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Notes on C^0 sufficiency of quasijets

Dedicated to Professor Masahisa Adachi on his 60th birthday

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Let $\mathcal{E}_{[k]}(n, 1)$ be the set of C^k function germs: $(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$ for $k=1, 2, \cdots, \infty, \omega$, and let $\mathcal{H}(n, 1)$ be the set of holomorphic function germs: $(\boldsymbol{C}^n, 0)\to(\boldsymbol{C}, 0)$. If for two function germs $f, g\in\mathcal{E}_{[k]}(n, 1)$ (resp. $\mathcal{H}(n, 1)$) there exists a local homeomorphism $\sigma:(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^n, 0)$ (resp. $\sigma:(\boldsymbol{C}^n, 0)\to(\boldsymbol{C}^n, 0)$) such that $f=g\circ\sigma$, we say that f is *C^0-equivalent* to g and write $f\sim g$. We shall not distinguish between germs and their representatives.

Consider the polynomial function $f:(\boldsymbol{R}^2, 0)\to(\boldsymbol{R}, 0)$ defined by

$$f(x, y)=x^3+3xy^{20}+y^{29}.$$

Then we see that

$$x^3+3xy^{20}\overset{\text{(i)}}{\sim}x^3+3xy^{20}+y^{29}\overset{\text{(ii)}}{\sim}x^3+y^{29}.$$

Here we interpret the above equivalences as follows (see [**6**], Example 4.3 also):

(i) Put $w=j^{21}f(0)=x^3+3xy^{20}$. Then w is C^0-equivalent to f. This follows from the Kuiper-Kuo theorem (see Lemma 5 in § 3).

(ii) Put $z=x^3+y^{29}$. Then z is C^0-equivalent to f. Since z is weighted homogeneous of type (1/3, 1/29) with a finite codimension and the weight of the term $3xy^{20}$ is $1/3+20/29>1$ (see V. I. Arnol'd [**1**]).

In the complex case, the equivalence (i) does not hold. For w is weighted homogeneous of type (1/3, 1/30) with an isolated singularity and the weight of the term y^{29} is $29/30<1$. Furthermore $y^{29}\notin\mathfrak{M}(\partial w/\partial x, \partial w/\partial y)$. Therefore w is not C^0-equivalent to $w+y^{29}=f$ (see M. Suzuki [**16**] or A. N. Varčenko [**18**]). (Of course, we can also see this directly by considering the C^0-type of $w^{-1}(0)$ and $f^{-1}(0)$, as germs at $0\in\boldsymbol{C}^2$.) Even in the real case, the equivalence (i) does not hold, if we replace plus by minus (i. e. $w=x^3-3xy^{20}$).

PROBLEM. *Is there a unified discription for explaining the above interpretations?*

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The purpose of this paper is to give a weighted form of the Kuiper-Kuo type theorem for real analytic functions of at most three variables as an answer to the above problem, and the corresponding result for holomorphic functions of general n variables ($n \neq 3$). We shall describe the results and corollaries in § 1, and prove them in § 3. In § 2, we shall apply the real result for the above example, and explain our results.

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§ 1. Main results.

Let $\boldsymbol{Q}^+$ (resp. $\boldsymbol{R}^+$) denote the set of positive rational numbers (resp. positive real numbers). For $\alpha=(\alpha_1, \cdots, \alpha_n) \in \boldsymbol{Q}^+ \times \cdots \times \boldsymbol{Q}^+$ with $\min_{1 \leq j \leq n} \alpha_j = 1$, we define the subset of $\boldsymbol{Q}^+$ by

$$I(\alpha)=\left\{\sum_{j=1}^{n} \alpha_j \beta_j \mid \beta_j \in \boldsymbol{N} \cup \{0\} \ (1 \leqq j \leqq n),\ \beta_1+\cdots+\beta_n \geqq 1\right\}.$$

Which can be expressed as

$$I(\alpha)=\{\alpha(1), \cdots, \alpha(N), \cdots\}, \qquad \alpha(i)<\alpha(i+1).$$

We put $\delta_N=\alpha(N+1)-\alpha(N)>0$. Note that δ_N is constant for all large N.

Let $\alpha(N) \in I(\alpha)$, and f, g be C^ω functions where

$$f(x)=\sum_{\beta \in \mathfrak{B}} A_\beta x_1^{\beta_1} \cdots x_n^{\beta_n} \qquad \text{and} \qquad g(x)=\sum_{\gamma \in \Gamma} B_\gamma x_1^{\gamma_1} \cdots x_n^{\gamma_n}.$$

