J. Math. Soc. Japan
Vol. 42, No. 1, 1990

A remark on the existence of a diffusion process with non-local boundary conditions

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(Received Jan. 9, 1989)

Introduction.

It is known that a diffusion process on a domain D with smooth boundary is determined by a pair of analytical data (A, L), where A is a second order differential operator of elliptic type (possibly degenerate) and L is a Wentzell's boundary condition which consists of the sum of a second order differential operator and non-local terms. (For the precise definition see § 1.) The problem of constructing the diffusion from a pair (A, L) has been discussed by many authors. Analytically, K. Sato and T. Ueno [**12**] laid a fundamental route and following it, Bony-Courrège-Priouret [**2**] and Taira [**14**] succeeded in very general cases. In their manner, one constructs a Feller semigroup (and hence the transition function) on $\bar{D}$ via Hille-Yosida semigroup theory to dispose the diffusion.

On the other hand, the construction of a semigroup can be carried out directly by probabilistic methods, which have an advantage to permit the degeneracy of A. That is, by using stochastic calculus or the martingale method. See Ikeda [**9**], Watanabe [**17**], Stroock-Varadhan [**13**], Anderson [**1**] and Cattiaux [**5**, **6**]. Apart from this, a direct construction of path functions (and hence the diffusion process) by using the notion of Poisson point process of Brownian excursion was succeeded by Watanabe [**18**]. See also Ikeda-Watanabe [**10**] and Takanobu-Watanabe [**16**].

Although we can construct diffusions as the functionals on Wiener-Poisson space as above, we have another task left to verify regularity results, for example, statements about transition functions. Returning to the viewpoint of analysis, one way to treat this problem in the case with non-local boundary conditions will be the use of the theory of pseudodifferential operators developed by Hörmander [**8**], Boutet de Monvel [**3**] et al. It is natural to make such a study since the class of pseudodifferential operators includes a wide class of significant non-local operators (cf. Cancelier [**4**]).

Here we shall afford a concrete example in this framework. That is, the pair (A, L) is given by a second order differential operator of uniformly elliptic type and a Wentzell's boundary condition having "non-local" terms of the form the generator of stable processes. In fact, we discuss the case where the boundary condition L possesses two non-local terms, one corresponds to the Cauchy process on the boundary and the other to a stable process of order $\beta \in (0, 1) \cap \boldsymbol{Q}$ having inward jumps from the boundary (see Section 1). The reason why we confine ourselves to the generator of stable processes is that it is a typical example in the class of stationary Lévy processes and that the form of its symbol is simple and well-known through several studies (e. g. Komatsu [**11**]). Under a supplementary condition and a transversality condition, we shall show analytically the existence and uniqueness of the Feller semigroup, and hence the diffusion.

We sum up the content of this paper briefly. In Section 1, we state a general existence theorem for a Feller semigroup on $\bar{D}$ (Theorem 1), and by utilizing it we state our main result (Theorem 2) under a supplementary condition (A). Note that our assumption (A) is a natural extension to the non-local case of the corresponding hypothesis in Taira [**15**]. In Sections 2 and 3, we prove our main theorem. We mimic the way of proof in [**14**]. That is, the existence of the semigroup is reduced to the hypoellipticity of a boundary operator and we solve it via index argument. Uniqueness follows from the maximum principle for operators A and L. The fundamental *a priori* estimate (Proposition 2.1) is proved separately in Section 4 because of the length of its proof.

The author wishes to express his hearty thanks to Professor K. Taira for having a seminar.

§1. Statement of the result.

Let D be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary ∂D. $\bar{D}=D\cup\partial D$. Let

$$Au(x)=\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x)+\sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i}+c(x)u(x)$$

be a differential operator in D whose coefficients satisfying

$$(1.1)\qquad \begin{cases} 1^\circ \quad a^{ij}\in C^\infty(\bar{D}), \qquad a^{ij}=a^{ji} \qquad \text{and} \\ \qquad \sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geqq C|\xi|^2, \qquad x\in\bar{D},\ \xi\in\boldsymbol{R}^N,\ C>0, \\ 2^\circ \quad b^i\in C^\infty(\bar{D}), \\ 3^\circ \quad c\in C^\infty(\bar{D}), \qquad c(x)\leqq 0 \qquad \text{in } D. \end{cases}$$

Let L be a "Wentzell's boundary condition" on ∂D given by

$$Lu(x')=\sum_{i,j=1}^{N-1}\alpha^{ij}(x')\frac{\partial^2 u}{\partial x_i\partial x_j}(x')+\sum_{i=1}^{N-1}\beta^i(x')\frac{\partial u}{\partial x_i}+\gamma(x')u(x')$$
$$+\mu(x')\frac{\partial u}{\partial\nu}(x')-\delta(x')Au(x')+s(x')\int_{\partial D}[u(y')-u(x')]\frac{dy'}{\|y'-x'\|^N}$$
$$+\lambda(x')\int_D[u(y)-u(x')]\frac{dy}{|y-x'|^{N+\beta}},$$

