J. Math. Soc. Japan
Vol. 41, No. 2, 1989

An elementary proof of Yoshida's inequality for block designs which admit automorphism groups

By Tsuyoshi ATSUMI

(Received July 9, 1987)
(Revised Dec. 16, 1987)

1. Introduction.

The main purpose of this paper is to give an elementary proof of Yoshida's inequality [**5**]. An incidence structure is a triple $D=(X, B, \mathcal{I})$, where X is a set of points, B is a set of blocks and $\mathcal{I}$ is a relation of incidence between points and blocks. A 2-(v, k, λ) design is an incidence structure $(X, B, \mathcal{I})$ satisfying the following requirements:

(1) $|X|=v$.
(2) Each block is incident with k points.
(3) Any 2 points are incident with λ blocks.

A 2-(v, k, λ) design is often called a block design. Let b be the total number of blocks. Note that each point of X is contained in exactly r blocks. We set $n=r-\lambda$, and we call n the order of the 2-design $(X, B, \mathcal{I})$. These parameters satisfy the following relations:

$$vr=bk, \qquad (v-1)\lambda=r(k-1). \tag{1}$$

The incidence matrix A of a block design $(X, B, \mathcal{I})$ is the $v\times b$ matrix whose rows are indexed by points and whose columns are indexed by blocks, with the entry in row x and column β being 1 if $x\mathcal{I}\beta$ and 0 otherwise. (The notation "$x\mathcal{I}\beta$" means that x is incident with β.) The conditions that $(X, B, \mathcal{I})$ is a block design can be expressed in terms of A:

$$AJ=rJ, \qquad JA=kJ, \tag{2}$$
$$AA^t=nI+\lambda J. \tag{3}$$

(Here, throughout this paper, I is the identity matrix and J the matrix with every entry 1 of appropriate size.) From (3) it follows that if $\lambda<r$, then

$$\det(AA^t)=rkn^{v-1}\neq 0,$$

This research was partially supported by Grant-in-Aid for Scientific Research (No. 62540057), Ministry of Education, Science and Culture.

which gives Fisher's inequality:

$$v \leqq b .$$

A block design satisfying $v=b$ is called symmetric. Suppose $D=(X, B, \mathcal{I})$ is a symmetric 2-(v, k, λ) design. Then the dual structure $\bar{D}=(B, X, \bar{\mathcal{I}})$ is a symmetric 2-(v, k, λ) design, where the incidence relation $\bar{\mathcal{I}}$ is defined by

$$\beta \bar{\mathcal{I}} x \quad \text{if and only if} \quad x \mathcal{I} \beta .$$

An automorphism group G of $(X, B, \mathcal{I})$ is a group satisfying the following: (i) G acts on X, (ii) $\beta g \in B$ for all $g \in G$, $\beta \in B$, (iii) if $x \mathcal{I} \beta$, then $xg \mathcal{I} \beta g$.

Suppose that a finite group G acts on Ω and that P is a normal subgroup of G. Let Ω^P denote the set of points in Ω fixed by P. Then Ω^P is G-invariant. Ω/G denotes the set of orbits of G on Ω.

Yoshida [**5**] proved the following:

THEOREM. *Suppose that $(X, B, \mathcal{I})$ is a block design which admits an automorphism group G. Let p be a prime which does not divide n, the order of the design, and let P be a normal p-subgroup of G. Then the following hold:*

(1) (a) *If $p \nmid r$, then $|X^P/G| \leqq |B^P/G|$.*
(b) *If $p \mid r$, then $|X^P/G| - 1 \leqq |B^P/G|$.*
(c) *If $p \nmid b$, then $|X^P/G| \leqq |B^P/G|$.*
(d) *If the design $(X, B, \mathcal{I})$ is symmetric, then $|X^P/G| = |B^P/G|$.*

(2) (a) $|(X \times B)^P/G| \leqq |(B \times B)^P/G|$.
(b) $|(X \times X)^P/G| - 1 \leqq |(X \times B)^P/G|$.
(c) *If $p \nmid r$ or $p \nmid b$, then*

$$|(X \times X)^P/G| \leqq |(X \times B)^P/G| .$$

(d) *If the design is symmetric, then*

$$|(X \times X)^P/G| = |(X \times B)^P/G| = |(B \times B)^P/G| .$$

Unfortunately, Yoshida's proof of the theorem need a lot of knowledge of category theory and Burnside rings. In this paper we will prove the above theorem by using a proposition of D. G. Higman [**2**] and analyzing the matrix AA^t. In the course of proof of the theorem we will establish the following proposition which is a slight extension of (1.c) of Theorem.

PROPOSITION 1. *Under the same notation as in Theorem, we have the following:*

If there exists a G-orbit on B whose length is relatively prime to p, then $|X^P/G| \leqq |B^P/G|$.

