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# Besov spaces and analytic semigroups of linear operators

Dedicated to Professor Hiroshi Fujita on his 60th birthday

By Tosinobu MURAMATU

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## Introduction and main results.

It is known that for a strongly continuous function $f$ on an interval $I=(a, b)$ with values in a Banach space $X$ and an analytic semigroup $e^{-tA}$ of linear operators in $X$ the function

$$F(t)=\int_{a}^{t} e^{-(t-s)A} f(s) ds \tag{1}$$

is not strongly differentiable in $I$ except that $X$ has a special property or $A$ is bounded in $X$ (see Baillon [**1**]). To guarantee strong differentiability of $F$ we have to assume more smoothness condition on $f$ than strong continuity. Crandall-Pazy [**2**] proved that $F$ is strongly differentiable in $I$ if

$$\int_{0}^{\delta} \sup\{\|f(t)-f(s)\|_{X} ;\quad t, s\in K,\ |t-s| \leqq h\} \frac{dh}{h}<\infty \tag{2}$$

for any compact interval $K$ contained in $I$. In particular, if $f$ is Hölder continuous, then $F$ is strongly differentiable. The aim of this note is to give an improvement on this result, that is,

THEOREM A. *Let $X$ be a Banach space, $e^{-tA}$, $t\geqq 0$, an analytic semigroup of linear operators in $X$, $I=(a, b)$ a finite open interval $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, and let $\sigma$ be a real number. Assume that $f$ belongs to $B_{p,q}^{\sigma}(I ; X)_{\mathrm{loc}}\cap L_{1}(I ; X)$. Then the function $F$ defined by* (1) *belongs to $B_{p,q}^{\sigma+1}(I ; X)_{\mathrm{loc}}$.*

Here, for a function space $\mathcal{F}(I ; X)$ we denote by $\mathcal{F}(I ; X)_{\mathrm{loc}}$ the space of functions $f$ which have property that $\phi f\in\mathcal{F}(I ; X)$ for any $\phi\in C_{0}^{\infty}(I)$, and by $B_{p,q}^{\sigma}$ we denote Besov spaces (Lipschitz spaces) whose definition will be given in §1.

To treat the inhomogeneous equation

$$\frac{du}{dt}(t)+Au(t)=f(t), \qquad a<t<b, \tag{3}$$

$$u(a)=x, \tag{4}$$

we shall need the following result;

THEOREM B. *Let $X$, $e^{-tA}$, $p$, $q$ and $\sigma$ be as in Theorem* A. *Assume that* $f\in B^{\sigma}_{p,q}(I;X)_{\text{loc}}\cap L_1(I;X)$, *and that one of the conditions*

$$\text{(a)}\quad \sigma>\frac{1}{p},\qquad \text{(b)}\quad \sigma=\frac{1}{p},\ q=1$$

*holds, and define $F$ by* (1). *Then $F(t)\in D(A)$ for $t$ in $I$, $F$ is strongly differentiable in $I$, and*

(5) $$\frac{dF}{dt}(t)+AF(t)=f(t),\qquad a<t<b,$$

*holds.*

We shall use the following notations;

In this note by $X$ we always denote a Banach space.

$D(A)$: the domain of an operator $A$.

$L_p(\Omega)$: the $L_p$ space with respect to the Lebesgue measure $dx$.

$L^*_p(\Omega)$: the $L_p$ space with respect to the measure $|x|^{-n}dx$.

$L_p(\Omega;X)$: the space of strongly measurable $X$-valued functions defined in $\Omega$ such that $\|f(t)\|_X\in L_p(\Omega)$.

$L^*_p(\Omega;X)$: the space analogous to $L_p(\Omega;X)$ except that $L_p$ replaced by $L^*_p$.

Here $\Omega$ is an open set in $\boldsymbol{R}^n$.

**1. Definition of Besov spaces.** Let $\Omega$ be an open set in $\boldsymbol{R}^n$, $m$ an integer, and let $1\leqq p,\ q\leqq\infty$. As usual, for $m\geqq 0$ $W^m_p(\Omega;X)$ is the space of all functions $f$ whose all derivatives of order up to $m$ belong to $L_p(\Omega;X)$, and its norm is defined by

(1) $$\|f\|_{W^m_p(\Omega;X)}=\sum_{|\alpha|\leqq m}\|\partial^\alpha_x f\|_{L_p(\Omega;X)}.$$

For $m<0$ $W^m_p(\Omega;X)$ consists of all distributions $f$ of the form

(2) $$f(x)=\sum_{|\alpha|\leqq -m}\partial^\alpha_x f_\alpha(x),\qquad f_\alpha\in L_p(\Omega;X),$$

and its norm is defined by

(3) $$\|f\|_{W^m_p(\Omega;X)}=\inf\sum_{|\alpha|\leqq -m}\|f_\alpha\|_{L_p(\Omega;X)},$$

where the infimum is taken over all expressions of the form (2).

For a real number $\sigma$ the Besov space $B^\sigma_{p,q}(\Omega;X)$ is defined as follows. Let $\sigma=m+\theta$, $m$ an integer, $0<\theta\leqq 1$.

