J. Math. Soc. Japan
Vol. 42, No. 1, 1990

Harmonic mappings from $\boldsymbol{R}^m$ into an Hadamard manifold

By Atsushi TACHIKAWA

(Received Dec. 26, 1988)

0. Introduction.

Let M, N be Riemannian manifolds. Given a C^1-mapping $U: M \to N$ we define the *energy density* $e(U)(x)$ at $x \in M$ by

$$e(U)(x) = \frac{1}{2}|dU(x)|^2,$$

where $|\quad|$ denotes the norm induced from the tensor product norm on $T_x^*M \otimes T_{U(x)}N$. For a bounded domain Ω of M, we define the energy of $U: M \to N$ on Ω by

$$E(U;\ \Omega) = \int_\Omega e(U)d\mu,$$

where $d\mu$ stands for the volume element on M. A mapping $U: M \to N$ is said to be *harmonic* if it is of class C^2 and satisfies the Euler-Lagrange equation of the energy functional E.

The notion of harmonic mappings is an extension of the one of harmonic functions. Therefore it is natural to expect that Liouville type theorems are valid also for harmonic mappings. In fact, by S. Hildebrandt-J. Jost-K.-O. Widman [**3**] it has been shown that a harmonic mapping $U: M \to N$ must be a constant mapping if M is *simple* and the image $U(M)$ is contained in a geodesic ball $B_r \subset N$ with $r < \pi/(2\sqrt{\kappa})$, where κ denotes the maximum of the sectional curvatures of N. Here, a Riemannian manifold is said to be *simple*, if it is topologically a Euclidean m-space $\boldsymbol{R}^m$ and furnished with a metric for which the associated Laplace-Beltrami operator is uniformly elliptic on $\boldsymbol{R}^m$. If N has nonpositive sectional curvatures then we can take arbitrary $r > 0$. (See also [**1**] and [**6**].) For the case that N has nonpositive sectional curvatures another type of non-existence result is shown by L. Karp [**5**], who proved that a non-constant harmonic mapping $U: M \to N$ defined on a complete, noncompact Riemannian manifold M satisfies a certain growth order condition. This implies non-existence of harmonic mappings under some growth order condition.

In this paper we prove non-existence of harmonic mappings U from $\boldsymbol{R}^m$ ($m\geqq 2$) to an *Hadamard n-manifold* N^n with negative sectional curvatures under a kind of nondegeneracy condition on U. In contrast with the above results, our non-existence result requires neither conditions about growth order of U nor boundedness of U. Here a manifold is said to be an *Hadamard manifold* if it is a complete, simply connected Riemannian manifold with nonpositive sectional curvatures.

Throughout this paper we use the following notations: We use a standard coordinate system $x=(x^1, \cdots, x^m)$ on $\boldsymbol{R}^m$ and a normal coordinate system $u=(u^1, \cdots, u^n)$ centered at some fixed point $P_0\in N^n$ on N^n. We shall write $(g_{ij}(u))$ for the metric tensor on N^n with respect to the normal coordinate system $(u^1, \cdots, u^n)$, $(g^{ij}(u))$ for the inverse of $(g_{ij}(u))$, and the Christoffel symbols of the first and second kind of the Levi-Civita connection on N^n will be denoted by Γ_{ijk} and Γ^i_{jk}. K_N denotes the supremum of the sectional curvatures of N^n. $\|\ \|$ and $\langle\ ,\ \rangle$ denote the Euclidean norms and inner products respectively. The Einstein summation convention is used unless otherwise mentioned.

For the representation $u(x)$ of U: $\boldsymbol{R}^m\to N^n$ with respect to the coordinate system $(x^1, \cdots, x^m)$ and $(u^1, \cdots, u^n)$ the energy on $\Omega\subset\boldsymbol{R}^m$ can be written as

$$E(u\,;\,\Omega)=\int_\Omega \frac{1}{2}\sum_{\alpha=1}^m g_{ij}(u)\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\alpha}dx$$

and the Euler-Lagrange equations can be written as

$$\Delta u^i+\sum_{\alpha=1}^m \Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\alpha}=0 \qquad \text{for } 1\leqq i\leqq n, \tag{0.1}$$

where Δ denotes the standard Laplacian on $\boldsymbol{R}^m$, i.e. $\Delta=\sum_{\alpha=1}^m(\partial/\partial x^\alpha)^2$.