It is immediate that (1.c) of Theorem follows from this proposition.

2. Contractions of matrices.

We will follow the first section of Higman [2]. Let R be a commutative ring with identity, and X, Y, Z be finite non-empty sets. We define $M_R(X, Y)$ to be the set of maps $A: X\times Y\rightarrow R$ and we call A an X by Y matrix over R. If $A\in M_R(X, Y)$ and $B\in M_R(Y, Z)$, then $AB\in M_R(X, Z)$ is defined by

$$AB(x, z)=\sum_{y\in Y}A(x, y)B(y, z) \qquad (x\in X, z\in Z).$$

If $\mathcal{P}, Q$ are partitions of X, Y, respectively, then we say that $A\in M_R(X, Y)$ has property $(\mathcal{P}, Q)$ if for all $S\in\mathcal{P}$ and $T\in Q$,

$$\sum_{t\in T}A(s, t) \text{ is independent of } s\in S.$$

If $A\in M_R(X, Y)$ has property $(\mathcal{P}, Q)$, and if $S\in\mathcal{P}$, $T\in Q$, we set $\delta(A)(S, T)=\sum_{t\in T}A(s, t)$ for some $s\in S$. Higman [2] proved the following:

PROPOSITION 2. *If $A\in M_R(X, Y)$ has property $(\mathcal{P}, Q)$ and $B\in M_R(Y, Z)$ has property $(Q, \mathcal{U})$, then $AB\in M_R(X, Z)$ has property $(\mathcal{P}, \mathcal{U})$ and $\delta(AB)=\delta(A)\delta(B)$.*

We note that Theorem 5 of [3] is the special case of Proposition 2.

If $\mathcal{P}=\{S_1, \cdots, S_l\}$ is a partition of X or Y, define an $l\times l$ matrix $D(\mathcal{P})$ as follows: Let $S_i, S_j\in\mathcal{P}$,

$$D(\mathcal{P})(S_i, S_j)=\begin{cases}|S_i| & \text{if } S_i=S_j,\\ 0 & \text{otherwise.}\end{cases}$$

Both Proposition 4 of [3] and a result in [4, pp. 96-98] are the special cases of the following proposition.

PROPOSITION 3. *Suppose that $\mathcal{P}=\{S_1, \cdots, S_l\}$ and $Q=\{T_1, \cdots, T_m\}$ are partitions of X and Y, respectively. If A and A^t (the transpose of A) have $(\mathcal{P}, Q)$ property and $(Q, \mathcal{P})$ property, respectively, then $D(Q)\delta(A^t)=\delta(A)^tD(\mathcal{P})$.*

PROOF. Essentially our proof is similar to that of Proposition 4 of [3]. Let $T_i\in Q$ and $S_j\in\mathcal{P}$. Then

$$\begin{aligned}D(Q)\delta(A^t)(T_i, S_j)&=\sum_{s\in S_j}|T_i|A^t(t, s), \qquad \text{where } t\in T_i\\ &=\sum_{s\in S_j}|T_i|A(s, t)=\sum_{t\in T_i}\sum_{s\in S_j}A(s, t)\\ &=\sum_{t\in T_i}A(s, t)|S_j|=\delta(A)^tD(\mathcal{P})(T_i, S_j).\end{aligned}$$

This completes the proof of Proposition 3.

3. Proof of Theorem.

To prove Theorem we need the following lemmas.

LEMMA 1 ([5]). *Let $(X, B, \mathcal{I})$ be a block design with parameters (v, b, r, k, λ) and let p be a prime such that $p \nmid n$. Then $p \mid r$ implies $p \nmid v$. Furthermore. $p \nmid r$ implies $p \nmid k$.*

LEMMA 2. *Assume that a finite group G acts on sets X, Y and let $x \in X$, $y \in Y$. Then the following hold:*

$$|X/G_y| = |(X \times yG)/G|, \qquad |Y/G_x| = |(xG \times Y)/G|.$$

PROOF. We will establish a 1-1 correspondence between X/G_y and $(X \times yG)/G$. If Δ is a G-orbit on $X \times yG$. Put $\Delta(y) = \{x \mid (x, y) \in \Delta\}$. Then $\Delta(y)$ is an element of X/G_y, and this establishes the correspondence. Indeed if Γ' $(=xG_y)$ is an element of X/G_y, then $\Gamma = (x, y)G$ is the unique element of $(X \times yG)/G$ such that $\Gamma(y) = \Gamma'$. Similarly we can prove that $|Y/G_x| = |(xG \times Y)/G|$. The proof of this lemma is completed.