When $m\geqq 0$, $B^\sigma_{p,q}(\Omega;X)$ consists of all functions $f$ in $W^m_p(\Omega;X)$ for which the seminorm

(4) $$|f|_{B^{\sigma}_{p,q}(\Omega;X)}=\sum_{|\alpha|=m}\||y|^{-\theta}\{\|\Delta^{k}_{y}\partial^{\alpha}_{x}f(x)\|_{L_p(\Omega_{k,y};X)}\}\|_{L^{*}_{p}(R^n)}$$

is finite, where $k=1$ or 2 according as $0<\theta<1$ or $\theta=1$,

(5) $$\Omega_{k,y}=\bigcap_{j=0}^{k}(\Omega-jy)=\{x;\ \ x+jy\in\Omega\ \text{for}\ j=0,\cdots,k\},$$

$\Delta^{k}_{y}$ is the difference operator with increment $y$ of $k$-th order, e.g.

$$\Delta_{y}f(x)=f(x+y)-f(x),\qquad \Delta^{2}_{y}f(x)=f(x+2y)-2f(x+y)+f(x),$$

and its norm is defined by

(6) $$\|f\|_{B^{\sigma}_{p,q}(\Omega;X)}=\|f\|_{W^{m}_{p}(\Omega;X)}+|f|_{B^{\sigma}_{p,q}(\Omega;X)}.$$

For $m<0$ $B^{\sigma}_{p,q}(\Omega;X)$ is the space of all distributions $f$ of the form (2) with $f_{\alpha}\in B^{\theta}_{p,q}(\Omega;X)$, and its norm is defined by

(7) $$\|f\|_{B^{\sigma}_{p,q}(\Omega;X)}=\inf\sum_{|\alpha|\leqq -m}\|f_{\alpha}\|_{B^{\theta}_{p,q}(\Omega;X)},$$

where the infimum is taken over all expressions of the form (2).

It is known that Sobolev spaces and Besov spaces are Banach spaces.

**2.** Here we describe one of our main tools, i.e., *integral representation by means of regularization.* For simplicity's sake we consider only the formulas for distributions on an interval $I$.

In what follows we denote by $t$, $s$ and $r$ real variables, and by $\mathcal{K}_0(I)$ the set of all $C^{\infty}$-functions $\phi(t,s)$ for which there exist a compact set $K_0$ and $0<c_0\leqq\infty$ satisfying the conditions that

(i) the function $\phi(t,s)$ of $s$ vanishes outside $K_0$ for every $t$,

(ii) $\bigcup$\{the support of the function $\phi(t,(t-s)/\tau)$ of $s$; $t\in K$, $0<\tau\leqq c_0$\} is contained in a compact set in $I$ for any compact set $K$ in $I$.

For positive integer $m$ $\mathcal{K}_m(I)$ is the set of $m$-th derivatives $\partial^{m}_{s}\phi(t,s)$ of functions in $\mathcal{K}_0(I)$. Here $\partial_s=\partial/\partial s$.

LEMMA 1. *Let $I=(a,b)$ be a finite open interval, and $\phi\in\mathcal{K}_0(I)$. Put $\int\phi(t,s)ds=\eta(t)$, and $\psi(t,s)=\partial_s\{s\phi(t,s)\}$. Then for any $f$ in $\mathcal{D}'(I;X)$ and a positive number $c\leqq c_0$*

(1) $$\eta(t)f(t)=\int_{0}^{c}\Big\langle\frac{1}{\tau}\psi\Big(t,\frac{t-s}{\tau}\Big),f(s)\Big\rangle_{s}\frac{d\tau}{\tau}+\Big\langle\frac{1}{c}\phi\Big(t,\frac{t-s}{c}\Big),f(s)\Big\rangle_{s},$$

*holds. Here the integral with respect to $\tau$ converges in the topology of $\mathcal{D}'$, and $\langle\ ,\ \rangle$ means the duality on $\mathcal{D}(\boldsymbol{R})\times\mathcal{D}'(\boldsymbol{R};X)$.*

PROOF. The lemma is a direct consequence of Lemma 2.1 and Lemma 3.1 in [**4**].

Now, let $I$ be a finite open interval, and choose a $C_0^\infty$-function $\phi_0$ so that its support is contained in $I$, and $\int\phi_0(t)dt=1$. For a positive integer $m$ put

(2) $$e_m(t, s)=\sum_{k=0}^{m-1}\partial_s^k\left\{\frac{1}{k!}s^k\phi_0(t-s)\right\}.$$

Then $\int e_m(t, s)ds=1$, and

(3) $$\partial_s\{se_m(t, s)\}=M(t, s)=\frac{m}{m!}\partial_s^m\{s^m\phi_0(t-s)\},$$

so we have the *first integral representation formula*:

(4) $$f(t)=\int_0^c\left\langle\frac{1}{\tau}M\left(t, \frac{t-s}{\tau}\right), f(s)\right\rangle_s\frac{d\tau}{\tau}+f_c(t),$$

where $f_c(t)=(1/c)\langle e_m(t, (t-s)/c), f(s)\rangle_s$ and $c$ is an arbitrary positive number less than 1. Substituting (4) into the first term in the right-hand side of (4), we have

$$f(t)=\int_0^c\frac{d\tau}{\tau}\left\{\int_0^c\left\langle\frac{1}{\tau}M\left(t, \frac{t-s}{\tau}\right), \left\langle\frac{1}{\tau'}M\left(s, \frac{s-r}{\tau'}\right), f(r)\right\rangle_r\right\rangle_s\frac{d\tau'}{\tau'}\right\}$$
$$+\int_0^c\left\langle\frac{1}{\tau}M\left(t, \frac{t-s}{\tau}\right), f_c(s)\right\rangle_s\frac{d\tau}{\tau}+f_c(t).$$

The identity (4) with $f$ replaced by $f_c$ also gives

$$\int_0^c\left\langle\frac{1}{\tau}M\left(t, \frac{t-s}{\tau}\right), f_c(s)\right\rangle_s\frac{d\tau}{\tau}+f_c(t)=2f_c(t)-\frac{1}{c}\left\langle e_m\left(t, \frac{t-s}{c}\right), f_c(s)\right\rangle_s$$
$$=\frac{1}{c}\left\langle e_m^*\left(t, \frac{t-s}{c}\right), f(s)\right\rangle_s,$$

where

(5) $$e_m^*(t, s)=2e_m(t, s)-\int e_m(t, r)e_m(t-cr, s-r)dr.$$

It is easy to see that $e_m^*(t, s)\in\mathcal{K}_0(I)$. Set

(6) $$\phi(t, s)=\frac{m}{m!}s^m\phi_0(t-s),$$

and let $m=l+h$, where $l$ and $h$ are non-negative integers. Then, $M(t, s)=\partial_s^m\phi(t, s)=\partial_s^l\phi_{0,h}(t, s)$. Here and hereafter we make use of the notation $\phi_{j,k}(t, s)=\partial_t^j\partial_s^k\phi(t, s)$. Hence we have for any $g$ in $\mathcal{D}'(I; X)$