We say that f and g are *$\alpha(N)$-equivalent*, if

$$\sum A_\beta x_1^{\beta_1} \cdots x_n^{\beta_n}=\sum B_\gamma x_1^{\gamma_1} \cdots x_n^{\gamma_n}$$

where the summation in the left side member (resp. the right side member) is taken over all elements $\beta \in \mathfrak{B}$ with $\alpha_1\beta_1+\cdots+\alpha_n\beta_n \leqq \alpha(N)$ (resp. all elements $\gamma \in \Gamma$ with $\alpha_1\gamma_1+\cdots+\alpha_n\gamma_n \leqq \alpha(N)$). This relation $\underset{\alpha(N)}{\sim}$ is an equivalence relation. Then we denote by $J_\alpha^{\alpha(N)}(n, 1)$ the quotient set of $\mathcal{E}_{[\omega]}(n, 1)$ and by $j_\alpha^{\alpha(N)} f(0)$ the equivalence class of f by the relation $\underset{\alpha(N)}{\sim}$. We identify $j_\alpha^{\alpha(N)} f(0)$ with its polynomial representative

$$\sum_\beta A_\beta x_1^{\beta_1} \cdots x_n^{\beta_n} \qquad \text{where } \beta \in \mathfrak{B},\ \alpha_1\beta_1+\cdots+\alpha_n\beta_n \leqq \alpha(N).$$

Let $f \in \mathcal{E}_{[\omega]}(n, 1)$ such that $0 \in \boldsymbol{R}^n$ is a singular point of f. For $\alpha(N) \in I(\alpha)$ with $\alpha_j<\alpha(N)$ $(1 \leqq j \leqq n)$, we define

$$\tilde{\partial}_{\alpha(N)} f(x)=\left(\left|\frac{\partial f}{\partial x_1}(x)\right|^{(\alpha(N)-1)/(\alpha(N)-\alpha_1)}, \cdots, \left|\frac{\partial f}{\partial x_n}(x)\right|^{(\alpha(N)-1)/(\alpha(N)-\alpha_n)}\right)$$

near $0 \in \boldsymbol{R}^n$.

REMARK 1. If $0 \in \boldsymbol{R}^n$ is an isolated singular point of $H=j_\alpha^{\alpha(N)} f(0)$, then we have $\alpha_j<\alpha(N)$ $(1\leqq j\leqq n)$. Therefore $\tilde{\partial}_{\alpha(N)}H(x)$ and $\tilde{\partial}_{\alpha(N)}f(x)$ are defined.

NOTATION. Let $\alpha=(\alpha_1, \cdots, \alpha_n)\in \boldsymbol{Q}^+\times\cdots\times\boldsymbol{Q}^+$. We put

$$|x|_\alpha=\sqrt{|x_1|^{2/\alpha_1}+\cdots+|x_n|^{2/\alpha_n}}$$

for $x=(x_1, \cdots, x_n)\in\boldsymbol{R}^n$.

Let $J\ (\supset[0, 1])$ be an open interval. Given a germ of C^ω function $F:(\boldsymbol{R}^n\times J, \{0\}\times J)\to(\boldsymbol{R}, 0)$. Let us consider the family of germs $f_s:(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$, $s\in J$ where $f_s(x)=F(x, s)$. We say the family $\{f_s|s\in J\}$ or $F:(\boldsymbol{R}^n\times J, \{0\}\times J)\to(\boldsymbol{R}, 0)$ has *no coalescing of critical points,* if there exists $a>0$ such that

$$|\mathrm{grad}\, f_s(x)|\neq 0 \qquad \text{for } 0<|x|<a \text{ and } s\in J.$$

Put $\Lambda=\{0\}\times J\subset\boldsymbol{R}^n\times J$. Suppose F is non-singular outside Λ. Then the pair $(\boldsymbol{R}^n\times J-\Lambda, \Lambda)$ is called *(a_F)-regular at* $(0, s)\in\Lambda$, if for any sequence of points $\{p_m\}$ in $\boldsymbol{R}^n\times J-\Lambda$ tending to $(0, s)\in\Lambda$ such that the plane $T_{p_m}F^{-1}(F(p_m))$ tends to τ, we have $\tau\supset\Lambda$. The pair $(\boldsymbol{R}^n\times J-\Lambda, \Lambda)$ is called *(a_F)-regular,* if it is (a_F)-regular at any $(0, s)\in\Lambda$. In the complex case, we can define the concept of *no coalescing of critical points* and *(a_F)-regularity* similarly.

Now we state the main results in this paper.

PROPOSITION. *Let $H\in J_\alpha^{\alpha(N)}(n, 1)$, $G\in\mathcal{E}_{[\omega]}(n, 1)$ with $j_\alpha^{\alpha(N)}G(0)=0$, and let $F(x, s)=H(x)+sG(x)$ for $s\in J$. If there exist C, $a>0$ and $\varepsilon>1-\delta_N$ such that*

$$|\tilde{\partial}_{\alpha(N)}H(x)|\geqq C|x|_\alpha^{\alpha(N)-\varepsilon} \qquad \text{for} \quad |x|<a,$$

then F has no coalescing of critical points. Furthermore the pair $(\boldsymbol{R}^n\times J-\Lambda, \Lambda)$ is (a_F)-regular.