We will prove Theorem. Since P is a normal subgroup of G, X^P and B^P are G-invariant. Hence we see that

$$X^P = x_1 G \cup x_2 G \cup \cdots \cup x_m G, \qquad B^P = \beta_1 G \cup \beta_2 G \cup \cdots \cup \beta_l G,$$

where $x_i G$ and $\beta_j G$ are the G-orbits of x_i and β_j, respectively. Since $X - X^P$ and $B - B^P$ are G-invariant, we have the following:

$$X - X^P = x_{m+1} G \cup x_{m+2} G \cup \cdots \cup x_{m+m'} G \qquad (x_{m+i} \notin X^P),$$
$$B - B^P = \beta_{l+1} G \cup \beta_{l+2} G \cup \cdots \cup \beta_{l+l'} G \qquad (\beta_{l+j} \notin B^P).$$

Clearly $x_i G$ $(m+1 \leqq i \leqq m+m')$ is a union of P-orbits and so is $\beta_j G$ $(l+1 \leqq j \leqq l+l')$.

Now we note the following fundamental lemma.

LEMMA 3. *$p \,\big|\, |xP|$ for all $x \in X - X^P$ and $p \,\big|\, |\beta P|$ for all $\beta \in B - B^P$.*

Hence we see that

$$p \,\big|\, |x_i G| \qquad (m+1 \leqq i \leqq m+m'). \tag{4}$$

Also, we get

$$p \,\big|\, |\beta_j G| \qquad (l+1 \leqq j \leqq l+l'). \tag{5}$$

Let A be the incidence matrix of the block design $(X, B, \mathcal{I})$. The following lemma plays an important part in the proof of the theorem.

LEMMA 4. *The number of 1's in every row of the submatrix $A|_{x_i G \times \beta_j G}$ $(1 \leqq i \leqq m,\ l+1 \leqq j \leqq l+l')$ is a multiple of p.*

PROOF. The number of 1's in a row x of the submatrix $A|_{x_iG\times\beta_jG}$ is equal to the number of elements of $\{\beta' | x\mathcal{I}\beta', \beta'\in\beta_jG\}$. Since $x\in X^P$, the set $\{\beta' \mid x\mathcal{I}\beta', \beta'\in\beta_jG\}$ is a union of nontrivial P-orbits. Thus this lemma follows immediately from Lemma 3.

Similarly the following holds:

LEMMA 5. *The number of* 1'*s in every row of the submatrix* $A^t|_{\beta_jG\times x_iG}$ $(1\leqq j\leqq l,\ m+1\leqq i\leqq m+m')$ *is a multiple of* p.

Let $\mathcal{P}=\{x_1G, x_2G, \cdots, x_{m+m'}G\}$ and $Q=\{\beta_1G, \beta_2G, \cdots, \beta_{l+l'}G\}$. Then $\mathcal{P}$ and Q are partitions of X and B, respectively. It is easy to prove the following:

LEMMA 6. *The incidence matrix* A *of the block design* $(X, B, \mathcal{I})$ *has property* $(\mathcal{P}, Q)$. *Also* A^t *has property* $(Q, \mathcal{P})$.

By the above lemma we may apply δ in Proposition 2 to A and A^t. To emphasize the dependence of this δ on the automorphism group G, we write δ_t for δ.

From now on we consider integral matrices as ones with entries in the p-element field F_p. Let $\mathcal{P}_1=\{x_1G, \cdots, x_mG\}$, $\mathcal{P}_2=\{x_{m+1}G, \cdots, x_{m+m'}G\}$, $Q_1=\{\beta_1G, \cdots, \beta_lG\}$ and $Q_2=\{\beta_{l+1}G, \cdots, \beta_{l+l'}G\}$. From Lemma 4, $\delta_G(A)$ has the form

$$\delta_G(A)=\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22}\end{pmatrix}, \tag{6}$$

where A_{11} is a $\mathcal{P}_1$ by Q_1 matrix, and A_{22} is a $\mathcal{P}_2$ by Q_2 matrix. From Lemma 5, $\delta_G(A^t)$ has the form

$$\delta_G(A^t)=\begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22}\end{pmatrix}, \tag{7}$$

where B_{11} is a Q_1 by $\mathcal{P}_1$ matrix, and B_{22} is a Q_2 by $\mathcal{P}_2$ matrix. Put $f_i=|x_iG|$, $g_j=|\beta_jG|$ $(1\leqq i\leqq m+m',\ 1\leqq j\leqq l+l')$. By applying Proposition 2 to (3), we obtain that AA^t has property $(\mathcal{P}, \mathcal{P})$, and

$$\delta_G(A)\delta_G(A^t)=\delta_G(AA^t)\,. \tag{8}$$

Since $f_i=0$ in F_p $(i>m)$, we have that $\delta_G((r-\lambda)I+\lambda J)$ has the form