$$\left\langle\frac{1}{\tau}M\left(t, \frac{t-s}{\tau}\right), g(s)\right\rangle_s=\tau^l\left\langle\frac{1}{\tau}\phi_{0,h}\left(t, \frac{t-s}{\tau}\right), \partial_s^l g(s)\right\rangle_s$$
$$=\tau^h\sum_{j=0}^h(-1)^{h-j}\binom{h}{j}\partial_t^j\left\langle\frac{1}{\tau}\phi_{h-j,l}\left(t, \frac{t-s}{\tau}\right), g(s)\right\rangle_s.$$

Therefore we obtain the *second integral representation formula*;

(7) $$f(t)=\int_0^c\left\langle\frac{1}{\tau}\phi_{0,h}\left(t, \frac{t-s}{\tau}\right)u^l(\tau, s)\right\rangle_s\frac{d\tau}{\tau}$$
$$+\sum_{j=0}^{h}\int_0^c\left\langle\frac{1}{\tau}\phi_{0,m+j}\left(t, \frac{t-s}{\tau}\right)u_{jh}(\tau, s)\right\rangle_s\frac{d\tau}{\tau}+\frac{1}{c}\left\langle e_m^*\left(t, \frac{t-s}{c}\right), f(s)\right\rangle_s,$$

(8) $$u^l(\tau, t)=\int_\tau^c\left[\frac{\tau}{\tau'}\right]^l\sum_{k=0}^{l}\binom{l}{k}\tau'^k\left\langle\frac{1}{\tau'}\phi_{k,m+l-k}\left(t, \frac{t-s}{\tau'}\right), f(s)\right\rangle_s\frac{d\tau'}{\tau'},$$

(9) $$u_{jh}(\tau, t)=(-\tau)^{h-j}\binom{h}{j}\int_0^\tau\left[\frac{\tau'}{\tau}\right]^h\left\langle\frac{1}{\tau'}\phi_{h-j,l}\left(t, \frac{t-s}{\tau'}\right), f(s)\right\rangle_s\frac{d\tau'}{\tau'}.$$

(For detail see pp. 329-350 in [**4**] or pp. 219-231 in [**5**].)

REMARK 1. The functions $e_m$ and $\phi$ given above belong not only to $\mathcal{K}_0(I)$, but also to $\mathcal{K}_0(J)$ for any open interval $J$ containing $I$.

**3. Characterization of Besov spaces.** The basic lemma in our study of Besov spaces is the following

LEMMA 2. *Let $X$ be a Banach space, $I$ an open interval, $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, $\sigma$ a real number, $\phi\in\mathcal{K}_0(I)$, and $0<c\leqq c_0$.*

(I) *If $m$ and $l$ are integers, then the operator*

(1) $$f\longrightarrow\frac{1}{c}\left\langle\phi\left(t, \frac{t-s}{c}\right), f(s)\right\rangle_s$$

*is bounded from $W_p^m(I; X)$ into $W_p^l(I; X)$.*

(II) *Let $m$ be a non-negative integer with $m>\sigma$, and assume that $\phi\in\mathcal{K}_m(I)$. Then the operator*

(2) $$f\longrightarrow\tau^{-\sigma}\left\langle\frac{1}{\tau}\phi\left(t, \frac{t-s}{\tau}\right), f(s)\right\rangle_s$$

*is bounded from $B_{p,q}^\sigma(I; X)$ into $L_q^*((0, c); L_p(I; X))$.*

(III) *Let $l$ be a non-negative integer with $-l<\sigma$, and assume that $\phi\in\mathcal{K}_l(I)$. Then the operator*

(3) $$u\longrightarrow\int_0^c\frac{d\tau}{\tau}\int\frac{1}{\tau}\phi\left(t, \frac{t-s}{\tau}\right)\tau^\sigma u(\tau, s)ds$$

*is bounded from $L_q^*((0, c); L_p(I; X))$ into $B_{p,q}^\sigma(I; X)$.*

PROOF. See [**4**] pp. 350-356.

This lemma and the integral representation (2.7) immediately give the following *characterization theorem of Besov spaces*:

THEOREM 1. *Let $X$, $I$, $p$, $q$ and $\sigma$ be as in Lemma 2, $0<c<c_0$, and let $m$ be a non-negative integer such that $\sigma<m$. Then an $X$-valued distribution $f$ on $I$*

*belongs to* $B^{\sigma}_{p,q}(I;X)$ *if and only if*

(i) $\left\langle \phi\left(t, \frac{t-s}{c}\right), f(s)\right\rangle_s \in L_p(I;X)$ *for any* $\phi \in \mathcal{K}_0(I)$, *and*

(ii) $\tau^{-\sigma}\left\langle \frac{1}{\tau}\phi\left(t, \frac{t-s}{\tau}\right), f(s)\right\rangle_s \in L^*_q((0, c); L_p(I;X))$ *for any* $\phi \in \mathcal{K}_m(I)$.

