\boldsymbol{A}_q(\Omega, \Gamma)$ denote the space of cusp forms for Γ of weight $(-2q)$ (or cusp q-forms, for short). For $A\in\Gamma$, a loxodromic (including hyperbolic) element with attractive fixed point α and repulsive fixed point β, we introduce the relative Poincaré series

$$\varphi_A(z)=\sum_{\gamma\in\Gamma_0\backslash\Gamma} g_{\alpha,\beta}^q(\gamma(z))\gamma'(z)^q, \qquad z\in\Omega, \tag{0.2}$$

where $\Gamma_0=\langle A\rangle$, the cyclic group generated by A. It was shown in [**K3**] that $\varphi_A\in\boldsymbol{A}_q(\Omega, \Gamma)$.

Assume now that Γ is a b-group and Δ is a simply connected invariant component of Γ (that is, of $\Omega(\Gamma)$). If B is a loxodromic element of Γ with attractive fixed point a and repulsive fixed point b, then the *period* $L_B(\varphi)$ *of* $\varphi\in\boldsymbol{A}_q(\Omega, \Gamma)$ *along* B is defined by

$$L_B(\varphi)=\int_{z_0}^{Bz_0} g_{a,b}^{1-q}(z)\varphi(z)dz. \tag{0.3}$$

The integral is independent of the point z_0 in Δ as long as the path of integration is restricted to lie in Δ. The period of φ depends, of course, only on $\varphi|\Delta$ (the space of restrictions of cusp forms to Δ will be denoted by $\boldsymbol{A}_q(\Delta, \Gamma)$).

The periods are conjugation invariant in the following sense. Let C_1 and C_2 be two arbitrary elements of $PSL(2, \boldsymbol{C})$ with the property $C_1\Gamma C_1^{-1}=C_2\Gamma C_2^{-1}$,

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then

$$L_B(\varphi_A) = L_{C_1 \circ B \circ C_1^{-1}}(\varphi_{C_2 \circ A \circ C_2^{-1}}).$$

The starting point of this investigation is the following theorem which will be proved in § 2.

THEOREM. *Let Γ be a finitely generated quasi-Fuchsian group of the first kind. For A a loxodromic element of Γ, the complex number $L_A(\varphi_A)$ is independent of the component of $\Omega(\Gamma)$ used to define L_A.*

REMARK. The linear functional L_B is defined, of course, on the larger space of holomorphic q-forms for Γ on Δ. See § 4.

Let Δ and $\tilde{\Delta}$ be the two invariant components of the finitely generated quasi-Fuchsian group Γ (of the first kind). One of the motivations behind our study is to determine whether it is possible for $\varphi_A|\Delta=0$ and $\varphi_A|\tilde{\Delta}\neq 0$. (This cannot, of course, occur for Fuchsian groups. If $\Gamma\subset PSL(2, \boldsymbol{R})$, then $\varphi_A(\bar{z})=\overline{\varphi_A(z)}$, all $z\in\Omega$.) Our theorem gives evidence to the claim that φ_A cannot vanish on only one component; but does not, of course, establish this claim. See [**K3**, § 4.5] for more on this question. The theorem does, however, establish the following

COROLLARY. *If $L_A(\varphi_A)\neq 0$, then φ_A does not vanish on either component of Γ.*

§ 1. Deformation spaces.

As before Γ is a finitely generated non-elementary Kleinian group with region of discontinuity Ω and limit set Λ. We assume that Γ has been normalized so that

$$\{0, 1, \infty\} \subset \Lambda.$$

Let $M(\Gamma)$ be the space of Beltrami coefficients for Γ; that is, $M(\Gamma)$ is the open unit ball in

$$L^\infty(\Gamma) = \{\mu\in L^\infty(\boldsymbol{C};\ \boldsymbol{C});\quad (\mu\circ\gamma)\bar{\gamma}'=\mu\gamma',\ \text{all}\ \gamma\in\Gamma\}.$$