where $x'=(x_1,\cdots,x_{N-1})\in\partial D$, satisfying

(1.2)
$$\begin{cases} 1^\circ & \alpha^{ij} \text{ are the components of a } C^\infty \text{ symmetric contravariant} \\ & \text{tensor of type } \binom{2}{0} \text{ on } \partial D \text{ satisfying} \\ & \sum_{i,j=1}^{N-1}\alpha^{ij}(x)\xi'_i\xi'_j\geqq 0,\ x'\in\partial D,\ \xi'=\sum_{j=1}^{N-1}\xi'_j dx_j\in T^*_{x'}(\partial D) \\ & \text{where } T^*_{x'}(\partial D) \text{ is the cotangent space of at } x'. \\ 2^\circ & \beta^i\in C^\infty(\partial D). \\ 3^\circ & \gamma\in C^\infty(\partial D),\quad \gamma(x')\leqq 0 \quad \text{on } \partial D. \\ 4^\circ & \mu\in C^\infty(\partial D),\quad \mu(x')\geqq 0 \quad \text{on } \partial D. \\ 5^\circ & \delta\in C^\infty(\partial D),\quad \delta(x')\geqq 0 \quad \text{on } \partial D. \\ 6^\circ & \lambda,\ s\in C^\infty(\partial D),\quad \lambda(x')\geqq 0,\quad s(x')\geqq 0 \quad \text{on } \partial D. \\ 7^\circ & \beta\in(0,1). \end{cases}$$

Here ν denotes the unit interior normal to ∂D at x', and $\|y'-x'\|$ denotes the length of $y'-x'$ with respect to the metric induced on ∂D by the Riemannian metric (a^{ij}) of $\boldsymbol{R}^N$.

The terms $\sum_{i,j}\alpha^{ij}(\partial^2u/\partial x_i\partial x_j)+\sum_i\beta^i(\partial u/\partial x_i)$, γu, $\mu(\partial u/\partial\nu)$ and δAu of L correspond to the diffusion along the boundary, absorption, reflection and viscosity phenomena respectively. The sixth and the last terms of L correspond to jump phenomena governed by the generator of Cauchy process on ∂D (a symmetric stable process of order 1), and the generator of the stable process of order β which possesses the inward jump from the boundary respectively.

We assume that $\beta\in\boldsymbol{Q}$. This assumption is necessary so that the pseudodifferential operator $T(\alpha)$ induced by L and A can be written as a pseudodifferential operator of polyhomogeneous type (cf. Theorem 4.2 below).

DEFINITION 1.1. A Wentzell's boundary condition L is said to be *transversal* on ∂D if

$$\mu(x')+\delta(x')>0 \qquad \text{on } \partial D.$$

Intuitively this implies that either reflection or viscosity phenomenon occurs on ∂D.

We can prove the following

THEOREM 1. *Let the differential operator A satisfy* (1.1) *and let the boundary* *condition* L *satisfy* (1.2) *and be transversal on* ∂D. *Suppose that the following conditions are satisfied:*

[I] (Existence) *For some constants* $\alpha \geqq 0$ *and* $\lambda \geqq 0$, *the boundary value problem*

$$(*) \qquad \begin{cases} (\alpha - A)u = 0 & \text{in } D \\ (\lambda - L)u = \varphi & \text{on } \partial D \end{cases}$$

has a solution $u \in C^\infty(\bar{D})$ *for any* $\varphi \in C^\infty(\partial D)$.

[II] (Uniqueness) *For some constant* $\alpha > 0$, *we have*

$$\begin{cases} u \in C(\bar{D}), \quad (\alpha - A)u = 0 \quad \text{in } D, \qquad Lu = 0 \quad \text{on } \partial D \\ \Longrightarrow u = 0 \quad \text{in } D. \end{cases}$$

Then there exists a Feller semigroup $(T_t)_{t \geq 0}$ *on* $\bar{D}$ *whose infinitesimal generator* $\mathfrak{a}$ *is characterized as follows:*

$$(1.3) \qquad \begin{cases} \text{(a)} & \text{the domain } \mathcal{D}(\mathfrak{a}) \text{ of } \mathfrak{a} \text{ is} \\ & \mathcal{D}(\mathfrak{a}) = \{u \in C(\bar{D}),\ Au \in C(\bar{D}),\ Lu = 0\} \\ \text{(b)} & \mathfrak{a}u = Au \qquad \text{for } u \in \mathcal{D}(\mathfrak{a}). \end{cases}$$

We omit the proof, since one can lead to the result easily following the way of proof of Theorem 1 in [**14**] and referring to results in [**12**]. The main point of the proof is that, under the conditions [I], [II], one can construct a Feller semigroup $(T_t)_{t \geq 0}$ on $\bar{D}$ by making use of a Feller semigroup $(S_t^\alpha)_{t \geq 0}$ on ∂D, and that its generator $\mathfrak{b}: C(\partial D) \to C(\partial D)$ is bijective if the boundary condition L is transversal. (See [**14**, Section 3.3].)

Next we shall state our main theorem.

To state a hypothesis on the boundary condition L, we introduce some notation. We say that a tangent vector $v = \sum_{j=1}^{N-1} v^j (\partial / \partial x_j) \in T_{x'}(\partial D)$ is *subunit* for the operator $L^0 = \sum_{i,j=1}^{N-1} \alpha^{ij} (\partial^2 / \partial x_i \partial x_j)$ if it satisfies

$$\left(\sum_{j=1}^{N-1} v^j \eta_j \right)^2 \leqq \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \eta_i \eta_j, \qquad \eta = \sum_{j=1}^{N-1} \eta_j dx_j \in T_{x'}^*(\partial D).$$

The fundamental hypothesis (A) we impose on the boundary condition L is

(A) *There exist constants* $\varepsilon \in (0, 1]$ *and* $C > 0$ *such that for all sufficiently small* $\rho > 0$

$$B_E(x', \rho) \subset B_{L^0}(x', C\rho^\varepsilon)$$

on $M = \{x' \in \partial D;\ \mu(x') = 0$ *and* $s(x') = 0\}$.