REMARK 2. Let $I$ be a finite open interval and let $h$ and $l$ be non-negative integers such that $-h<\sigma<l$. Define $\phi(t, s)$ by (2.6), $e^*_m(t, s)$ by (2.5) with $m=h+l$, and set

$$\phi_k(t, s) = \partial^k_t \partial^{l-k}_s e^*_m(t, s), \qquad k=0, \cdots, l.$$

Then a distribution $f$ belongs to $B^{\sigma}_{p,q}(I;X)$ if the conditions:

$$\tau^{-\sigma}\left\langle \frac{1}{\tau}\phi_{k,m+l-k}\left(t, \frac{t-s}{\tau}\right), f(s)\right\rangle_s \in L^*_q((0, c); L_p(I;X)) \quad \text{for } k=0, \cdots, l,$$

$$\tau^{-\sigma}\left\langle \frac{1}{\tau}\phi_{h-j,l}\left(t, \frac{t-s}{\tau}\right), f(s)\right\rangle_s \in L^*_q((0, c); L_p(I;X)) \quad \text{for } j=0, \cdots, h,$$

$$\left\langle \phi_k\left(t, \frac{t-s}{c}\right), f(s)\right\rangle_s \in L_p(I;X) \quad \text{for } k=0, \cdots, l,$$

are satisfied and the norm of $f$ in $B^{\sigma}_{p,q}(I;X)$ is equivalent with the sum of the corresponding norms of the above functions.

Note that $\phi$ belongs not only to $\mathcal{K}_0(I)$ but also to $\mathcal{K}_0(J)$ for any open interval $J$ containing $I$.

Many properties of Besov spaces can be shown straightforward by making use of the characterization theorem of Besov spaces. Here we shall describe some of them which will be used later on.

THEOREM 2 (Approximation theorem). *Let* $X$, $p$, $q$ *and* $\sigma$ *be as in Lemma* 1, $I$ *a finite open interval,* $h$ *and* $l$ *non-negative integers such that* $-h<\sigma<l$, *and assume that* $q$ *is finite. Put* $f_n(t)=\langle ne^*_m(t, n(t-s)), f(s)\rangle_s$, $m=k+l$, *where* $e^*_m(t, s)$ *is given by* (2.5). *Then* $f_n \to f$ *in* $B^{\sigma}_{p,q}(I;X)$ *as* $n\to\infty$.

Note that $f_n$ in Theorem 2 belongs to $C^\infty([a, b];X) \cap B^{\sigma}_{p,q}(I;X)$.

THEOREM 3 (Imbedding theorem). *Let* $X$ *and* $I=(a, b)$ *be as in Theorem* 2.

(I) *Let* $\sigma$ *and* $\tau$ *be real numbers,* $1\leqq p_1 \leqq p_2 \leqq \infty$, $1\leqq q_1\leqq\infty$, *and* $1\leqq q_2\leqq\infty$. *Assume that*

(a) $\sigma-\frac{1}{p_1}>\tau-\frac{1}{p_2}$, *or* (b) $\sigma-\frac{1}{p_1}=\tau-\frac{1}{p_2}$ *and* $q_1\leqq q_2$

*holds. Then* $B^{\sigma}_{p_1,q_1}(I;X)\subset B^{\tau}_{p_2,q_2}(I;X)$ *and the inclusion operator is continuous.*

(II) $B^m_{p,1}(I;X) \subset W^m_p(I;X) \subset B^m_{p,\infty}(I;X)$ *for any integer* $m$.

*In particular,* $B^m_{\infty,1}(I;X)\subset C^m([a, b];X)$ *for any non-negative integer* $m$.

**4.** Theorem A is an easy consequence of the following

THEOREM 4. *Let* $-A$ *be the generator of an analytic semigroup* $e^{-tA}$ *of linear operators on* $X$, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, $\sigma \geqq 0$, *and* $-\infty < a < a_1 < b < \infty$. *Put* $I=(a, b)$ *and* $I_1=(a_1, b)$.

*Then for any* $f \in B_{p,q}^{\sigma}(I ; X) \cap L_1(I ; X)$

$$F(t)=\int_a^t e^{-(t-s)A} f(s) ds \in B_{p,q}^{\sigma+1}(I_1 ; X), \tag{1}$$

*and there is a constant* $C$ *such that*

$$\|F\|_{B_{p,q}^{\sigma+1}(I_1 ; X)} \leqq C\|f\|_{B_{p,q}^{\sigma}(I ; X)}+C\|f\|_{L_1(I ; X)} . \tag{2}$$

To prove Theorem 4 we need the following three lemmas:

LEMMA 3. *Let* $I, I_1$ *and* $p$ *be as in Theorem* 4, *and assume that* $1 \leqq p_1 \leqq p$, $0<s'<s \leqq b-a$, *and* $\phi \in \mathcal{K}_0(I_1)$. *If* $f \in L_{p_1}(I ; X)$, *then*

$$\left\|\int_a^{b-s} \frac{1}{\tau} \phi\left(t-s', \frac{t-s-r}{\tau}\right) f(r) dr\right\|_{L_p(I ; X)} \leqq C \tau^{-1/p_1+1/p}\|f\|_{L_{p_1}(I ; X)} . \tag{3}$$

PROOF. Setting $1/q_2=1/p_1-1/p$, $1/q_3=1-1/p_1$, $1/\rho=1/p+1q_3$, $K(t, r)=(1/\tau) \phi(t-s',(t-s-r)/\tau)$, by Hölder's inequality we get

$$\begin{aligned} & \left\|\int_a^b K(t, r) f(r) dr\right\|_X \tag{4} \\ & \leqq\left\{\int_a^b|K(t, r)|^\rho\|f(r)\|_X^{p_1} dr\right\}^{1/p}\|f\|_{L_{p_1}(I ; X)}^{p_1/q_2}\left\{\int_a^b|K(t, r)|^\rho dr\right\}^{1/q_3}, \end{aligned}$$

which gives (3). In fact, since the function $\phi(t-s', s)$ of $s$ vanishes outside some finite closed interval $[-\gamma, \gamma]$, we have

$$\int\left|\frac{1}{\tau} \phi\left(t-s', \frac{t-s-r}{\tau}\right)\right|^\rho dr \leqq C \int_{|t-s-r| \leqq \gamma\tau} \tau^{-\rho} dr=2 C \gamma \tau^{1-\rho}, \tag{5}$$

$$\int\left|\frac{1}{\tau} \phi\left(t-s', \frac{t-s-r}{\tau}\right)\right|^\rho dt \leqq C \int_{|t-s-r| \leqq \gamma\tau} \tau^{-\rho} dt=2 C \gamma \tau^{1-\rho} . \tag{6}$$