For $\mu\in M(\Gamma)$, let w^μ be the unique normalized (fixing $0, 1, \infty$) μ-conformal (satisfying the Beltrami equation $w_{\bar{z}}=\mu w_z$) automorphism of $\boldsymbol{C}\cup\{\infty\}$. The *deformation space* $T(\Gamma)$ is defined as the set of restrictions to Λ of mappings w^μ with $\mu\in M(\Gamma)$. For $\mu\in M(\Gamma)$, $[\mu]$ will denote its image in $T(\Gamma)$; that is, $[\mu]=w^\mu|\Lambda$. It is well known ([**B**], [**M**] and [**K1**]) that $T(\Gamma)$ is a complex manifold of the same dimension as $A_2(\Omega, \Gamma)$.

The *Bers fiber space* $F(\Gamma)$ is defined as

$$F(\Gamma) = \{([\mu], z)\ ;\ \mu\in M(\Gamma),\ z\in w^\mu(\Omega)\}.$$

It is a complex, not necessarily connected, manifold with a natural projection onto $T(\Gamma)$. The fiber over $[\mu]\in T(\Gamma)$ is $w^{\mu}(\Omega)=\Omega(\Gamma^{\mu})$, where $\Gamma^{\mu}=w^{\mu}\Gamma(w^{\mu})^{-1}$.

The group Γ acts on $F(\Gamma)$ in a fiber preserving manner by the rule

$$\gamma([\mu], z)=([\mu], \gamma^{\mu}(z)),$$

where $\gamma\in\Gamma$, $\mu\in M(\Gamma)$, $z\in\Omega^{\mu}$, $\gamma^{\mu}=w^{\mu}\circ\gamma\circ(w^{\mu})^{-1}$. The action of Γ on $F(\Gamma)$ is holomorphic. For $z\in\Lambda$, $z\neq\infty$, the mapping

$$M(\Gamma)\ni\mu\longmapsto w^{\mu}(z)\in \boldsymbol{C}$$

is holomorphic and defines a function (also holomorphic) on $T(\Gamma)$. Using this observation, we can extend g of (0.1) to be a holomorphic function on $F(\Gamma)$ by defining

$$G_{\alpha,\beta}([\mu], z)=g_{w^{\mu}(\alpha), w^{\mu}(\beta)}(z).$$

Observe that $w^{\mu}(\alpha)$ and $w^{\mu}(\beta)$ are well defined and hence holomorphic functions of $[\mu]\in T(\Gamma)$. It is then easy to show, using standard L^1 estimates, that

$$\Phi_A([\mu], z)=\sum_{\gamma\in\Gamma_0\backslash\Gamma}G_{\alpha,\beta}(\gamma([\mu], z))(\gamma^{\mu})'(z)^q=\varphi_{A^{\mu}}(z), \quad \mu\in M(\Gamma),\ z\in w^{\mu}(\Gamma),$$

defines a holomorphic (cusp) form for the action of Γ on $F(\Gamma)$; that is,

$$\Phi_A(\gamma([\mu], z))(\gamma^{\mu})'(z)^q=\Phi_A([\mu], z), \quad \text{all } \gamma\in\Gamma, \text{ all } \mu\in M(\Gamma), \text{ all } z\in\Omega^{\mu}.$$

This construction extends φ_A of (0.2) to $F(\Gamma)$.