THEOREM 1. *Let $H\in J_\alpha^{\alpha(N)}(n, 1)$ where $n\leqq 3$. If there exist C, $a>0$ and $\varepsilon>1-\delta_N$ such that*

$$|\tilde{\partial}_{\alpha(N)}H(x)|\geqq C|x|_\alpha^{\alpha(N)-\varepsilon} \qquad \text{for} \quad |x|<a,$$

then for any $G\in\mathcal{E}_{[\omega]}(n, 1)$ with $j_\alpha^{\alpha(N)}G(0)=0$, $H+G$ is C^0-equivalent to H.

REMARK 2. (1) If $0\in\boldsymbol{R}^n$ is a regular point of H, then for any $G\in\mathcal{E}_{[\omega]}(n, 1)$ with $j_\alpha^{\alpha(N)}G(0)=0$, $H+G$ is C^0-equivalent to H (any $\alpha\in\boldsymbol{Q}^+\times\cdots\times\boldsymbol{Q}^+$ and $\alpha(N)\in I(\alpha)$). This follows from the implicit function theorem.

(2) Since $1\geqq\delta_N>0$, we have $\varepsilon>1-\delta_N\geqq 0$.

(3) In [7], the author proposed a partition problem on real analytic functions. In attempt to solve it, we have obtained this theorem.

(4) Thanks to J. Bochnak-J.-J. Risler [2] and T.C. Kuo [12], we think the system of weights in rational numbers only.

PROBLEM. *Does the above theorem also hold for $n\geqq 4$?* (See Remark 5 in §3 also.)

Let [ ] denote the Gauss symbol.

COROLLARY 1. *Suppose* $H\in J_{\alpha}^{\alpha(N)}(n, 1)$ $(n\leqq 3)$ *satisfies the hypothesis of Theorem* 1. *Then* $H+G$ *is* C^0*-equivalent to* H *for any* $G\in\mathcal{E}_{[[\alpha(N)]+1]}(n, 1)$ *with* $j_{\alpha}^{\alpha(N)}G(0)=0$.

REMARK 3. *If* $\alpha_1\beta_1+\cdots+\alpha_n\beta_n\leqq\alpha(N)$, then $\beta_1+\cdots+\beta_n\leqq[\alpha(N)]$. Therefore we can define $J_{\alpha}^{\alpha(N)}(n, 1)$ in $\mathcal{E}_{[k]}(n, 1)$ for $k=[\alpha(N)], [\alpha(N)]+1, \cdots$.

In this paper, a polynomial $H(x_1, \cdots, x_n)=\sum_{\beta\in\mathfrak{B}}C_{\beta}x_1^{\beta_1}\cdots x_n^{\beta_n}$ is called weighted homogeneous of type $(\alpha_1/r, \cdots, \alpha_n/r)$ where r, α_j $(1\leqq j\leqq n)\in\boldsymbol{Q}^+$, if

$$\min_{1\leq j\leq n}\alpha_j=1 \qquad \text{and} \qquad \frac{\alpha_1}{r}\beta_1+\cdots+\frac{\alpha_n}{r}\beta_n=1$$

for any multiindex $\beta=(\beta_1, \cdots, \beta_n)\in\mathfrak{B}$.

COROLLARY 2. *Let* $H:(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$ $(n\leqq 3)$ *be a weighted homogeneous polynomial of type* $(\alpha_1/r, \cdots, \alpha_n/r)$ *with an isolated singularity, and let* $G\in\mathcal{E}_{[[r]+1]}(n, 1)$ *such that* $j^{[r]+1}G(0)=\sum_{\beta\in\mathfrak{B}}A_{\beta}x_1^{\beta_1}\cdots x_n^{\beta_n}$. *If* $\sum_{j=1}^{n}(\alpha_j/r)\beta_j>1$ (*resp.* $\sum_{j=1}^{n}(\alpha_j/r)\beta_j\geqq 1$) *for any* $\beta\in\mathfrak{B}$, *then* $H+G$ *is* C^0*-equivalent to* H (*resp. then there exists* $\varepsilon>0$ *such that* $H+sG$ *is* C^0*-equivalent to* H *for* $|s|<\varepsilon$).

REMARK 4 (J. Damon-T. Gaffney, [**4**], Corollary 5). If H has an *algebraically* isolated singularity, then the above corollary holds for general n variables case. (See the proof of Corollary 1 in §3 also.)

In the complex case, we define $I(\alpha)$, δ_N, $J_{\alpha}^{\alpha(N)}(n, 1)$, $\tilde{\partial}_{\alpha(N)}$, and $|\ \ |_{\alpha}$ for a given system of the weights $\alpha=(\alpha_1, \cdots, \alpha_n)$ with $\min_{1\leq j\leq n}\alpha_j=1$, similarly as in the real case. Then we have the corresponding complex result to Theorem 1.