LEMMA 4. *Let* $I, I_1$, *and* $\phi$ *be as in Lemma* 3. *Then for any* $f \in L_1(I ; X)$, *any* $t \in I_1$ *and for any* $0 \leqq s \leqq a_1-a$

$$\begin{aligned} & \int_a^{b-s} \frac{1}{\tau} \phi\left(t, \frac{t-s-r}{\tau}\right) f(r) dr \tag{7} \\ & \quad=\sum_{j=0}^2 \frac{s^j}{j!} u_j(\tau, t-s)+\frac{s^3}{2} \int_0^1 \eta^2 d \eta \int_a^b \frac{1}{\tau} \phi_{3,0}\left(t-\eta s, \frac{t-s-r}{\tau}\right) f(r) dr, \end{aligned}$$

*where*

(8) $$u_j(\tau, t)=\int\frac{1}{\tau}\phi_{j,0}\Big(t, \frac{t-r}{\tau}\Big)f(r)dr.$$

PROOF. The identity (7) follows from Taylor's formula

(9) $$\phi(t, s')=\sum_{j=0}^{2}\frac{s^j}{j!}\phi_{j,0}(t-s, s')+\frac{s^3}{2}\int_0^1\eta^2\phi_{3,0}(t-\eta s, s')d\eta$$

and the fact that for any $t\in I_1$ the support of $\phi(t, (t-s-r)/\tau)$ with respect to $r$ is contained in $(a_1-s, b-s)\subset(a, b-s)$.

LEMMA 5. *Let $I=(a, b)$ and $A$ be as in Theorem 4. Assume that $f\in C^1([a, b]; X)$ and define $F$ by (1). Then $F$ belongs to $C^1([a, b]; X)$, $F(t)$ belongs to $D(A)$ for any point $t$ in $I$, and*

(10) $$\frac{dF}{dt}(t)=f(t)-AF(t)=e^{-(t-a)A}f(a)+\int_a^t e^{-(t-s)A}f'(s)ds$$

*holds, where $f'$ is the derivative of $f$.*

PROOF. Let $\delta$ be a positive number and put

$$F_\delta(t)=\int_a^{t-\delta}e^{-(t-s)A}f(s)ds.$$

Then we have

$$AF_\delta(t)=\int_a^{t-\delta}Ae^{-(t-s)A}f(s)ds=\int_a^{t-\delta}\Big(\frac{\partial}{\partial s}e^{-(t-s)A}\Big)f(s)ds$$
$$=e^{-\delta A}f(t-\delta)-e^{-(t-a)A}f(a)-\int_a^{t-\delta}e^{-(t-s)A}f'(s)ds.$$

Since $A$ is closed and $e^{-(t-s)A}f'(s)$ is uniformly continuous, by letting $\delta\to 0$ we get $F(t)\in D(A)$ and

$$AF(t)=f(t)-e^{-(t-a)A}f(a)-\int_a^t e^{-(t-s)A}f'(s)ds.$$

Also, since $e^{-(t-s)A}f(s)$ is continuously differentiable at any point in $(a, t-\delta)$, we get

$$\frac{d}{dt}F_\delta(t)=e^{-\delta A}f(t-\delta)-\int_a^{t-\delta}Ae^{-(t-s)A}f(s)ds=e^{-\delta A}f(t-\delta)-AF_\delta(t).$$

Letting $\delta\to 0$, this gives (10).

**5.** PROOF OF THEOREM 4. It suffices to consider only the case where $\sigma<2+1/p$. In fact, if $\sigma\geqq 2+1/p$, then by Theorem 3 we have $B_{p,q}^\sigma(I; X)\subset B_{\infty,1}^1(I; X)\subset C^1([a, b]; X)$, and so by (4.10) we can reduce the case $\sigma\geqq 2+1/p$ to this case.

Now assume that $0\leqq\sigma<2+1/p$. With the aid of Theorem 1, the theorem is established if we prove the following inequalities;

(i) $\left\|\int \frac{1}{c}\phi\left(t, \frac{t-s}{c}\right)F(s)ds\right\|_{L_p(I_1;X)} \leqq C\|f\|_{L_1(I;X)}$ for any $\phi\in\mathcal{K}_0(I_1)$,

(ii) $\left\|\tau^{-\sigma-1}\int \frac{1}{\tau}\phi\left(t, \frac{t-s}{\tau}\right)F(s)ds\right\|_{L_q^*((0,c);L_p(I_1;X))} \leqq C\|f\|_{B_{p,q}^{\sigma}(I;X)}+C\|f\|_{L_1(I;X)}$

for any $\phi\in\mathcal{K}_8(I_1)\cap\mathcal{K}_8(I)$ (see Theorem 1 and Remark 2).

The part (i) is a direct consequence of Lemma 3, since

$$\int \frac{1}{c}\phi\left(t, \frac{t-s}{c}\right)F(s)ds = \int_0^{b-a} e^{-sA}ds\int_a^{b-s}\frac{1}{c}\phi\left(t, \frac{t-s-r}{c}\right)f(r)dr.$$