REMARK. Assume that the loxodromic element $A\in\Gamma$ has multiplier K with $0<|K|<1$. Assume that $q=2$. It follows that

$$\iint_{\Omega_0/\Gamma_0}|g^2_{\alpha,\beta}(z)dz\wedge d\bar{z}|=-4\pi\log|K|,$$

where $\Omega_0=\boldsymbol{C}\cup\{\infty\}-\{\alpha, \beta\}$ and $\Gamma_0=\langle A\rangle$. Thus

$$\|\varphi_A\|=\iint_{\Omega/\Gamma}|\varphi_A(z)dz\wedge d\bar{z}|\leqq -4\pi\log|K|,$$

and

$$\|\varphi_{A^{\mu}}\|=\iint_{\Omega^{\mu}/\Gamma^{\mu}}|\varphi_{A^{\mu}}(z)dz\wedge d\bar{z}|\leqq -4\pi\log|K^{\mu}|,$$

where K^{μ} is the multiplier of A^{μ}. Since

$$M(\Gamma)\ni\mu\longmapsto K^{\mu}\in\{z\in\boldsymbol{C};\ 0<|z|<1\}$$

is a holomorphic map, hence distance decreasing (in the Poincaré metric), it follows that

$$|K|^{\kappa}\leqq|K^{\mu}|\leqq|K|^{1/\kappa},$$

where

$$\kappa=\frac{1+\|\mu\|_{\infty}}{1-\|\mu\|_{\infty}} \quad \text{and} \quad \|\mu\|_{\infty} \text{ is the } L^{\infty}\text{-norm of } \mu.$$

We conclude that

$$\|\varphi_{A^\mu}\| \leqq -4\pi\kappa \log|K|\,.$$

PROPOSITION. *Let Δ be a simply connected component of Ω. Let $B\in\Gamma$ be loxodromic with attractive fixed point a and repulsive fixed point b (as before) and $B(\Delta)=\Delta$ (that is, $B\in\Gamma_\Delta$, stabilizer of Δ in Γ). Then*

$$T(\Gamma)\ni[\mu]\longmapsto L_{B^\mu}(\varphi_{A^\mu})\in \boldsymbol{C}$$

is a holomorphic function on the deformation space.

PROOF. We begin by examining

$$L_{B^\mu}(\varphi_{A^\mu})=\int_{z_0}^{B^\mu(z_0)} g^{1-q}_{w^\mu(a),\,w^\mu(b)}(z)\varphi_{A^\mu}(z)dz\,,$$

and observe that the integrand (as previously remarked) and the upper limit of integration are holomorphic functions on the deformation space as long as $z_0\in w^\mu(\Delta)$. This latter condition can be achieved by choosing $z_0\in w^{\mu_0}(\Delta)$ for some $\mu_0\in M(\Gamma)$, and restricting μ to lie in a sufficiently small neighborhood of μ_0. We note that although $\mu\mapsto w^\mu(z_0)$ is not a well defined function on $T(\Gamma)$, $\mu\mapsto B^\mu(z_0)$ is well defined and hence holomorphic. Alternatively, it suffices for our purposes to consider $[\mu]$ in a neighborhood of zero (by right translation). Local coordinates on $T(\Gamma)$ in a neighborhood of $\mu=0$ may be obtained by considering *harmonic* Beltrami coefficients; that is, elements of the form $=\lambda^{-2}\bar{\varphi}$, where λ is the Poincaré metric on Ω and $\varphi\in A_2(\Omega,\Gamma)$.

§ 2. Periods of cusp forms.

In this section we prove the theorem of the introduction. Let Γ be a finitely generated Fuchsian group of the first kind acting on the upper half plane U. (Hence also on the lower half plane U^*.) In this case,

$$\overline{\varphi_A(\bar{z})}=\varphi_A(z),\qquad \text{all } z\in U\cup U^*,\ \text{all hyperbolic } A\in\Gamma\,.$$

It was shown in [**K3**], that

$$L_A(\varphi_A)=\frac{1}{2\pi}\frac{(2q-2)(2q-4)\cdots4\cdot2}{(2q-3)(2q-5)\cdots3\cdot1}\iint_{\Delta/\Gamma}\lambda(z)^{2-2q}|\varphi_A(z)|idz\wedge d\bar{z}\,,\tag{2.1}$$

where we can use for Δ *either* U^* or U in defining $L_A(\varphi_A)$. (As before, λ is the Poincaré metric on Δ.)