Our main result is stated as follows.

THEOREM 1. *Let N^n be an Hadamard n-manifold and K_N the supremum of the sectional curvatures of N^n. Assume that $K_N<0$. Then there exists no harmonic mapping U: $\boldsymbol{R}^m\to N^n$ ($m\geqq 2$) which is defined on the whole $\boldsymbol{R}^m$ and whose coordinate representation $u(x)=(u^1(x), \cdots, u^n(x))$ with respect to the normal coordinate system centered at $U(0)$ satisfies the following condition: There exists a positive constant ε, such that*

$$\left\|D\Big(\frac{u(x)}{\|u(x)\|}\Big)\right\|^2=\sum_{\alpha=1}^m\sum_{i=1}^n\Big(\frac{\partial}{\partial x^\alpha}\Big(\frac{u^i(x)}{\|u(x)\|}\Big)\Big)^2\geqq\frac{\varepsilon}{\|x\|^2} \tag{0.2}$$

for all $x\in\{x\in\boldsymbol{R}^m: u(x)\neq(0, \cdots, 0)\}$, where $\|u\|=(\sum_{i=1}^n(u^i)^2)^{1/2}$, and $\|x\|=(\sum_{\alpha=1}^m(x^\alpha)^2)^{1/2}$.

In [8], the author has proved the above theorem for the case that N^n is the hyperbolic n-space with constant negative curvature $\boldsymbol{H}^n$ and $\varepsilon=n-1$.

REMARK 1. Denoting $\phi^i=u^i/\|u\|$, $\omega^\alpha=x^\alpha/\|x\|$ and $r=\|x\|$, we have

$$\left\| D\left(\frac{u^i(x)}{\|u(x)\|}\right)\right\|^2 = \sum_{\alpha=1}^{m}\sum_{i=1}^{n}\left(\frac{\partial\phi^i}{\partial\omega^\alpha}\right)^2 \frac{1}{\|x\|^4}(\|x\|^2-(x^\alpha)^2)+\sum_{i=1}^{n}\left(\frac{\partial\phi^i}{\partial r}\right)^2.$$

Thus we can see that the condition

$$\min_{\alpha,i}\left(\frac{\partial\phi^i}{\partial\omega^\alpha}\right)^2 \geqq \delta \qquad \text{for some } \delta>0 \text{ and for all } x\in \boldsymbol{R}^m$$

is a sufficient condition for (0.2). This implies that the condition (0.2) can be considered as a kind of nondegeneracy condition.

REMARK 2. From Remark 1, it is easy to see that a rotationally symmetric mapping u from $\boldsymbol{R}^n$ into a warped product manifold $\boldsymbol{R}_+\times_f S^{n-1}$ defined as $u(x) :=(\rho(\|x\|), x/\|x\|)\in \boldsymbol{R}_+\times_f S^{n-1}$ ($\rho: \boldsymbol{R}_+\to\boldsymbol{R}_+$) satisfies the condition (0.2). Therefore Theorem 1 implies non-existence of rotationally symmetric harmonic mappings from $\boldsymbol{R}^n$ to an warped product n-manifold whose sectional curvatures are bounded from above by a negative constant. This contrasts with the result of [7] which asserts the existence of rotationally symmetric harmonic mappings from $\boldsymbol{R}^n$ onto a warped product manifold $\boldsymbol{R}_+\times_f S^{n-1}$ whose sectional curvatures $K(P)$ at a point P tend to 0 sufficiently rapidly as the distance from the origin to the point P tends to infinity.

ACKNOWLEDGEMENT. The author would like to thank Professor Morio Obata and Professor Seiki Nishikawa for their useful advice.

1. Differential inequality for $\|u\|$.

In this section we show that if $K_N<0$ then for a harmonic mapping u: $\boldsymbol{R}^m\to N^n$, $\|u\|$ satisfies an elliptic differential inequality which enables us to use a comparison theorem for elliptic equations.

First of all we need the following differential geometric estimates.