Let us next consider (ii). Let $0<\tau\leqq c$. In view of Lemma 3 we may assume that $0<c<a_1-a$. We can write as $\phi(t, s)=\partial_s^5\psi(t, s)$ with $\psi\in\mathcal{K}_3(I_1)\cap\mathcal{K}_3(I)$, since $\phi\in\mathcal{K}_8(I_1)\cap\mathcal{K}_8(I)$, so by integration by parts we get

$$\text{(1)}\qquad \int_\tau^{b-r}\frac{1}{\tau}\phi\left(t, \frac{t-s-r}{\tau}\right)e^{-sA}ds = \sum_{k=0}^{4}\psi_{0,k}\left(t, \frac{t-\tau-r}{\tau}\right)(-\tau A)^{4-k}e^{-\tau A}$$
$$-\int_\tau^{b-r}\tau^4\psi\left(t, \frac{t-s-r}{\tau}\right)A^5e^{-sA}ds,$$

where $\psi_{j,k}(t, s)=\partial_t^j\partial_s^k\psi(t, s)$. Substituting (1) into the second part of the following integral with respect to $s$, we get

$$\begin{aligned}
U(\tau, t) &= \int \frac{1}{\tau}\phi\left(t, \frac{t-s}{\tau}\right)F(s)ds\\
&= \left\{\int_0^\tau+\int_\tau^{b-a}\right\}e^{-sA}ds\int_a^{b-s}\frac{1}{\tau}\phi\left(t, \frac{t-s-r}{\tau}\right)f(r)dr\\
&= \int_0^\tau e^{-sA}ds\int_a^{b-s}\frac{1}{\tau}\psi_{0,5}\left(t, \frac{t-s-r}{\tau}\right)f(r)dr\\
&\quad+\sum_{k=0}^{4}\tau(-\tau A)^{4-k}e^{-\tau A}\int_a^{b-\tau}\frac{1}{\tau}\psi_{0,k}\left(t, \frac{t-\tau-r}{\tau}\right)f(r)dr\\
&\quad-\tau^5\int_\tau^{b-a}A^5e^{-sA}ds\int_a^{b-s}\frac{1}{\tau}\psi\left(t, \frac{t-s-r}{\tau}\right)f(r)dr\\
&= U_1(\tau, t)+\sum_{k=0}^{4}V_k(\tau, t)+U_2(\tau, t).
\end{aligned}$$

Putting

$$\text{(2)}\qquad v_{jk}(\tau, t)=\int \frac{1}{\tau}\psi_{j,k}\left(t, \frac{t-s}{\tau}\right)f(s)ds,$$

by Lemma 4 we have

$$U_1(\tau, t)=\int_0^\tau e^{-sA}ds\left\{\sum_{j=0}^{2}\frac{s^j}{j!}v_{j5}(\tau, t-s)\right.$$
$$\left.+\frac{s^3}{2}\int_0^1\eta^2d\eta\int_a^b\frac{1}{\tau}\psi_{3,5}\left(t-\eta s, \frac{t-s-r}{\tau}\right)f(r)dr\right\},$$

$$V_k(\tau, t) = \tau(-\tau A)^{4-k} e^{-\tau A}\Big\{ \sum_{j=0}^{2} \frac{\tau^j}{j!} v_{jk}(\tau, t-\tau) + \frac{\tau^3}{2}\int_0^1 \eta^2 d\eta \int_a^b \frac{1}{\tau}\phi_{3,k}\Big(t-\eta\tau, \frac{t-\tau-r}{\tau}\Big) f(r)dr \Big\},$$
$$\text{for } k=0, \cdots, 4,$$

$$U_2(\tau, t) = -\tau^5 \int_{a_1-a}^{b-a} A^5 e^{-sA} ds \int_a^{b-s} \frac{1}{\tau}\phi\Big(t, \frac{t-s-r}{\tau}\Big) f(r)dr$$
$$-\tau^5 \int_\tau^{a_1-a} A^5 e^{-sA} ds \Big\{ \sum_{j=0}^{2} \frac{s^j}{j!} v_{j0}(\tau, t-s) + \frac{s^3}{2}\int_0^1 \eta^2 d\eta \int_a^b \frac{1}{\tau}\phi_{3,0}\Big(t-\eta s, \frac{t-s-r}{\tau}\Big) f(r)dr \Big\}.$$

Thus, by Lemma 3, the estimate

$$\|s^j A^j e^{-sA}\| \leqq M_j \quad \text{for } 0<s\leqq b-a,\ j=0, 1, 2, \cdots, \tag{3}$$

and the inequality

$$\|v_{jk}(\tau, \cdot-s)\|_{L_p(I_1; X)} \leqq \|v_{jk}(\tau, \cdot)\|_{L_p(I; X)}, \quad j, k=0, 1, \cdots, \tag{4}$$

we get

$$\|U(\tau, t)\|_{L_p(I_1; X)} \leqq C\tau \sum_{j=0}^{2} \tau^j \sum_{k=0}^{5} \|v_{jk}(\tau, \cdot)\|_{L_p} + C\tau^{3+1/p}\|f\|_{L_1},$$

which, with the aid of Lemma 2, verifies the part (ii).

**6.** To prove Theorem B we need the following theorem:

THEOREM 5. *Let $X$, $A$, $I$, $p$ and $q$ be as in Theorem 4, and assume that "$\sigma>1/p$" or "$\sigma=1/p$ and $q=1$" holds. Suppose that $f\in B^{\sigma}_{p,q}(I; X)$, and define $F(t)$ by (4.1). Then $F$ belongs to $C^1(I; X)\cap C^0(I; D(A))$ and*

$$\frac{dF}{dt}(t) = f(t) - AF(t) \tag{1}$$

*holds for any $t$ in $I$.*

PROOF. Note first that $B^{\sigma}_{p,q}(I; X)\subset B^{0}_{\infty,1}(I; X)\subset C^0([a, b]; X)$ if $\sigma>1/p$ or if $\sigma=1/p$ and $q=1$ (see Theorem 3). Let $f\in B^{\sigma}_{p,q}(I; X)$, $a<a_1<b$, and set $I_1=(a_1, b)$. By Theorem 2 we know that there is a sequence $\{f_n\}$ in $C^1([a, b]; X)\cap B^{\sigma}_{p,q}(I; X)$ such that $f_n\to f$ in $B^{\sigma}_{p,q}(I; X)$. Note that $f_n\to f$ in $L_p(I; X)\subset L_1(I; X)$. If $F_n$ is defined by (4.1) with $f=f_n$, then by Theorem 4 we have $F\in B^{\sigma+1}_{p,q}(I_1; X)\subset C^1([a_1, b]; X)$ and $F_n\to F$ in $B^{\sigma+1}_{p,q}(I_1; X)$ as $n\to\infty$, since

$$\|F_n-F\|_{B^{\sigma+1}_{p,q}(I_1;X)} \leqq C\|f_n-f\|_{B^{\sigma}_{p,q}(I;X)}+C\|f_n-f\|_{L_1(I;X)} \to 0.$$

On the other hand it follows from Lemma 5 that $F_n(t) \in D(A)$ and the formula (1) with $f$ replaced by $f_n$ holds, and so $AF_n=(f_n-f)-(F_n'-F')+f-F' \to f-F'$ in $B^{\sigma}_{p,q}(I_1;X)$ as $n\to\infty$. Thus we have $F\in C^0([a_1, b]; D(A))$ and $AF(t)=f(t)-F'(t)$. Since $a_1$ can be chosen arbitrarily close to $a$, this completes the proof of the theorem.