Here $B_E(x', \rho)$ denotes the ordinary Euclidian ball of radius ρ about x', and $B_{L^0}(x', \rho)$ denotes the set of all points $y \in \partial D$ which can be joined to x' by a Lipschitz path: $[0, \rho] \to \partial D$ for which the tangent vector $\dot{v}(t)$ of ∂D at $v(t)$ is subunit for L^0 for almost all t.

The notion of subunit trajectory was first introduced by Fefferman-Phong [7]. The intuitive meaning of (A) is that a particle starting at any point of the set M, where no reflection nor jumps along the boundary occur, can exit M in finite time.

Now we can state our main result.

THEOREM 2. *Let the differential operator A satisfy* (1.1) *and let the boundary condition L satisfy* (1.2) *and be transversal on ∂D. Suppose that hypothesis* (A) *is satisfied. Then we have the conclusion of Theorem* 1.

REMARK 1.2. It is known that the infinitesimal generator $\mathfrak{a}$ coincides with the minimal closed extension in $C(\bar{D})$ of the restriction of A to the space $\{u\in C^2(\bar{D});\ Lu=0\}$.

§ 2. Preliminaries.

We start by formulating the local coordinate system. Choose an open neighborhood W of ∂D in D and a C^∞-diffeomorphism (a collar) φ of $\partial D\times[0, 1)$ onto W. Choose for each point x' of ∂D a neighborhood U of x' in $\boldsymbol{R}^N$ and (under the mapping φ) a local coordinate system $(x_1, \cdots, x_{N-1}, x_N)$ on U such that

1° $x\in U\cap D \iff x\in U,\ \ x_N(x)>0,$

$x\in U\cap\partial D \iff x\in U,\ \ x_N(x)=0.$

2° The functions $(x_1, \cdots, x_{N-1})$, restricted on $U\cap\partial D$, form a local coordinate system of D on $U\cap\partial D$.

We may take $x_N(x)=\mathrm{dist}(x, \partial D)$, $x\in\boldsymbol{R}^N$. Then we have

$$\mathrm{grad}\, x_N(x')=\nu(x') \quad \text{and hence} \quad \frac{\partial}{\partial\nu}=\frac{\partial}{\partial x_N}.$$

We divide the last term of L (the inward jump term)

$$\lambda(x')\int_D[u(y)-u(x')]\frac{dy}{|y-x'|^{N+\beta}}$$

into the two sum

$$\lambda(x')\int_D[(\theta.\ u)(y)-(\theta.\ u)(x')]\frac{dy}{|y-x'|^{N+\beta}}+\lambda(x')\int_D(1-\theta)(y)u(y)\frac{dy}{|y-x'|^{N+\beta}}$$
$$:=\lambda(x')K_1u(x')+\lambda(x')K_2u(x') \qquad \text{(say)},$$

where $(\theta.\ u)(y)=\theta(y)u(y)$. Here $\theta\in C^\infty(\boldsymbol{R}^N)$ is a function such that $\mathrm{supp}\,\theta\subset W$, $\theta(x')=1$ on ∂D and $\mathrm{supp}(1-\theta)\subset\{x_N\geqq 1/2\}$ in the local coordinate. We remark that K_2 forms a compact operator on $C(\bar{D})$ in view of the Ascoli-Arzela theorem.

As stated in Section 1, we shall reduce the problem $(*)$ to that of an operator on the boundary. It is known that if the differential operator A satisfy (1. 1) and $\alpha \geqq 0$, then the Dirichlet problem

$$\begin{cases} (A-\alpha)u = 0 & \text{in } D \\ u|_{\partial D} = \varphi & \text{on } \partial D \end{cases}$$

has a unique solution $u \in H^t(D)$ for any $\varphi \in H^{t-1/2}(\partial D)$. Here $H^s(D)$ (resp. $H^s(\partial D)$) denotes the Sobolev space of order s on D (resp. ∂D).

Put the harmonic operator $H_\alpha : H^{t-1/2}(\partial D) \to H^t(D)$ by $u = H_\alpha \varphi$. Then (cf. [**14**]) H_α is an isomorphism

$$H^{t-1/2}(\partial D) \longrightarrow \{u \in H^t(D)\,;\, (A-\alpha)u = 0 \text{ in } D\}$$

and its inverse is the trace operator on ∂D.

Put the operator $S(\alpha): C^\infty(\partial D) \to C^\infty(\partial D)$ by $\varphi \mapsto LH_\alpha \varphi$. Then $S(\alpha)$ can be written

$$S(\alpha) = Q(\alpha) + \mu \Pi(\alpha) + s\Xi_1(\alpha) + \lambda \Xi_2(\alpha) + \lambda \Xi_3(\alpha)\,,$$

where

$$Q(\alpha)\varphi = \sum_{i,j=1}^{N-1} \alpha^{ij}(x') \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{N-1} \beta^i(x') \frac{\partial \varphi}{\partial x_i} + (\gamma - \alpha\delta)\varphi\,,$$

$$\Pi(\alpha)\varphi = \frac{\partial}{\partial \nu}[H_\alpha \varphi]|_{\partial D}, \qquad \Xi_1(\alpha)\varphi = \int_{\partial D} [\varphi(y') - \varphi(x')] \frac{dy'}{\|y'-x'\|^N}\,,$$

and

$$\Xi_2(\alpha)\varphi = K_1 H_\alpha \varphi\,, \qquad \Xi_3(\alpha)\varphi = K_2 H_\alpha \varphi\,.$$

We put $T(\alpha) = S(\alpha) - \Xi_3(\alpha)$. Then $T(\alpha)$ is a second order pseudodifferential operator on ∂D (cf. Bouted de Monvel [**3**, p. 32]), and its symbol is given by

$$\sigma(T(\alpha)) = \sigma(Q) + \mu\sigma(\Pi) + s\sigma(\Xi_1) + \lambda\sigma(\Xi_2)\,.$$