LEMMA 1. *Let $(u^1, \cdots, u^n)$ be a normal coordinate system on N^n. Assume that*

$$K_N = -\kappa^2 < 0 \qquad (\kappa>0). \tag{1.1}$$

Then we have

$$g_{ij}(u)(X^iX^j+u^k\Gamma^i_{kl}(u)X^lX^j) \geqq \|\xi\|^2+\left(\frac{1}{2\kappa\|u\|}\sinh(2\kappa\|u\|)\right)\|\zeta\|^2 \tag{1.2}$$

for all u and $X\in\boldsymbol{R}^n$, where $\xi=\langle X, u\rangle u/\|u\|^2$ and $\zeta=X-\xi$.

PROOF. Since we are using the normal coordinate system, we have the following relations (cf. [4]).

$$g_{ij}(u)u^i = u^j, \tag{1.3a}$$

(1.3b)
$$\Gamma^i_{kl}u^k u^l = 0,$$

(1.3c)
$$g_{ij}(u)u^k\Gamma^i_{kl}(u)u^j = \Gamma_{kjl}(u)u^k u^j = -\Gamma_{klj}(u)u^k u^j + u^k u^j \frac{\partial g_{jl}}{\partial u^k}$$
$$= 0 + u^k \frac{\partial}{\partial u^k}(g_{jl}u^j) - g_{jl}\delta^j_k u^k = u^k \frac{\partial u^l}{\partial u^k} - u^l = 0.$$

Therefore the following equality holds.

(1.4)
$$g_{ij}(u)(X^i X^j + u^k\Gamma^i_{kl}(u)X^l X^j) = g_{ij}(u)\xi^i\xi^j + g_{ij}(u)(\zeta^i\zeta^j + u^k\Gamma^i_{kl}(u)\zeta^l\zeta^j).$$

On the other hand, using Rauch's comparison theorem, we have the following estimates as in Lemma 6 of [**4**].

(1.5)
$$g_{ij}(u)(\zeta^i\zeta^j + u^k\Gamma^i_{kl}(u)\zeta^l\zeta^j) \geqq \kappa\|u\|\coth(\kappa\|u\|)g_{ij}\zeta^i\zeta^j,$$

(1.6)
$$g_{ij}(u)\zeta^i\zeta^j \geqq \frac{\sinh^2(\kappa\|u\|)}{\kappa^2\|u\|^2}\|\zeta\|^2.$$

Combining (1.4), (1.5) and (1.6), we obtain (1.2). $\square$

Let $u: \boldsymbol{R}^m \to N^n$ be a harmonic mapping. Then u satisfies the following equation of weak form

(1.7)
$$\int_{\boldsymbol{R}^m} \sum_{\alpha=1}^m g_{ij}(u)(D_\alpha u^i D_\alpha \phi^j + \phi^k \Gamma^j_{kl} D_\alpha u^l D_\alpha u^i)dx = 0$$

for all $\phi \in C_0^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. Here and in the sequel we write D_α for $\partial/\partial x^\alpha$.

Now we can prove the following proposition which, with the condition (0.2), implies that $\|u\|$ is a subsolution of an elliptic differential equation and enables us to apply a comparison theorem.

PROPOSITION 1. *Assume that* $K_N = -\kappa^2 < 0$ *and let* $u: \boldsymbol{R}^m \to N^n$ *be a harmonic mapping. Then* $\|u\|$ *satisfies the following differential inequality*

(1.8)
$$\Delta\|u\| - \frac{\sinh(2\kappa\|u\|)}{2\kappa\|u\|^2}(\|Du\|^2 - \|D\|u\|\|^2) \geqq 0$$

on $\Omega = \{x \in \boldsymbol{R}^m : u(x) \neq 0\}$, *where* $\|Du\|^2 = \sum_{i=1}^n \sum_{\alpha=1}^m (D_\alpha u^i)^2$ *and* $\|D\|u\|\|^2 = \sum_{\alpha=1}^m (D_\alpha\|u\|)^2$.

PROOF. Taking $\phi = u\eta$, $\eta \in C_0^\infty(\boldsymbol{R}^m, \boldsymbol{R})$, in (1.7), we get

(1.9)
$$\int \sum_{\alpha=1}^m g_{ij}(u)\{u^j D_\alpha u^i D_\alpha \eta + (D_\alpha u^i D_\alpha u^j + u^k \Gamma^i_{kl}(u) D_\alpha u^l D_\alpha u^i)\eta\}dx = 0.$$

Using (1.3a), from (1.9) we get

(1.10)
$$\int \sum_{\alpha=1}^m \left\{\frac{1}{2}D_\alpha\|u\|^2 D_\alpha \eta + g_{ij}(u)(D_\alpha u^i D_\alpha u^j + u^k \Gamma^j_{kl} D_\alpha u^l D_\alpha u^i)\eta\right\}dx = 0$$

for all $\eta \in C_0^\infty(\boldsymbol{R}^m, \boldsymbol{R})$.