**7.** Proof of Theorem A. Let $f\in B^{\sigma}_{p,q}(I;X)_{\mathrm{loc}}\cap L_1(I;X)$, and let $\phi$ be a $C^\infty$-function with support contained in $I_1=[a_1, b_1]\subset I$. Let $a<a_3<a_2<a_1<b_1<b_2<b_3<b$, choose a $C^\infty$-function $\psi_1$ so that $0\leqq\psi_1(t)\leqq 1$, $\psi_1(t)=1$ on $[a_2, b_2]$, $\psi_1(t)=0$ outside $(a_3, b_3)$, and set $\psi_0(t)=1-\psi_1(t)$. Then

$$(1)\qquad F(t)=\int_a^t e^{-(t-s)A}f(s)ds=\int_a^t e^{-(t-s)A}\psi_0(s)f(s)ds+\int_a^t e^{-(t-s)A}\psi_1(s)f(s)ds$$
$$=F_0(t)+F_1(t).$$

Since $\psi_1 f\in B^{\sigma}_{p,q}(I;X)\cap L_1(I;X)$, by Theorem 4 we have $F_1\in B^{\sigma+1}_{p,q}(I_1;X)$, and hence $\phi F_1\in B^{\sigma+1}_{p,q}(I;X)$. On the other hand the function $\phi(t)e^{-(t-s)A}\psi_0(s)f(s)$ is $C^\infty$ with respect to $t$ for any fixed point $s$ and $X$-norm of its all derivatives are majorized by a constant multiple of $\|f(s)\|_X$. Hence

$$(2)\qquad \phi(t)F_0(t)=\int_a^{a_2}\phi(t)e^{-(t-s)A}\psi_0(s)f(s)ds\in C_0^\infty(I;X),$$

$$(3)\qquad \frac{d}{dt}\phi(t)F_0(t)=-\int_a^{a_2}\phi(t)Ae^{-(t-s)A}\psi_0(s)f(s)ds+\phi'(t)F_0(t).$$

Consequently we have $\phi F=\phi F_0+\phi F_1\in B^{\sigma+1}_{p,q}(I;X)$.

**8.** Proof of Theorem B. Let $f$, $\phi$, $a_1$, $b_1$, $a_2$, $b_2$, $a_3$, $b_3$, $\psi_1$, $\psi_0$, $F_1$ and $F_0$ be as in §7. Assume that "$\sigma>1/p$" or "$\sigma=1/p$, $q=1$". Then by Theorem 4 and Theorem 5 we have $F_1(t)\in D(A)$ for $t\in I$, $AF_1\in B^{\sigma}_{p,q}(I_1;X)$ and

$$(1)\qquad F_1'(t)+AF_1(t)=\psi_1(t)f(t).$$

Since it follows from Theorem 3 that $f\in C^0([a, b]; X)$, the function $A\phi(t)e^{-(t-s)A}\psi_0(s)f(s)$ is strongly continuous with respect to $s$ in $I$ and an $X$-valued integrable function. But $A$ is closed, we get $\phi(t)F_0(t)\in D(A)$, and

$$(2)\qquad A\phi(t)F_0(t)=\int_a^{a_2}A\phi(t)e^{-(t-s)A}\psi_0(s)f(s)ds.$$

Finally, combining (7.3), (1) and (2), we have

$$\begin{aligned}\frac{d}{dt}(\phi F)(t) &= \phi(t)F_1'(t)+\phi'(t)F_1(t)+\frac{d}{dt}(\phi F_0)(t)\\ &= -\phi(t)AF_1(t)+\phi(t)\phi_1(t)f(t)-A\phi(t)F_0(t)+\phi'(t)F(t)\\ &= -A\phi(t)F(t)+\phi(t)f(t)+\phi'(t)F(t).\end{aligned}$$

This implies that $F'(t)+AF(t)=f(t)$ for $t\in I$, since we can choose $\phi\in C_0^\infty(I)$ so that $\phi$ is identically equal to 1 near any fixed point $t$ in $I$. This completes the proof of Theorem B.

**9.** We shall finish this note by giving a lemma which shows our result include an improvement on that of Crandall-Pazy [2].

LEMMA 6. *Let $f(t)$ be a strongly measurable, essentially bounded $X$-valued function defined in an open interval $I$ such that*

$$\int_0^\delta \text{ess.}\sup\{\|f(t)-f(s)\|_X\,;\ t,\ s\in I,\ |t-s|\leqq h\}\frac{dh}{h}<\infty.$$

*Then $f$ belongs to $B^0_{\infty,1}(I\,;X)$.*

PROOF. Let $\phi\in\mathcal{K}_2(I)$. Then we may assume that for some $\gamma>0$ $\phi(t,s)=0$ when $|s|>\gamma$. Therefore, setting

$$\omega(h)=\text{ess.}\sup\{\|f(t)-f(s)\|_X\,;\ t,\ s\in I,\ |t-s|\leqq h\},$$

we have

$$\begin{aligned}\Big\|\int\frac{1}{\tau}\phi\Big(t,\frac{t-s}{\tau}\Big)f(s)ds\Big\|_X &= \Big\|\int\frac{1}{\tau}\phi\Big(t,\frac{t-s}{\tau}\Big)\{f(s)-f(t)\}ds\Big\|_X\\ &\leqq \int|\phi(t,s)|ds\,\omega(\gamma\tau)\leqq C\omega(\gamma\tau)\in L_1^*((0,\delta/\gamma)).\end{aligned}$$

This together with Theorem 1 shows that $f$ belongs to $B^0_{\infty,1}(I\,;X)$.