Here,

1° $\sigma(Q) = -\sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j + \sqrt{-1}\sum_{i=1}^{N-1} \beta^i(x')\xi_i + \text{terms of order} \leqq 0.$

2° $\sigma(\Pi) = \sqrt{-1}\,\xi_N^+ + \text{terms of order} \leqq 0,$ where

$$\xi_N^+ = \frac{-a_1(x', \xi') - \sqrt{-1}[4A_2(x')a_0(x', \xi') - a_1(x', \xi')^2]^{1/2}}{2A_2(x')}\,.$$

Here, writing the operator A in the form

$$A(x, D) = A_2(x)D_N^2 + A_1(x, D_{x'})D_N + A_0(x, D_{x'})\,,$$

$a_1(x, \xi')$, $a_0(x, \xi')$ denote the principal symbols of $A_1(x, D'_x)$ and $A_0(x, D'_x)$ respectively.

We write $\sqrt{-1}\,\xi_N^+ = p_1(x', \xi') + \sqrt{-1}\,q_1(x', \xi')$, where

$$p_1(x', \xi') = \frac{[4A_2(x')a_0(x', \xi') - a_1(x', \xi')^2]^{1/2}}{2A_2(x')}, \quad q_1(x', \xi') = \frac{-a_1(x', \xi')}{2A_2(x')}.$$

Note that $p_1(x', \xi')<0$ on $T^*(\partial D)-\{0\}$.

3° $\sigma(\Xi_1)=-C|\xi'|$. Here $|\xi'|$ denotes the length of $\xi'=(\xi_1, \cdots, \xi_{N-1})$ with respect to the Riemannian metric induced on ∂D by the Riemannian metric (a_{ij}) (the inverse matrix of (a^{ij})) of $\boldsymbol{R}^N$.

4° $\sigma(\Xi_2) = -C_\beta[\phi(x', \xi')]+$terms of order $\leqq 0$.

Here $\phi(x', \xi')$ is a symbol derived from the last term of L and given by $\phi(x', \xi')=\psi(\xi', \xi_N^+(x', \xi'))$,

$$\psi(\xi) = \psi(\xi', \xi_N) = \int_{S^{N-1}\cap\{\theta_N>0\}} |\langle\theta, \xi\rangle|^\beta (1-\sqrt{-1}(\tan \beta\pi/2)\cdot \mathrm{sgn}\langle\theta, \xi\rangle) d\theta$$

(cf. Komatsu [11]). Note that $\mathrm{Re}\,\phi(x', \xi')>0$ on $T^*(\partial D)-\{0\}$.

Since $T(\alpha)$: $C^\infty(\partial D)\to C^\infty(\partial D)$ extends to a continuous linear operator $T(\alpha)$: $H^s(\partial D)\to H^{s-2}(\partial D)$ for all $s\in\boldsymbol{R}$ (cf. [15, Section 10.2]), we extend $T(\alpha)$ to a densely defined, closed linear operator $\tau(\alpha)$ as follows;

$$\tau(\alpha)\colon H^{s-5/2+\kappa}(\partial D) \longrightarrow H^{s-5/2}(\partial D),$$

$$\begin{cases} \mathcal{D}(\tau(\alpha)) = \{\varphi\in H^{s-5/2+\kappa}(\partial D);\ \ T(\alpha)\varphi\in H^{s-5/2}(\partial D)\} \\ \tau(\alpha)\varphi = T(\alpha)\varphi, \qquad \varphi\in\mathcal{D}(\tau(\alpha)), \end{cases}$$

where κ is a constant and will be fixed later on.

Then we have

PROPOSITION 2.1. *Let A and L be as in Theorem 2 and suppose that hypothesis* (A) *is satisfied. Then there exists a constant $0<\kappa\leqq 1$ such that for any $s\in\boldsymbol{R}$ we have*

$$\varphi \in \mathcal{D}'(\partial D), \quad \tau(\alpha)\varphi \in H^s(\partial D) \Longrightarrow \varphi \in H^{s+\kappa}(\partial D).$$

Furthermore, for any $t<s+\kappa$ there exists a constant $C_{s,t}>0$ such that

$$|\varphi|_{H^{s+\kappa}(\partial D)} \leqq C_{s,t}(|T(\alpha)\varphi|_{H^s(\partial D)}+|\varphi|_{H^t(\partial D)})$$

holds.

Proposition 2.1 is the essential step in the proof of Theorem 2 and will be proved in the last section.

Lastly, we remark that $\Xi_3(\alpha)$ extends to a compact operator on $C(\partial D)$ as a composition of continuous and compact operators, which also extends to a compact operator

$$\Xi_3(\alpha)\colon H^{s-5/2+\kappa}(\partial D) \xrightarrow{\Xi_3(\alpha)} H^{s-5/2+\kappa}(\partial D) \xrightarrow[\text{injection}]{\text{natural}} H^{s-5/2}(\partial D)$$

if $s>2-\kappa$ (cf. [3, p. 25]).

§ 3. Proof of Theorem 2.

Due to Theorem 1 we have only to show that existence [I] and uniqueness [II] results follow under the given condition.