Now, using Lemma 1, we obtain from (1.10) an inequality for $\|u\|$, (1.11) $$\int\Big\{\sum_{\alpha=1}^{m}\frac{1}{2}D_\alpha\|u\|^2 D_\alpha\eta+\Big(\|\xi\|^2+\frac{1}{2\kappa\|u\|}\sinh(2\kappa\|u\|)\|\zeta\|^2\Big)\eta\Big\}dx\leqq 0$$

for any nonnegative function $\eta\in C_0^\infty(\boldsymbol{R}^m, \boldsymbol{R})$, where we are writing

$$\xi=(\xi_\alpha^i)=\Big(\frac{\langle u, D_\alpha u\rangle}{\|u\|^2}u^i\Big) \quad\text{and}\quad \zeta=(\zeta_\alpha^i)=(D_\alpha u^i)-(\xi_\alpha^i),$$

and therefore

$$\|\xi\|^2=\sum_{\alpha=1}^{m}\sum_{i=1}^{n}(\xi_\alpha^i)^2=\frac{\|D\|u\|^2\|^2}{4\|u\|^2},$$

$$\|\zeta\|^2=\sum_{\alpha=1}^{m}\sum_{i=1}^{n}(\zeta_\alpha^i)^2=\|Du\|^2-\frac{\|D\|u\|^2\|^2}{4\|u\|^2}.$$

Thus from (1.11), we can see that $\|u\|$ satisfies the following differential inequality on Ω;

(1.12) $$\frac{1}{2}\Delta\|u\|^2-\frac{\|D\|u\|^2\|^2}{4\|u\|^2}-\frac{1}{2\kappa\|u\|}\sinh(2\kappa\|u\|)\Big(\|Du\|^2-\frac{\|D\|u\|^2\|^2}{4\|u\|^2}\Big)\geqq 0.$$

Now, from (1.12) we obtain (1.8). □

2. Proof of Theorem 1.

In this section we prove Theorem 1 using a comparison theorem for elliptic equations.

The following lemma gives a supersolution of (1.8).

LEMMA 2. *Let $\rho_\delta(t)=\log(1+t^\delta)-\log(1-t^\delta)$. Let B_s denote a ball $\{x\in\boldsymbol{R}^m: \|x\|<s\}$ and $\phi_{\delta,\kappa,R}(x)$: $B_{R/\kappa}\to\boldsymbol{R}$ be a function defined by*

$$\phi_{\delta,\kappa,R}(x)=\frac{1}{\kappa}\rho_\delta\Big(\frac{\kappa\|x\|}{R}\Big).$$

For any fixed $\varepsilon>0$, if $\delta>0$ is sufficiently small, then $\phi_{\delta,\kappa,R}(x)$ satisfies

(2.1) $$\Delta\phi_{\delta,\kappa,R}(x)-\frac{\varepsilon}{2\kappa\|x\|^2}\sinh(2\kappa\phi_{\delta,\kappa,R}(x))\leqq 0 \qquad \text{on } B_{R/\kappa}-\{0\},$$

for all $R>0$ and $\kappa>0$.

PROOF. Denoting $r=\|x\|$ and $t=\kappa r/R$, we have

$$\Delta\phi_{\delta,\kappa,R}(x)-\frac{\varepsilon}{2\kappa\|x\|^2}\sinh(2\kappa\phi_{\delta,\kappa,R}(x))=\frac{\kappa}{R^2}\rho_\delta''(t)+\frac{m-1}{Rr}\rho_\delta'(t)-\frac{\varepsilon}{2\kappa r^2}\sinh(2\rho_\delta(t))$$

$$=\frac{\kappa}{R^2}\Big\{\rho_\delta''(t)+\frac{R(m-1)}{\kappa r}\rho_\delta'(t)-\frac{\varepsilon R^2}{2\kappa^2 r^2}\sinh(2\rho_\delta(t))\Big\}$$

$$=\frac{\kappa}{R^2}\Big\{\rho_\delta''(t)+\frac{m-1}{t}\rho_\delta'(t)-\frac{\varepsilon}{2t^2}\sinh(2\rho_\delta(t))\Big\}$$

$$=\frac{2\kappa}{R^2(1-t^{2\delta})^2}[t^{\delta-2}\{\delta(\delta-1)+\delta(m-1)-\varepsilon\}+t^{3\delta-2}\{\delta(\delta+1)-\delta(m-1)-\varepsilon\}].$$

Thus, for any $\varepsilon>0$, if we take $\delta>0$ sufficiently small we obtain the inequality (2.1). □

Using the above lemma, we can prove Theorem 1 by a comparison theorem for elliptic equations.