EXAMPLE. Let $\alpha$ be a positive number and put

$$f_\alpha(t)=|(\log|t|)|^\alpha\psi(t)\quad\text{when } t\neq 0,\ \text{and } f_\alpha(0)=0.$$

Then $f_\alpha$ belongs to $B^{1/p}_{p,1}(\boldsymbol{R})$ for $1\leqq p\leqq\infty$. Here $\psi$ is a $C^\infty$-function which satisfies the conditions that

$$\begin{aligned}&\psi(t)=1\quad\text{when } |t|\leqq 1/4,\\ &\psi(t)=0\quad\text{when } |t|\geqq 1/2,\\ &0\leqq\psi(t)\leqq 1.\end{aligned}$$

It is obvious that $f_\alpha$ belongs to $L_p$.

Consider the case where $1<p\leqq\infty$. Let $\phi\in C_0^\infty(\boldsymbol{R})$. Then,

$$U(\tau, t)=\int\frac{1}{\tau}\phi'\Big(\frac{t-s}{\tau}\Big)f_\alpha(s)ds=\int_0^\infty\Big\{\frac{1}{\tau}\phi'\Big(\frac{t-s}{\tau}\Big)+\frac{1}{\tau}\phi'\Big(\frac{t+s}{\tau}\Big)\Big\}f_\alpha(s)ds$$
$$=\int_0^\infty\Big\{\phi\Big(\frac{t-s}{\tau}\Big)-\phi\Big(\frac{t+s}{\tau}\Big)\Big\}f'_\alpha(s)ds=\Big\{\int_0^\tau+\int_\tau^\infty\Big\}\Big\{\phi\Big(\frac{t-s}{\tau}\Big)-\phi\Big(\frac{t+s}{\tau}\Big)\Big\}f'_\alpha(s)ds$$
$$=U_1(\tau, t)+U_2(\tau, t).$$

Let $0<\tau\leqq 1/4$. Since $\phi(t)$ is identically equal to 1 if $0<t\leqq\tau\leqq 1/4$, we have

$$\|tf'_\alpha(t)\|_{L_p([0,\tau])}=\|\alpha|\log t|^{-\alpha-1}\|_{L_p([0,\tau])}\leqq\alpha|\log\tau|^{-\alpha-1}\tau^{1/p}.$$

Hence, in view of the facts that

$$\int_0^\tau\Big|\frac{1}{s}\Big\{\phi\Big(\frac{t-s}{\tau}\Big)-\phi\Big(\frac{t+s}{\tau}\Big)\Big\}\Big|ds=\int_0^\tau\Big|\frac{1}{\tau}\int_{-1}^1\phi'\Big(\frac{t-\eta s}{\tau}\Big)d\eta\Big|ds\leqq 2\|\phi'\|_{L_\infty},$$

$$\int_0^\infty\Big|\frac{1}{s}\Big\{\phi\Big(\frac{t-s}{\tau}\Big)-\phi\Big(\frac{t+s}{\tau}\Big)\Big\}\Big|dt\leqq\int_0^\infty\Big|\int_{-1}^1\phi'\Big(t-\frac{\eta s}{\tau}\Big)d\eta\Big|dt\leqq 2\|\phi'\|_{L_1},$$

we get (cf. Proof of Lemma 3)

$$\|U_1(\tau, t)\|_{L_p}\leqq C'\|tf'_\alpha(t)\|_{L_p([0,\tau])}\leqq C'\alpha|\log\tau|^{-\alpha-1}t^{1/p}.$$

On the other hand, we have

$$\|f'_\alpha(t)\|_{L_p([\tau,\infty))}\leqq C_1+\|\{\alpha t^{-1}|\log t|^{-\alpha-1}\|_{L_p([\tau,1/2])}$$
$$\leqq C_1+C_2|\log\tau|^{-\alpha-1}\|t^{-1}\|_{L_p([\tau,\sqrt{\tau}])}+C_3\|t^{-1}\|_{L_p([\sqrt{\tau},1])}\leqq C|\log\tau|^{-\alpha-1}\tau^{1/p-1},$$

which implies

$$\|U_2(\tau, t)\|_{L_p}\leqq C''\|\tau f'_\alpha(t)\|_{L_p([\tau,\infty))}\leqq C''C|\log\tau|^{-\alpha-1}\tau^{1/p},$$

since

$$\int\Big|\frac{1}{\tau}\Big\{\phi\Big(\frac{t-s}{\tau}\Big)-\phi\Big(\frac{t+s}{\tau}\Big)\Big\}\Big|ds\leqq C'',\qquad\int\Big|\frac{1}{\tau}\Big\{\phi\Big(\frac{t-s}{\tau}\Big)-\phi\Big(\frac{t-s}{\tau}\Big)\Big\}\Big|dt\leqq C''.$$

Therefore we get

$$\tau^{-1/p}\|U(\tau, t)\|_{L_p}\leqq C(C'+C'')|\log\tau|^{-\alpha-1}\in L_1^*([0, 1/4]).$$

Similarly, we get

$$\tau^{-1}\Big\|\int\frac{1}{\tau}\phi''\Big(\frac{t-s}{\tau}\Big)f_\alpha(s)ds\Big\|_{L_1}\leqq C'''|\log\tau|^{-\alpha-1}\in L_1^*([0, 1/4]).$$

Note that the function $f_\alpha$ does not satisfy the condition in Lemma 6 when $\alpha\leqq 1$.

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Tosinobu MURAMATU
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305
Japan