3.1. First we prove the uniqueness [II].

PROPOSITION 3.1. *Let the differential operator A satisfy* (1.1) *and let the boundary condition L satisfy* (1.2) *and be transversal on ∂D. Then for any $\alpha>0$ we have*

$$\begin{cases} u \in C^2(D), \quad (A-\alpha)u \geqq 0 \quad \text{in } D, \qquad Lu \geqq 0 \quad \text{on } \partial D \\ \Longrightarrow u \leqq 0 \quad \text{on } \bar{D}. \end{cases}$$

PROOF. If u is constant in D, then

$$0 \leqq (A-\alpha)u = (c-\alpha)u$$

and hence u is $\leqq 0$ in D, since $c \leqq 0$ in D and $\alpha>0$.

Thus we may assume that u is not constant in D. Suppose that $\max_{x\in\bar{D}} u(x) >0$. Then it follows from the weak maximum principle for $A-\alpha$ (cf. [**14**, Theorem 2.7]) that there exists a point x_0' of ∂D such that

$$\begin{cases} u(x_0') = \max\limits_{x\in\bar{D}} u(x) > 0, \qquad u(x) < u(x_0') \quad \text{in } D \\ \dfrac{\partial u}{\partial \nu}(x_0') < 0 . \end{cases}$$

Further note that

$$\frac{\partial u}{\partial x_i}(x_0') = 0 \quad (1 \leqq i \leqq N-1), \qquad Au(x_0') \geqq \alpha u(x_0') > 0$$

and that $\sum_{i,j=1}^{N-1} \alpha^{ij}(x_0')(\partial^2 u/\partial x_i \partial x_j)(x_0') \leqq 0$, since the matrices $(\alpha^{ij}(x_0'))$ and $(-(\partial^2 u/\partial x_i \partial x_j)(x_0'))$ are nonnegative definite.

In the expression

$$\begin{aligned} Lu(x_0') = &\sum_{i,j=1}^{N-1} \alpha^{ij}(x_0') \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0') + \gamma(x_0')u(x_0') \\ &+ s(x_0')\int_{\partial D} [u(y') - u(x_0')] \frac{dy'}{\|y'-x_0'\|^N} + \mu(x_0')\frac{\partial u}{\partial \nu}(x_0') \\ &- \delta(x_0')Au(x_0') + \lambda(x_0')\int_D [u(y) - u(x_0')] \frac{dy}{|y-x_0'|^{N+\beta}}, \end{aligned}$$

the first and second terms are $\leqq 0$, since $\gamma(x_0') \leqq 0$. The third term is $\leqq 0$ since $u(y') - u(x_0') \leqq 0$ for $y' \in \partial D$ and $s(x_0') \geqq 0$. For the fourth, fifth and sixth terms, we note that $(\partial u/\partial \nu)(x_0') < 0$, $-Au(x_0') < 0$, and $\int_D ([u(y) - u(x_0')]/|y-x_0'|^{N+\beta}) dy < 0$.

Hence we have that the fourth+fifth+sixth term<0 since $\mu(x')+\delta(x')>0$ on ∂D, and hence $Lu(x_0')<0$. This contradicts the assumption $Lu\geqq 0$ on ∂D.

The uniqueness [II] follows immediately from this proposition.

3.2. Next we prove the existence [I].

Let $S=\boldsymbol{R}/2\pi Z$. We consider the following boundary value problem

$$(*)\qquad \begin{cases} \left(A+\dfrac{\partial^2}{\partial y^2}\right)\tilde{u}=0 & \text{in } D\times S \\ L\tilde{u}=\tilde{\varphi} & \text{on } \partial D\times S\,. \end{cases}$$

Put $\Lambda=A+\partial^2/\partial y^2$. As before we define a harmonic operator $\tilde{H}_\alpha$ as follows; The Dirichlet problem

$$\begin{cases} \Lambda\tilde{w}=0 & \text{in } D\times S \\ \tilde{w}\,|_{\partial D}=\tilde{\varphi} & \text{on } \partial D\times S \end{cases}$$

has a unique solution $\tilde{w}\in H^t(D\times S)$ for any $\tilde{\varphi}\in H^{t-1/2}(\partial D\times S)$. Define $\tilde{H}_\alpha$: $H^{t-1/2}(\partial D\times S)\to H^t(D\times S)$ by $\tilde{w}=\tilde{H}_\alpha\tilde{\varphi}$. Then $\tilde{H}_\alpha$ is an isomorphism $H^{t-1/2}(\partial D\times S)$ $\to\{u\in H^t(D\times S);\ \Lambda\tilde{u}=0\}$ and its inverse is the trace operator on $\partial D\times S$.

We put the operator

$$\tilde{T}(\alpha)\colon C^\infty(\partial D\times S)\longrightarrow C^\infty(\partial D\times S) \quad \text{by} \quad \tilde{\varphi}\longmapsto(L-K_2)\tilde{H}_\alpha\tilde{\varphi}\,.$$

As in Section 2, $\tilde{T}(\alpha)$ is a pseudodifferential operator on $\partial D\times S$ of second order with symbol $\sigma(\tilde{T}(\alpha))$ given by

$$\sigma(\tilde{T}(\alpha))=\sigma(\tilde{Q})+\mu\sigma(\tilde{\Pi})+s\sigma(\tilde{\Xi}_1)+\lambda\sigma(\tilde{\Xi}_2)\,.$$

Here

$$\sigma(\tilde{Q})=-\sum_{i,j=1}^{N-1}\alpha^{ij}(x')\xi_i\xi_j-\delta(x')\eta^2+\sqrt{-1}\sum_{i=1}^{N-1}\beta^i(x')\xi_i+\text{terms of order}\leqq 0,$$

$$\sigma(\tilde{\Pi})=[\tilde{p}_1(x',\xi',y,\eta)+\sqrt{-1}\,\tilde{q}_1(x',\xi',y,\eta)]+\text{terms of order}\leqq 0\,,$$

where

$$\tilde{p}_1(x',\xi',y,\eta)=\frac{(4A_2(x')(a_0(x',\xi')-\eta^2)-a_1(x',\xi')^2)^{1/2}}{2A_2(x')}$$

and

$$\tilde{q}_1(x',\xi',y,\eta)=-a_1(x',\xi')/2A_2(x'),$$

$$\sigma(\tilde{\Xi}_1)=-C(|\xi'|\otimes 1)\,,$$

$$\sigma(\tilde{\Xi}_2)=-C_\beta[\phi(x',\xi')\otimes 1]+\text{terms of order}\leqq 0\,.$$