THEOREM 2. *Let* $H\in J_{\alpha}^{\alpha(N)}(n, 1)$ *where* $n\neq 3$. *If there exist* C, $a>0$ *and* $\varepsilon>1-\delta_N$ *such that*

$$|\tilde{\partial}_{\alpha(N)}H(x)|\geqq C|x|_{\alpha}^{\alpha(N)-\varepsilon} \qquad \textit{for} \quad |x|<a,$$

then for any $G\in\mathcal{H}(n, 1)$ *with* $j_{\alpha}^{\alpha(N)}G(0)=0$, $H+G$ *is* C^0*-equivalent to* H.

§2. Applications.

Let $f:(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$ be a C^{ω} function with an isolated singularity. Then, for any $\alpha=(\alpha_1, \cdots, \alpha_n)\in\boldsymbol{Q}^+\times\cdots\times\boldsymbol{Q}^+$ with $\min_{1\leq j\leq n}\alpha_j=1$, there exist $\alpha(N)\in I(\alpha)$, C, $a>0$, and $\varepsilon>1-\delta_N$ such that

$$|\tilde{\partial}_{\alpha(N)}H(x)|\geqq C|x|_{\alpha}^{\alpha(N)-\varepsilon} \qquad \text{for} \quad |x|<a,$$

where $H(x)=j_{\alpha}^{\alpha(N)}f(0)$. Therefore Theorem 1 asserts that given any system of weights, a C^{ω} function $(n\leqq 3)$ with an isolated singularity has C^0 finite determinacy related to it.

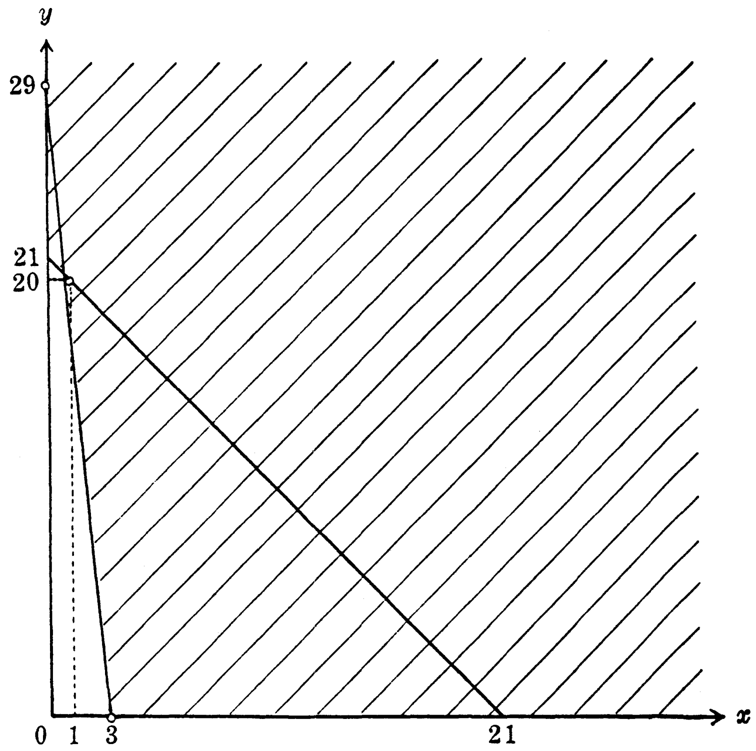

Figure.

Let us go back to the example in the introduction. We can interpret the equivalence (i) by applying Theorem 1 with the system of weights $\alpha=(1, 1)$ $(\alpha(N)=21)$, and the equivalence (ii) with $\alpha=(29/3, 1)$ $(\alpha(N)=29)$. Furthermore the theorem holds as $\varepsilon=1$ in both cases. For $\alpha=(1, 1)$ and $\alpha(N)=21$ (resp. $\alpha=(29/3, 1)$ and $\alpha(N)=29$), let $h\in\mathcal{E}_{[\omega]}(2, 1)$ with $j_\alpha^{\alpha(N-1)}h(0)=0$. Then it follows from the Kuiper-Kuo theorem and Corollary 2 that if all coefficients of terms of degree 21 (resp. 29) related to the weight $\alpha=(1, 1)$ (resp. $\alpha=(29/3, 1)$) in h are sufficiently small, then $f+h$ is C^0-equivalent to f. Namely, f is controling not only terms inside the shaded region in Figure, but also *local terms* on the boundary in the meaning of C^0-equivalence.

There are interesting works on the topological triviality of deformations of a complex function germ, assuring certain conditions on the Newton boundary (e.g. V.I. Arnol'd [**1**], A.G. Kouchnirenko [**8**], M. Oka [**15**], J. Damon-T. Gaffney [**4**], M. Buchner-W. Kucharz [**3**]). The works [**3**] and [**4**] contain the corresponding real results, too. On the other hand, E. Yoshinaga has established π-MAT (which is stronger than topological triviality) under assumptions on the Newton boundary in the real case ([**19**]). The approaches based on the Newton boundary are very effective for the problem of topological triviality in the complex case. In fact, this is intrinsic in the semi-quasihomogeneous case (V. I. Arnol'd [**1**], M. Suzuki [**16**], A. N. Varčenko [**18**]). But the above example shows that this is not necessarily so in the real case. Therefore our results on C^0 determinacy of analytic functions are formulated in terms of an arbitrary system of weights, without using the Newton boundary.