Now we view $L_{A^\mu}(\varphi_{A^\mu})$ as two functions on $T(\Gamma)$ as in the Proposition by considering first U and then U^* in defining the period. Observe that $T(\Gamma)$ is the space of quasi-Fuchsian groups and that the real points (see [**KM**]) in $T(\Gamma)$ are precisely the Fuchsian groups. To be more specific, we showed in [**KM**] that we can choose $d+3$ points in $\Lambda(\Gamma)$, ∞, 0, 1, $a_1,\cdots,a_d$, $d=\dim A_2(\Omega,\Gamma)$, so that the holomorphic map

$$\mu \longmapsto (w^{\mu}(a_1), \cdots, w^{\mu}(a_d))$$

establishes an isomorphism between the space of quasi-Fuchsian groups $T(\Gamma)$ and a domain D in $\boldsymbol{C}^d$. Hence we can identify $T(\Gamma)$ with its image D under this map. The space of Fuchsian groups (a real analytic model for Teichmüller space) can be identified with the points in D all of whose coordinates are real (for more details see [**KM**]). The two holomorphic functions on $T(\Gamma)$ (hence also on D) that we have constructed agree on the real points of D. Hence they agree everywhere on D (equivalently on $T(\Gamma)$).

REMARK. As above, the real points in $T(\Gamma)$ can be canonically identified with points in the Teichmüller space $T(p, n)$, where (p, n) is the type of Γ. This identification is real but not complex analytic. We shall henceforth make this identification whenever complex analyticity is not an issue.

Note that for Fuchsian Γ, we have, from (2.1), the equivalence

$$L_A(\varphi_A) = 0 \Longleftrightarrow \varphi_A = 0 .$$

Let us also assume that A is primitive. We have shown in [**K3**] that if $\gamma_1, \gamma_2, \gamma_3, \cdots$ is a set of coset representatives for $\Gamma_0 \backslash (\Gamma - \Gamma_0)$, then

$$L_A(\varphi_A) = \log K + \sum_{j=1}^{\infty} (h_j(Az_0) - h_j(z_0)),$$

where K is the multiplier of A, chosen so that $0<K<1$, $\log K \in \boldsymbol{R}$, and h_j is defined by

$$h'_j = g_{\alpha,\beta}^{1-q} g^q_{\gamma_j^{-1}(\alpha), \gamma_j^{-1}(\beta)}, \qquad h(\alpha) = 0 .$$

We have also shown in [**K3**] that the vanishing of φ_A is equivalent to the vanishing of a certain linear functional l_A on the space of parabolic cohomology classes $PH^1(\Gamma, \Pi_{2q-2})$, and that this latter condition is verifiable via linear algebra. Here, Π_{2q-2} is the vector space of polynomials of degree $\leqq 2q-2$ and Γ acts on Π_{2q-2} via the Eichler representation. For the convenience of the reader we give a definition of the functional l_A. Let χ be a cocycle representing a cohomology class in $PH^1(\Gamma, \Pi_{2q-2})$. Expand the polynomial $\chi(A) \in \Pi_{2q-2}$ using eigenvalues for the automorphism that A induces on Π_{2q-2}:

$$\chi(A)(z) = \sum_{j=0}^{2q-2} a_j(\alpha-\beta)^{-j}(z-\alpha)^j(z-\beta)^{2q-2-j}, \qquad z \in \boldsymbol{C};$$

here α and β are the attractive and repulsive fixed points on A, respectively (as before). The definition of the linear functional now reads

$$l_A(\chi) = a_{q-1} .$$

Our results of this paper when combined with the work in [**K3**] yield the following

THEOREM. *Let Γ be a finitely generated Fuchsian group of the first kind. Let $A \in \Gamma$ be primitive hyperbolic with multiplier K. Then following conditions are equivalent:*