We observe $\tilde{p}_1(x',\xi',y,\eta)<0$, $\sigma(\tilde{\Xi}_1)<0$ and $-C_\beta[\phi(x',\xi')\otimes 1]<0$ on $T^*((\partial D\times S)$ $-\{0\})$.

As before we extend $\tilde{T}(\alpha)$ to $\tilde{\tau}$: $H^{s-5/2+\kappa}(\partial D\times S)\to H^{s-5/2}(\partial D\times S)$ as follows:

$$\begin{cases} \mathcal{D}(\tilde{\tau})=\{\tilde{\varphi}\in H^{s-5/2+\kappa}(\partial D\times S);\ \tilde{T}(\alpha)\tilde{\varphi}\in H^{s-5/2}(\partial D\times S)\} \\ \tilde{\tau}\tilde{\varphi}=\tilde{T}(\alpha)\tilde{\varphi}, \qquad \tilde{\varphi}\in\mathcal{D}(\tilde{\tau})\,. \end{cases}$$

As in Section 1, we put $\tilde{L}^0=\sum_{i,j=1}^{N-1}a^{ij}(\partial^2/\partial x_i\partial x_j)+\delta(\partial^2/\partial y^2)$ and $\tilde{M}=\{(x', y)\in\partial D\times S;\ \mu(x')=0$ and $s(x')=0\}=M\times S$. Since $\delta(x')>0$ on $M=\{x'\in\partial D;\ \mu(x')=0$ and $s(x')=0\}$, we observe that if the condition (A) holds then the condition

($\tilde{\mathrm{A}}$) *There exists a constant* $\tilde{C}>0$ *such that for all sufficiently small* $\rho>0$ *we have*

$$B_E((x', y), \rho)\subset B_{\tilde{L}^0}((x', y), \tilde{C}\rho^{\varepsilon}), \qquad (x', y)\in\tilde{M}$$

holds.

Then just as that Proposition 2.1 follows under the assumption (A), we have the following

PROPOSITION 3.2. *Let A and L be as in Theorem 2 and suppose that hypothesis* (A) *is satisfied. Let κ be the same constant as in Proposition* 2.1. *Then for any $s\in\boldsymbol{R}$ we have:*

$$\tilde{\varphi}\in\mathcal{D}'(\partial D\times S),\ \tilde{T}\tilde{\varphi}\in H^{s-5/2}(\partial D\times S)\Longrightarrow\tilde{\varphi}\in H^{s-5/2+\kappa}(\partial D\times S).$$

Furthermore, for any $t<s-5/2+\kappa$ there exists a constant $\tilde{C}_{s,t}>0$ such that

$$|\tilde{\varphi}|_{H^{s-5/2+\kappa}(\partial D\times S)}\leqq\tilde{C}_{s,t}(|\tilde{T}(\alpha)\tilde{\varphi}|_{H^{s-5/2}(\partial D\times S)}+|\tilde{\varphi}|_{H^t(\partial D\times S)})$$

holds.

Hence, since the injection $H^{s-5/2+\kappa}(\partial D\times S)\hookrightarrow H^t(\partial D\times S)$ $(t<s-5/2+\kappa)$ is compact, it follows from a version of Peetre's lemma (cf. [**14**, Lemma 4.8]) that the dimension of the kernel $\mathcal{N}(\tilde{\tau})$ of $\tilde{\tau}$ is finite and that the range $\mathcal{R}(\tilde{\tau})$ of $\tilde{\tau}$ is closed in $H^{s-5/2}(\partial D\times S)$.

The codimension $\operatorname{codim}\mathcal{R}(\tilde{\tau})=\dim\mathcal{N}(\tilde{\tau}^*)$ of the range of $\tilde{\tau}$ is calculated to be finite by considering the adjoint $\tilde{\tau}^*$ of $\tilde{\tau}$ and use the similar estimate.

Hence we conclude that the index $\operatorname{ind}(\tilde{\tau})\equiv\dim\mathcal{N}(\tilde{\tau})-\operatorname{codim}\mathcal{R}(\tilde{\tau})$ of $\tilde{\tau}$ is finite. It is known (cf. [**14**]) that;

If the index of $\tilde{\tau}$ is finite, then for any $\alpha\geqq 0$ the index of $\tau(\alpha)$ is equal to zero.

Hence we have $\operatorname{ind}(\tau(\alpha))=0$. And so $\operatorname{ind}(\sigma(\alpha))=0$ by the stability theorem of index, where $\sigma(\alpha)=\tau(\alpha)+\Xi_3(\alpha)$. By the uniqueness [II] we observe that $\dim\mathcal{N}(\sigma(\alpha))=0$ for any $\alpha>0$, and this implies $\operatorname{codim}\mathcal{R}(\sigma(\alpha))=\dim\mathcal{N}(\sigma^*(\alpha))=0$. Combining this with the regularity result (Proposition 2.1) and the remark given at the end of Section 2, we have the existence result [I]. The proof of Theorem 2 is now complete.

§4. Proof of Proposition 2.1.