§ 3. Proof of the results.

PROOF OF PROPOSITION. Let

$$G(x)=\sum_{\beta\in\mathfrak{B}} A_\beta x_1^{\beta_1}\cdots x_n^{\beta_n}\in\mathcal{E}_{[\omega]}(n,1)$$

with $j_\alpha^{\alpha(N)}G(0)=0$.

LEMMA 1. *There exist $K>0$ and a neighborhood W of 0 in $\boldsymbol{R}^n$ such that*

$$|G(x)|\leqq K|x|_\alpha^{\alpha(N)+\delta_N}\qquad in\ W.$$

PROOF. Put $q=[\alpha(N)+\delta_N]$. Note that $q+1>\alpha(N)+\delta_N$ and $|x|_\alpha\geqq|x|$ for $|x|\leqq 1$. We denote by $G_1(x)$ the sum of all terms in $G(x)$ with $\beta_1+\cdots+\beta_n\leqq q$, and denote by $G_2(x)$ the sum of the remainder in $G(x)$. Then we have $G(x)=G_1(x)+G_2(x)$ and $j^qG_2(0)=0$. Thus there exist $K_2>0$ and a neighborhood W_2 of 0 in $\boldsymbol{R}^n$ such that

$$|G_2(x)|\leqq K_2|x|^{q+1}\leqq K_2|x|_\alpha^{q+1}\qquad \text{in}\ W_2\cap\{|x|\leqq 1\}. \tag{3.1}$$

Let $S(x)=x_1^{\beta_1}\cdots x_n^{\beta_n}$ where $\sum_{j=1}^n\alpha_j\beta_j\geqq\alpha(N)+\delta_N$. If we set $K_1=\max_{|x|_\alpha=1}|S(x)|>0$, then we have

$$|S(x)|\leqq K_1|x|_\alpha^{\alpha(N)+\delta_N}\qquad \text{for}\ |x|_\alpha\leqq 1. \tag{3.2}$$

Since the number of terms in $G_1(x)$ is finite, the statement of this lemma follows from (3.1) and (3.2).

Let $H\in J_\alpha^{\alpha(N)}(n,1)$ satisfy the hypothesis of Proposition.

LEMMA 2. *There exists $a_0>0$ such that*

$$|\operatorname{grad}(H+G)(x)|\geqq\frac{C}{2}|x|_\alpha^{\alpha(N)-\varepsilon}\qquad for\ |x|<a_0.$$

PROOF. Put

$$W(x)=\left(\left(\left|\frac{\partial H}{\partial x_1}(x)\right|+\left|\frac{\partial G}{\partial x_1}(x)\right|\right)^{(\alpha(N)-1)/(\alpha(N)-\alpha_1)}, \cdots,\left(\left|\frac{\partial H}{\partial x_n}(x)\right|+\left|\frac{\partial G}{\partial x_n}(x)\right|\right)^{(\alpha(N)-1)/(\alpha(N)-\alpha_n)}\right).$$

We first show the following

ASSERTION. *There exists* $a_1>0$ *such that*

$$|\tilde{\partial}_{\alpha(N)}(H+G)(x)| \geqq \frac{1}{2}|W(x)| \qquad \text{for} \quad |x|<a_1.$$

Assume that the assertion does not hold. Then, by the curve selection lemma ([**14**]), there exists a C^ω curve $\lambda: [0, \rho)\to \boldsymbol{R}^n$ ($\rho>0$) with $\lambda(0)=0$ such that

$$|\tilde{\partial}_{\alpha(N)}(H+G)(\lambda(t))| < \frac{1}{2}|W(\lambda(t))| \qquad \text{for} \quad t>0. \tag{3.3}$$

Let us write

$$\lambda_j(t) = a_j(1)t^{d_j(1)}+a_j(2)t^{d_j(2)}+\cdots,$$

where
$$a_j(i) \neq 0 \quad \text{if} \quad \lambda_j \not\equiv 0 \quad (\text{for any } i)$$
$$d_j(1) = \infty \quad \text{if} \quad \lambda_j \equiv 0 \qquad (1\leqq j\leqq n),$$

and put $d=\min_{1\leq j\leq n}\{d_j(1)/\alpha_j\}>0$.

By a similar argument as in Lemma 1, there exist K, $b>0$ such that

$$\left|\frac{\partial G}{\partial x_j}(x)\right| \leqq K|x|_\alpha^{\alpha(N)+\delta_N-\alpha_j} \qquad \text{for} \quad |x|<b \ (1\leqq j\leqq n).$$

Therefore we have

$$O\Big(\frac{\partial G}{\partial x_j}(\lambda(t))\Big) \geqq d(\alpha(N)+\delta_N-\alpha_j) \qquad (1\leqq j\leqq n), \tag{3.4}$$

where $O(Q(t))$ denotes the order of Q in t.