(a) $l_A \in PH^1(\Gamma, \Pi_{2q-2})^*$ *is the zero linear functional,*

(b) φ_A *vanishes on one component of* $\Omega(\Gamma)$,

(c) $\varphi_A = 0$,

(d) $L_A(\varphi_A) = 0$ *(using either component of* $\Omega(\Gamma)$*),*

(e) $L_A = 0$ *(on one, hence both, components of* $\Omega(\Gamma)$*),*

(f) $\log K + \sum_{j=1}^{\infty} (h_j(Az) - h_j(z)) = 0$, *all* $z \in \Omega$, *and*

(g) $\log K + \sum_{j=1}^{\infty} (h_j(Az_0) - h_j(z_0)) = 0$, *some* $z_0 \in \Omega$.

The reader is refered to [**K3**] for alternate definitions of l_A as well as additional properties of this linear functional, the cohomology space $PH^1(\Gamma, \Pi_{2q-2})$ and its dual space $PH^1(\Gamma, \Pi_{2q-2})^*$.

§3. Separating elements of Γ by cohomology classes in $H^1(\Gamma, \Pi_{2q-2})$.

Let Γ be an arbitrary Kleinian group. Let $A \in \Gamma$ be loxodromic or parabolic. Does there exist a cohomology class $\chi \in H^1(\Gamma, \Pi_{2q-2})$ that is non-trivial on A; that is, a χ such that $\chi | \langle A \rangle$ is not a coboundary? The question is equivalent to the non-triviality of the linear functional l_A (as defined in [**K2**] and [**K3**]). If A is parabolic and $q \geqq 3$ or $q=2$ and the fixed point a of A is cusped, then there exists a χ which is non-trivial on A if and only if a is q-admissible (see [**K2**]).

Next assume that A is loxodromic. Let $N(A)$ be the largest elementary subgroup of Γ that contains A. Then a necessary condition for the existence of a χ that is non-trivial on the element A is that $H^1(N(A), \Pi_{2q-2}) \neq \{0\}$. From [**K2**, Proposition 4.2], we see that $H^1(N(A), \Pi_{2q-2}) = \{0\}$ if $N(A)$ is isomorphic to $\boldsymbol{Z}_2 * \boldsymbol{Z}_2$ or to a double dihedral group and q is odd. A sufficient condition for the existence of a χ that is non-trivial on A is that $\varphi_A \neq 0$.

If the Bers map (see [**K2**], for example, for the definition)

$$\beta^* : A_q(\Omega, \Gamma) \longrightarrow PH^1(\Gamma, \Pi_{2q-2})$$

is subjective, then the existence of a $\chi \in PH^1(\Gamma, \Pi_{2q-2})$ that is non-trivial on A is equivalent to $\varphi_A \neq 0$.

Let Γ be a finitely generated Fuchsian group of the first kind, $\Gamma \subset PSL(2, \boldsymbol{R})$. We view the Teichmüller space of Γ as the real points in the deformation space $T(\Gamma)$; these correspond to symmetric Beltrami coefficients:

$$\{\mu \in M(\Gamma);\ \mu(\bar{z}) = \overline{\mu(z)},\ \text{almost all } z \in \boldsymbol{C}\}.$$

Let A be a hyperbolic element of Γ. We define the known and studied *length function* on the Teichmüller space. It is the restriction to the real points in $T(\Gamma)$ of the function

$$T(\Gamma)\ni[\mu]\overset{f_A}{\longmapsto}-\log K^{\mu}\in \boldsymbol{C}^*,$$

where K^{μ} is the multiplier of A^{μ}. We assume that the multiplier K of A and the branch of the logarithm have been chosen so that $0<K<1$ and $-\log K>0$. The function f_A is complex analytic on $T(\Gamma)$ and satisfies

$$\mathrm{Re}(-\log K^{\mu})>0, \qquad \text{all } [\mu]\in T(\Gamma).$$

If μ is symmetric (and thus represents a point in the Teichmüller space), then $-\log K^{\mu}\in \boldsymbol{R}^+$ is the length of the geodesic (closed curve) corresponding to the element $A\in\Gamma$ on the Riemann surface U/Γ^{μ}. Let

$$\dot{f}_A[\mu](\nu)=\lim_{t\to 0}\frac{f_A([\mu+t\nu])-f_A([\mu])}{t}, \qquad \mu\in M(\Gamma),\ \nu\in L^{\infty}(\Gamma).$$