PROOF OF THEOREM 1. Let $U: \boldsymbol{R}^m \to N^n$ be a harmonic mapping whose coordinate representation u with respect to the normal coordinate system centered at $U(0)$ satisfies (0.2). Then from (0.2) and (1.8) we can see that $\|u\|$ satisfies

$$\Delta\|u\|-\frac{\varepsilon}{2\kappa\|x\|^2}\sinh(2\kappa\|u\|)\geqq 0, \tag{2.2}$$

for the condition (0.2) yields that

$$\frac{\|Du\|^2-\|D\|u\|\|^2}{\|u\|^2}=\left\|D\left(\frac{u}{\|u\|}\right)\right\|^2\geqq\frac{\varepsilon}{\|x\|^2}.$$

Since (0.2) implies that u is not a constant mapping, we can choose a compact set $D\Subset\boldsymbol{R}^m-\{0\}$ on which $\|u\|\geqq\varepsilon_0$ for sufficiently small $\varepsilon_0>0$.

Now, take δ so small that $\phi_{\delta,\kappa,R}(x)=\kappa^{-1}\rho_\delta(\kappa\|x\|/R)$ satisfy (2.1) for all $R>0$ and $\kappa>0$, and take R_0 so large that

$$\phi_{\delta,\kappa,R}(x)<\varepsilon_0 \qquad \text{on } D \text{ for } R>R_0.$$

Remarking that $\phi_{\delta,\kappa,R}(0)=\|u(0)\|=0$ and that $\phi_{\delta,\kappa,R}\to\infty$ as $\|x\|\to R/\kappa$ while u remains to be bounded on every bounded set, we can see that for every $R\geqq R_0$ there exists a bounded domain $\Omega_R\supset D$, $\not\ni 0$ such that $\|u\|=\phi_{\delta,\kappa,R}(x)$ on $\partial\Omega_R$. Now, from (2.1) and (2.2) we can use a comparison theorem for elliptic equations (see for example [2], Theorem 10.1) to get $\|u\|\leqq\phi_{\delta,\kappa,R}(x)$ in Ω_R for all $R>R_0$. This implies that $\|u\|=0$ on D, which is a contradiction. Thus Theorem 1 is proved. □

References

[1] M. Giaquinta and S. Hildebrandt, A priori estimates for harmonic mappings, J. Reine Angew. Math., **336** (1982), 124-164.

[2] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order (second edition), Springer, 1983.

[3] S. Hildebrandt, J. Jost and K.-O. Widman, Harmonic mappings and minimal submanifolds, Invent. Math., **62** (1980), 269-298.

[4] S. Hildebrandt and H. Kaul, Two-dimensional variational problems with obstructions, and Plateau's problem for H-surfaces in a Riemannian manifold, Comm. Pure Appl. Math., **25** (1972), 187-223.

[5] L. Karp, The growth of harmonic functions and mappings, Differential Geometry Proceedings, Special Year, Maryland 1981-1982 (Progress in Mathematics, Vol. **32**),

Birkhäuser, 1983, pp. 153-161.
[6] A. Tachikawa, On interior regularity and Liouville's theorem for harmonic mappings, Manuscripta Math., **42** (1983), 11-40.
[7] A. Tachikawa, Rotationally symmetric harmonic maps from a ball into a warped product manifold, Manuscripta Math., **53** (1985), 235-254.
[8] A. Tachikawa, A non-existence result for harmonic mappings from $\boldsymbol{R}^n$ into $\boldsymbol{H}^n$, Tokyo J. Math., **11** (1988), 311-316.

Atsushi TACHIKAWA
Department of Mathematics
Faculty of Liberal Arts
Shizuoka University
Ohya, 836, Shizuoka 422
Japan