Furthermore it follows from the hypothesis of Proposition that

$$|\tilde{\partial}_{\alpha(N)}H(\lambda(t))| \geqq C|\lambda(t)|_\alpha^{\alpha(N)-\varepsilon} \qquad \text{for} \quad t\geqq 0.$$

Therefore, by the curve selection lemma, there exists k with $1\leqq k\leqq n$ such that

$$O\Big(\left|\frac{\partial H}{\partial x_k}(\lambda(t))\right|^{(\alpha(N)-1)/(\alpha(N)-\alpha_k)}\Big) \leqq d(\alpha(N)-\varepsilon).$$

Thus we have

$$O\Big(\frac{\partial H}{\partial x_k}(\lambda(t))\Big) \leqq d(\alpha(N)-\varepsilon)(\alpha(N)-\alpha_k)/(\alpha(N)-1). \tag{3.5}$$

NOTE. *For* $1\leqq j\leqq n$, *we have*

$$d(\alpha(N)-\varepsilon)(\alpha(N)-\alpha_j)/(\alpha(N)-1) < d(\alpha(N)+\delta_N-\alpha_j).$$

PROOF. Since $\varepsilon>1-\delta_N$ and $\alpha_j\geqq 1$, we have

$$(\alpha(N)-\varepsilon)/(\alpha(N)-1) < (\alpha(N)+\delta_N-1)/(\alpha(N)-1)$$
$$= 1+\delta_N/(\alpha(N)-1) \leqq 1+\delta_N/(\alpha(N)-\alpha_j) = (\alpha(N)+\delta_N-\alpha_j)/(\alpha(N)-\alpha_j).$$

It follows from Note and (3.4) that for any k satisfying the condition (3.5),

$$\begin{cases} O\Big(\Big|\dfrac{\partial H}{\partial x_k}(\lambda(t))+\dfrac{\partial G}{\partial x_k}(\lambda(t))\Big|\Big) \leqq d(\alpha(N)-\varepsilon)(\alpha(N)-\alpha_k)/(\alpha(N)-1), \\ O\Big(\Big|\dfrac{\partial H}{\partial x_k}(\lambda(t))\Big|+\Big|\dfrac{\partial G}{\partial x_k}(\lambda(t))\Big|\Big) \leqq d(\alpha(N)-\varepsilon)(\alpha(N)-\alpha_k)/(\alpha(N)-1). \end{cases}$$

Therefore we have

$$(3.6) \qquad \begin{cases} O\Big(\Big|\dfrac{\partial H}{\partial x_k}(\lambda(t))+\dfrac{\partial G}{\partial x_k}(\lambda(t))\Big|^{(\alpha(N)-1)/(\alpha(N)-\alpha_k)}\Big) \leqq d(\alpha(N)-\varepsilon), \\ O\Big(\Big(\Big|\dfrac{\partial H}{\partial x_k}(\lambda(t))\Big|+\Big|\dfrac{\partial G}{\partial x_k}(\lambda(t))\Big|\Big)^{(\alpha(N)-1)/(\alpha(N)-\alpha_k)}\Big) \leqq d(\alpha(N)-\varepsilon). \end{cases}$$

On the other hand, for any j not satisfying the condition (3.5), we have

$$O\Big(\frac{\partial H}{\partial x_j}(\lambda(t))\Big) > d(\alpha(N)-\varepsilon)(\alpha(N)-\alpha_j)/(\alpha(N)-1).$$

Therefore we have

$$(3.7) \qquad \begin{cases} O\Big(\Big|\dfrac{\partial H}{\partial x_j}(\lambda(t))+\dfrac{\partial G}{\partial x_j}(\lambda(t))\Big|^{(\alpha(N)-1)/(\alpha(N)-\alpha_j)}\Big) > d(\alpha(N)-\varepsilon), \\ O\Big(\Big(\Big|\dfrac{\partial H}{\partial x_j}(\lambda(t))\Big|+\Big|\dfrac{\partial G}{\partial x_j}(\lambda(t))\Big|\Big)^{(\alpha(N)-1)/(\alpha(N)-\alpha_j)}\Big) > d(\alpha(N)-\varepsilon). \end{cases}$$

The conditions (3.4), (3.6) and (3.7) contradict (3.3). This completes the proof of Assertion.

Now we show Lemma 2 by using this assertion. For $|x|<a_1$, we have

$$|\tilde{\partial}_{\alpha(N)}(H+G)(x)| \geqq \frac{1}{2}|W(x)| \geqq \frac{1}{2}|\tilde{\partial}_{\alpha(N)}H(x)| \geqq \frac{C}{2}|x|_\alpha{}^{\alpha(N)-\varepsilon}.$$

Thus there exists $a_0>0$ such that

$$|\mathrm{grad}(H+G)(x)| \geqq |\tilde{\partial}_{\alpha(N)}(H+G)(x)| \geqq \frac{C}{2}|x|_\alpha{}^{\alpha(N)-\varepsilon}$$

for $|x|<a_0$.