It is easy to check that $\dot{f}_A[\mu]=0$ if and only if $\dot{f}_{A^{\mu}}[0]=0$, where

$$\dot{f}_{A^{\mu}}[0](\nu)=\lim_{t\to 0}\frac{f_{A^{\mu}}([t\nu])-f_{A^{\mu}}([0])}{t}, \qquad \nu\in L^{\infty}(\Gamma^{\mu}).$$

It is also well known that (see [**G**], [**K3**], [**W**])

$$\dot{f}_{A^{\mu}}[0](\nu)=\frac{1}{2\pi}\iint_{\Omega^{\mu}/\Gamma^{\mu}}\varphi_{A^{\mu}}(z)\nu(z)i\,dz\wedge d\bar{z}, \qquad \nu\in M(\Gamma^{\mu}).$$

Thus the critical points of f_A are precisely those $[\mu]\in T(\Gamma)$ for which $\varphi_{A^{\mu}}=0$.

Wolpert [**W**] has shown that on the Teichmüller space, all the critical points of f_A are minima and a minimum occurs if and only if A is essential (the complement of the projection to U/Γ of the axis of A consists of discs and punctured discs). We have established the following

THEOREM. *Let $A\in\Gamma$ be a hyperbolic element of a finitely generated Fuchsian group Γ of the first kind. The following conditions are equivalent:*

(a) $l_A=0$ *on* $PH^1(\Gamma, \Pi_2)$,

(b) $\varphi_A=0$ (*for* $q=2$), *and*

(c) *the curve on U/Γ corresponding to A is essential and has shorter length on U/Γ than on all (nearby) Fuchsian groups.*

We also remark that as a consequence of our theorem, condition (c) is verifiable by purely algebraic methods. Thus linear algebra alone can be used to decide whether a curve is both essential and of minimal length over Teichmüller space. This seems rather surprising.

§ 4. Cohomological interpretation of the periods.

Let Γ be a finitely generated quasi-Fuchsian group of the first kind with Δ one of its invariant components. Let $A_q^+(\Delta, \Gamma)$ denote the space of holomorphic automorphic q-forms for Γ on Δ (these are allowed to have a finite limit at the cusps; whereas cusp forms vanish at the cusps). The Eichler period map

$$\mathcal{E}: A_q^+(\Delta, \Gamma) \longrightarrow H^1(\Gamma, \Pi_{2q-2})$$

has been studied in [**K3**, § 4], where we have shown that for loxodromic $A \in \Gamma$, we have

$$l_A(\mathcal{E}\varphi) = \frac{(-1)^{q-1}}{(2q-2)} \binom{2q-2}{q-1} L_A(\varphi), \qquad \text{all } \varphi \in A_q^+(\Delta, \Gamma).$$

A similar interpretation exists for parabolic $A \in \Gamma$ (implied by the results of [**K2**]). The corresponding statements for elliptic $A \in \Gamma$ reduce to trivialities.

For fixed loxodromic $B \in \Gamma$, the cohomology class $\mathcal{E}(\varphi_B | \Delta)$ depends on the choice of component Δ of Γ. Hence (by the results [**K3**]) so does the sequence of complex numbers $\{L_A(\varphi_B)\}$ as A varies over the (conjugacy classes of) loxodromic elements of Γ. However, $L_B(\varphi_B)$ is independent of which component is used.

We will continue to investigate the relations between periods of automorphic forms and Eichler cohomology in a forthcoming paper.

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