Consider again the pseudodifferential operator $T(\alpha)$. Its principal symbol is $-\sum_{i,j=1}^{N-1}a^{ij}(x')\xi_i\xi_j$ and the subprincipal symbol on the characteristic set

$$\Sigma = \{(x', \xi') \in T^*(\partial D) - \{0\} ; \sum_{i,j=1}^{N-1} \alpha^{ij}(x')\xi_i\xi_j = 0\}$$

is equal to

$$\mu(x')p_1(x', \xi') - s(x')C|\xi'|$$
$$+\sqrt{-1}\Big(\mu(x')q_1(x', \xi') + \sum_{i=1}^{N-1} \beta^i(x')\xi_i - (1/2)\sum_{i,j=1}^{N-1} \frac{\partial \alpha^{ij}}{\partial x_j}(x')\xi_i\Big).$$

The reason for the appearance of the term $-\sqrt{-1}(1/2)\sum_{i,j=1}^{N-1}(\partial\alpha^{ij}/\partial x_j)(x')\xi_i$ in the subprincipal part is deeply related to the theory of Weyl calculus of pseudodifferential operators. See [8], pages 83, 161; see also [**15**], p. 213.

For a pseudodifferential operator P with symbol $p(x', \xi')$, we shall denote by $P^{(j)}$ and $P_{(j)}$ $(1 \leqq j \leqq N-1)$ pseudodifferential operators with symbols $(\partial p/\partial \xi_j)(x', \xi')$ and $D_j p(x', \xi')$ respectively. Here $D_j = (1/\sqrt{-1})\partial/\partial x_j$.

We first prove a localized version of Proposition 2.1.

PROPOSITION 4.1. *Let A and L be as in Theorem 2 and suppose that hypothesis* (A) *is satisfied. Then for any point x_0' of ∂D, we can find a neighborhood $U(x_0')$ of x_0' such that for every compact set $K \subset U(x_0')$ there is a constant $\kappa = \kappa(K)$, $0 < \kappa(K) \leqq 1$, such that for any $s \in \boldsymbol{R}$ and $t < s + \kappa$ we have*

$$\sum_{j=1}^{N-1}(|T(\alpha)^{(j)}\phi|^2_{H^{s+\kappa/2}(\partial D)} + |T(\alpha)_{(j)}\phi|^2_{H^{s-1+\kappa/2}(\partial D)}) + |\phi|^2_{H^{s+\kappa}(\partial D)} \tag{4.1}$$
$$\leqq C_{K,s,t}(|T(\alpha)\phi|^2_{H^s(\partial D)} + |\phi|^2_{H^t(\partial D)}), \qquad \phi \in C_0^\infty(K)$$

with a constant $C_{K,s,t} > 0$.

PROOF OF PROPOSITION 4.1.

1° Let x_0' be a point of $M = \{x' \in \partial D ; \mu(x') = 0 \text{ and } s(x') = 0\}$. Then the assumption (A) implies that one can find a neighborhood $U(x_0')$ of x_0' such that for sufficiently small $\rho > 0$ we have $B_E(x', \rho) \subset B_{L^0}(x', 2C\rho^\varepsilon)$, $x' \in U(x_0')$.

In view of the form of the boundary condition L, we see that we are in just the same situation on K as in [**15**, Proposition 10.2.4], provided the estimate

$$\sum_{j=1}^{N-1}\Big(\Big|\sum_{i=1}^{N-1}\alpha^{ij}D_i\phi\Big|^2_{H^s(\partial D)} + \Big|\sum_{k,m=1}^{N-1}\frac{\partial\alpha^{km}}{\partial x_j}D_kD_m\phi\Big|^2_{H^{s-1}(\partial D)}\Big) \tag{4.2}$$
$$\leqq C_{K,s}(|T(\alpha)\phi|^2_{L^2(\partial D)} + |\phi|^2_{H^{2s}(\partial D)}), \qquad \phi \in C_0^\infty(K)$$

for some constant $C_{K,s} > 0$, where $s \geqq 0$. This estimate (4.2) follows as in the proof of [**15**, Proposition 10.2.3], if we note that non-local terms $\mu\Pi(\alpha)$, $s\Xi_1$ and $\lambda\Xi_2$ are pseudodifferential operators of order 1, 1 and β respectively whose real parts of principal symbols are all nonpositive, and use Gårding inequality. And so (4.1) follows with $\kappa(K) = \varepsilon$.

2° Secondly we prove Proposition 4.1 in case that $x_0' \notin M$ by using the following result due to Hörmander.

THEOREM 4.2 (Hörmander [8, Theorem 22.3.4]). *Assume that P is a properly supported, polyhomogeneous* (cf. [8, Definition 18.1.5]) *pseudodifferential operator in $X\subset\boldsymbol{R}^{N-1}$ of order m, and that the principal symbol p_m is nonnegative in a conic neighborhood of $(x_0, \xi_0)\in T^*(X)-\{0\}$ but vanishes at (x_0, ξ_0). Let Q be the Hessian of $p_m/2$ at (x_0, ξ_0), and assume that the subprincipal symbol p_{m-1}^s satisfies $p_{m-1}^s(x_0, \xi_0)+\mathrm{Tr}^+Q\notin\{t\in\boldsymbol{R}\,;\,t\leqq 0\}$.*

Then $u\in\mathcal{D}'(X)$, $Pu\in H^s$ at (x_0, ξ_0) implies $u\in H^{s+m-1}$ at (x_0, ξ_0). Here Tr^+ *stands for the trace with respect to positive eigenvalues.*

Let us apply this theorem to $P=-T(\alpha)$. It is seen that the principal symbol $p_2(x', \xi')=\sum_{i,j=1}^{N-1}\alpha^{ij}(x')\xi_i\xi_j$ satisfies $p_2(x', \xi')\geqq 0$ on $T^*(\partial D)-\{0\}$.