Let $J\ (\supset[0, 1])$ be an open interval. Consider the C^ω function $F: (\boldsymbol{R}^n\times J, \{0\}\times J)\to(\boldsymbol{R}, 0)$ defined by

$$F(x, s) = H(x)+sG(x).$$

Then we have

LEMMA 3. *There exists* $a_0>0$ *such that*

$$|\mathrm{grad}_{(x,s)}F(x, s)| \geqq \frac{C}{2}|x|_\alpha{}^{\alpha(N)-\varepsilon}$$

for $|x|<a_0$ *and* $s\in J$. *Furthermore* $\{f_s\}$ *has no coalescing of critical points.*

PROOF. By modifying the proof of Assertion, it is easy to see that there exists $a_1>0$ such that

$$|\tilde{\partial}_{\alpha(N)}(H+sG)(x)| \geqq \frac{1}{2}\left|\left(\left(\left|\frac{\partial H}{\partial x_1}(x)\right|+\left|\frac{\partial(sG)}{\partial x_1}(x)\right|\right)^{(\alpha(N)-1)/(\alpha(N)-\alpha_1)},\right.\right.$$
$$\left.\left.\cdots, \left(\left|\frac{\partial H}{\partial x_n}(x)\right|+\left|\frac{\partial(sG)}{\partial x_n}(x)\right|\right)^{(\alpha(N)-1)/(\alpha(N)-\alpha_n)}\right)\right|$$

for $|x|<a_1$ and $s\in J$. As in the proof of Lemma 2, there exists $a_0>0$ such that

$$|\mathrm{grad}_x F(x, s)| \geqq \frac{C}{2}|x|_\alpha{}^{\alpha(N)-\varepsilon} \qquad \text{for} \quad |x|<a_0 \text{ and } s\in J.$$

Therefore Lemma 3 follows immediately.

It follows from Lemmas 1, 3 that there exists $a_2>0$ such that

$$|G(x)|/|\mathrm{grad}_{(x,s)}F(x, s)| \leqq \frac{2K}{C}|x|_\alpha{}^{\varepsilon+\delta_N}$$

for $0<|x|<a_2$ and $s\in J$. Therefore the pair $(\boldsymbol{R}^n\times J-\Lambda, \Lambda)$ is (a_F)-regular. This completes the proof of Proposition.

REMARK 5. In general, it does not seem that the following properties hold.

(1) The pair $(F^{-1}(0)-\Lambda, \Lambda)$ is *Whitney* (b)*-regular* (see J. Mather [**13**]).

(2) Let us consider the *Kuo vector field* $\{v(x)\}$ using F in the above proof (see T. C. Kuo [**10**]). Then $\{v(x)\}$ is a *local Liapounov trivialization* along Λ at a point $(0, s)\in\Lambda$ (see T. C. Kuo [**11**], Appendix 2).

PROOF OF THEOREM 1. We recall King's results ([**5**]) on no coalescing of critical points:

LEMMA 4. *If* $f_s:(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$ $(n\leqq 3)$, $s\in\boldsymbol{R}^p$ *is a continuous family of germs of analytic functions with no coalescing of critical points, then there is a continuous family of homeomorphism germs* $h_s:(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^n, 0)$ *such that* $f_0=f_s\circ g_s$ *for all* $s\in\boldsymbol{R}^p$.

REMARK 6. (1) Lemma 4 for $n\geqq 5$ does not hold.

(2) Lemma 4 in the complex case holds for $n\neq 3$.

Let J $(\supset[0, 1])$ be an open interval, and consider the function $F:(\boldsymbol{R}^n\times J, \{0\}\times J)\to(\boldsymbol{R}, 0)$ defined by

$$F(x, s)=H(x)+sG(x)$$

as above. Then it follows from Proposition and Lemma 4 that $H+G$ is C^0-equivalent to H.

PROOF OF COROLLARY 1. We first recall the Kuiper-Kuo theorem (N. Kuiper [**9**], T. C. Kuo [**10**]):

LEMMA 5. *Let* $H \in J^r(n, 1)$. *If there exist* C, a, $\varepsilon > 0$ *such that*

$$|\operatorname{grad} H(x)| \geqq C|x|^{r-\varepsilon} \quad \text{for} \quad |x| < a,$$

then H *is* C^0*-sufficient in* $\mathcal{E}_{[r+1]}(n, 1)$ *i.e. for any* $G \in \mathcal{E}_{[r+1]}(n, 1)$ *with* $j^r G(0) = 0$, $H + G$ *is* C^0*-equivalent to* H.