On the other hand, the subprincipal symbol $p_1^s(x', \xi')$ on Σ is equal to

$$-\mu(x')p_1(x', \xi')+s(x')C|\xi'|$$
$$-\sqrt{-1}\Big(\mu(x')q_1(x', \xi')+\sum_{i=1}^{N-1}\beta^i(x')\xi_i-(1/2)\sum_{i,j=1}^{N-1}\frac{\partial\alpha^{ij}}{\partial x_j}(x')\xi_i\Big).$$

Here we observe that the condition of Theorem 4.2 is satisfied, since $-\mu(x')p_1(x', \xi')+s(x')C|\xi'|\geqq 0$ on $T^*(\partial D)-\{0\}$. Furthermore, if $x_0'\notin M$ we can find a neighborhood $U(x_0')$ of x_0' such that $\mu(x')+s(x')>0$ on $U(x_0')$, and hence

$$\mathrm{Re}\,p_1^s(x', \xi')>0 \qquad \text{in } U(x_0')\times(\boldsymbol{R}^{N-1}-\{0\}).$$

Therefore applying Theorem 4.2 to $-T(\alpha)$, we obtain that for every compact $K\subset U(x_0')$

$$|\phi|^2_{H^{s+1}(\partial D)}\leqq C(|T(\alpha)\phi|^2_{H^s(\partial D)}+|\phi|^2_{H^t(\partial D)}), \tag{4.3}$$

$\phi\in C_0^\infty(K)$ holds $(t<s+1)$.

Consequently, in case $\mu(x_0')+s(x_0')>0$, the estimate (4.1) with $\kappa(K)=1$ follows from (4.2) and (4.3) in the same way as in case $\mu(x_0')=s(x_0')=0$. This completes the proof of Proposition 4.1.

The argument to lead Proposition 2.1 from the estimate (4.1) is the same as in [**15**, Section 2-3)] and so we omit the proof. This completes the proof of Proposition 2.1.

§ 5. Concluding remarks.

In Section 1 we confined ourselves to the case when the fractional order β of the last term of L to the rational numbers in (0, 1), so as that the sum of their symbols are of polyhomogeneous type. The reason is that when it is irrational we do not know whether the conclusion of Theorem 4.2 applied to $-T(\alpha)$ holds or not.

The arguments above can be applied if we replace the boundary condition

L with $W=P+sT_1+\lambda T_\beta$, where P is a second order differential operator, T_1 and T_β are polyhomogeneous pseudodifferential operators of order 1, β ($\beta<1$) respectively.

References

[1] R. F. Anderson, Diffusions with second order boundary conditions, Part I, II, Indiana Univ. Math. J., **25** (1976), 367-395, 403-411.
[2] J.-M. Bony, P. Courrège et P. Priouret, Semi-group de Feller sur une varaété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principle du maximum, Ann. Inst. Fourier (Grenoble), **18** (1968), 369-521.
[3] L. Boutet de Monvel, Boundary problems for pseudo-differential operators, Acta Math., **126** (1971), 11-51.
[4] C. Cancelier, Problemes aux pseudo-differentiels donnant lieu au principle du maximium, Comm. P. D. E., **11** (1986), 1677-1726.
[5] P. Cattiaux, Regularite au bord pour les densites et les densites conditionnelles d'une diffusion reflechie hypoelliptique, Stochastics, **20** (1987), 309-340.
[6] P. Cattiaux, Stochastic calculus and degenerate boundary value problems, preprint, November 1987.
[7] C. Fefferman and D. H. Phong, Subelliptic eigenvalue problems, In Conference on Harmonic Analysis (1981. Chicago), Wadsworth, Belmont, 1983, pp. 590-606.
[8] L. Hörmander, The analysis of linear partial differential operators III, Springer, 1985.
[9] N. Ikeda, On the construction of two-dimensional diffusion processes satisfying Wentzell's boundary conditions and its application to boundary value problems, Mem. Coll. Sci. Univ. Kyoto, Ser. A, **33** (1961), 367-427.
[10] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland/Kodansha, 1981.
[11] T. Komatsu, On the martingale problem for generators of stable processes with perturbations, Osaka J. Math., **21** (1984), 113-132.
[12] K. Sato and T. Ueno, Multi-dimensional diffusion and the Markov process on the boundary, J. Math. Kyoto Univ., **4** (1965), 529-605.
[13] D. W. Stroock and S. R. S. Varadhan, On degenerate elliptic-parabolic operators of second order and their associated diffusions, Comm. Pure Appl. Math., **24** (1972), 651-713.
[14] K. Taira, Semigroups and boundary value problems, Duke Math. J., **49** (1982), 287-320.
[15] K. Taira, Diffusion processes and partial differential equations, Academic Press, 1988.
[16] S. Takanobu and S. Watanabe, On the existence and uniqueness of diffusion processes with Wentzell's boundary conditions, J. Math. Kyoto Univ., **28** (1988), 71-80.
[17] S. Watanabe, On stochastic differential equations for multi-dimensional diflusion processes with boundary conditions I, II, J. Math. Kyoto Univ., **11** (1971), 169-180, **11** (1971), 545-551.
[18] S. Watanabe, Construction of diffusion processes with Wentzell's boundary conditions by means of Poisson point processes of Brownian excursions, Probability Theory, Banach Center Publications, **5**, PWN-Polish Scientific Publishers, Warsaw, 1979, pp. 255-271.

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