FACT. Put $\varepsilon_0 = \varepsilon + \delta_N - 1 > 0$. Then we have

$$\alpha(N) - \varepsilon \leqq [\alpha(N)] - \varepsilon_0 .$$

Put $z = j^{[\alpha(N)]}(H+G)(0)$. Then, by Remark 3, we have $j_\alpha^{\alpha(N)} z(0) = H$. Therefore it follows from Theorem 1 that z is C^0-equivalent to H. On the other hand, by the proof of Lemma 2, we see that z satisfies the same hypothesis as H. Therefore it follows from Fact and Lemma 5 that z is C^0-equivalent to $H+G$. Thus $H+G$ is C^0-equivalent to H.

PROOF OF COROLLARY 2. Let $H: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}, 0)$ be a weighted homogeneous polynomial of type $(\alpha_1/r, \cdots, \alpha_n/r)$ with an isolated singularity. Then we can define

$$\tilde{\partial} H(x) = \left(\left| \frac{\partial H}{\partial x_1}(x) \right|^{(r-1)/(r-\alpha_1)}, \cdots, \left| \frac{\partial H}{\partial x_n}(x) \right|^{(r-1)/(r-\alpha_n)} \right)$$

near $0 \in \boldsymbol{R}^n$. The next lemma follows from the proof of Theorem 3.1 in V.I. Arnol'd [**1**].

LEMMA 6. *If* H *has an isolated singularity, then there exist* $C > 0$ *and a neighborhood* U *of* 0 *in* $\boldsymbol{R}^n$ *such that*

$$|\tilde{\partial} H(x)| \geqq C|x|_\alpha^{r-1} \quad \text{in} \quad U.$$

REMARK 7. Lemma 6 for homogeneous polynomials has been proved by N. Kuiper ([**9**], Theorem 4B).

Corollary 2 is an immediate consequence of Theorem 1 (Corollary 1) and Lemma 6. Here note that "Theorem 1 as $\varepsilon = 1$" holds in this case.

PROOF OF THEOREM 2. Let J ($\supset [0, 1]$) be an open interval, and consider the function $F: (\boldsymbol{C}^n \times J, \{0\} \times J) \to (\boldsymbol{C}, 0)$ defined by

$$F(x, s) = H(x) + sG(x).$$

Then we see that F has no coalescing of critical points, by using the same argument as in the proof of Proposition. Therefore it follows from Remark 6(2) that $H+G$ is C^0-equivalent to H in the case $n \neq 3$.

References

[1] V. I. Arnol'd, Normal forms of functions in the neighborhood of degenerate critical points, Uspekhi Mat. Nauk, **29** (1974), 11-49 = Russian Math. Surveys, **29** (1974), 19-48.

[2] J. Bochnak and J.-J. Risler, Sur les exposants de Łojasiewicz, Comment. Math. Helv., **50** (1975), 493-507.

[3] M. Buchner and W. Kucharz, Topological triviality of a family of zero-sets, Proc. Amer. Math. Soc., **102** (1988), 699-705.

[4] J. Damon and T. Gaffney, Topological triviality of deformations of functions and Newton filtrations, Invent. Math., **72** (1983), 335-358.

[5] H. King, Topological type in families of germs, Invent. Math., **62** (1980), 1-13.

[6] S. Koike, C^0-sufficiency of jets via blowing-up, J. Math. Kyoto Univ., **28** (1988), 605-614.

[7] S. Koike, A partition problem on analytic functions, Singularities and Differential Geometry, Hokkaido University Technical Report Series in Math., **7** (1988).

[8] A. G. Kouchnirenko, Polyhèdres de Newton et nombres de Milnor, Invent. Math., **32** (1976), 1-31.

[9] N. Kuiper, C^1-equivalence of functions near isolated critical points, Symposium on Infinite Dimensional Topology (Baton Rouge 1967), Ann. of Math. Stud., **69** (1972), 199-218.

[10] T. C. Kuo, On C^0-sufficiency of jets of potential functions, Topology, **8** (1969), 167-171.

[11] T. C. Kuo, Characterizations of v-sufficiency of jets, Topology, **11** (1972), 115-131.

[12] T. C. Kuo, Computation of Łojasiewicz exponent of $f(x, y)$, Comment. Math. Helv., **49** (1974), 201-213.

[13] J. Mather, Notes on topological stability, Harvard University notes, 1970.

[14] J. Milnor, Singular points of complex hypersurfaces, Ann. of Math. Stud., **61**, 1968.

[15] M. Oka, On the bifurcation of the multiplicity and topology of the Newton boundary, J. Math. Soc. Japan, **31** (1979), 435-450.

[16] M. Suzuki, The stratum with constant Milnor number of a mini-transversal family of a quasihomogeneous function of corank two, Topology, **23** (1983), 101-115.

[17] J. Timourian, The invariance of Milnor's number implies topological triviality, Amer. J. Math., **99** (1977), 437-446.

[18] A. N. Varčenko, A lower bound for the codimension of the stratum μ=const in terms of the mixed Hodge structure, Vestnik Moskov. Univ., **37** (1982), 28-31.

[19] E. Yoshinaga, The modified analytic trivialization of real analytic families via blowing-ups, J. Math. Soc. Japan, **40** (1988), 161-179.

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