$$\delta_G((r-\lambda)I+\lambda J)=\begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22}\end{pmatrix}, \tag{9}$$

where

$$C_{11}=\begin{pmatrix} r-\lambda+\lambda f_1, & \lambda f_2, & \cdots, & \lambda f_m \\ \lambda f_1, & r-\lambda+\lambda f_2, & \cdots, & \lambda f_m \\ & \cdots\cdots\cdots & & \\ \lambda f_1, & \cdots\cdots\cdots, & & r-\lambda+\lambda f_m \end{pmatrix},\quad C_{22}=\begin{pmatrix} r-\lambda, & & & 0 \\ & r-\lambda, & & \\ & & \ddots & \\ 0 & & & r-\lambda \end{pmatrix}.$$

From (6), (7), (8) and (9) we have

$$A_{11}B_{11}=C_{11}\,, \tag{10}$$

$$A_{22}B_{22}=C_{22}\,. \tag{11}$$

To compute the rank of $A_{11}B_{11}\,(=C_{11})$, we first subtract the first row of C_{11} from every other row and then we add every other column to the first column. This procedure gives

$$\begin{pmatrix} a, \lambda f_2, \cdots, \lambda f_m & & \\ & r-\lambda & 0 \\ & \ddots & \\ 0 & & r-\lambda \end{pmatrix},$$

where $a=r-\lambda+\lambda(f_1+f_2+\cdots+f_m)$.

Note that in F_p, $a=a+\sum_{m+1}^{m+m'}\lambda f_i=r-\lambda+\lambda v=kr$ by (4) and (1).

We will prove (1. a). Since $p\nmid n$, we have that if $p\nmid r$, then rank $A_{11}B_{11}$ is equal to m by Lemma 1. A_{11} must have rank at least m. Since A_{11} has size $m\times l$, we have

$$m\leqq l, \qquad \text{that is } |X^P/G|\leqq|B^P/G|.$$

We complete a proof of (1. a).

Here we note that

$$\operatorname{rank} A_{11}B_{11}=\begin{cases} m & \text{if } p\nmid r\,, \\ m-1 & \text{if } p\mid r\,. \end{cases} \tag{12}$$

We will prove (1. b). If $p\mid r$, then the rank of $A_{11}B_{11}$ is $m-1$ by (12). Similarly we obtain that $|X^P/G|-1\leqq|B^P/G|$, as claimed in (1. b).

We will prove Proposition 1. If $p\nmid r$, then Proposition 1 follows from (1. a). Thus we may assume that $p\mid r$. In this case we already proved that

$$\operatorname{rank} A_{11}B_{11}=m-1\,, \quad \text{and} \quad m-1\leqq l\,. \tag{13}$$

If $l\geqq m$, then our proof is done. By (13) we may assume that

$$l=m-1\,. \tag{14}$$

Recall that

$$\operatorname{rank} A_{11}B_{11}\leqq\operatorname{rank} B_{11}\,. \tag{15}$$

Since B_{11} has size $l\times m$,

$$\operatorname{rank} B_{11}\leqq l\,. \tag{16}$$

By (13), (14), (15) and (16) we see that

$$B_{11} \text{ has rank } l\,. \tag{17}$$

From (2) and $p\mid r$ it follows that

$$JA^t=rJ=0\,, \tag{18}$$

where 0 is a zero matrix. By applying Proposition 2 to (18) we obtain

$$\delta_G(J)\delta_G(A^t) = 0\,. \tag{19}$$

From Lemma 5, $\delta_G(J)$ has the form

$$\delta(J) = \begin{pmatrix} E_{11} & 0 \\ E_{21} & 0 \end{pmatrix},$$

where $E_{11}=\begin{pmatrix} g_1, g_2, \cdots, g_l \\ g_1, \cdots\cdots, g_l \\ \cdots\cdots \\ g_1, g_2, \cdots, g_l \end{pmatrix}$, and has size $m\times l$. By (7) and (19) we have

$$E_{11}B_{11} = 0\,, \tag{20}$$

where 0 is a zero matrix. If $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_l$ are the row vectors of B_{11}, then (20) gives the following:

$$g_1\boldsymbol{b}_1+g_2\boldsymbol{b}_2+\cdots+g_l\boldsymbol{b}_l = \boldsymbol{0} \quad \text{(zero vector)}.$$

Since B_{11} has rank l, $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_l$ are linearly independent, and so

$$g_1 = g_2 = \cdots = g_l = 0 \quad \text{in } F_p\,. \tag{21}$$

(21) and Lemma 3 contradict the assumption of Proposition 1. Hence $l\neq m-1$. This completes the proof of Proposition 1.

We will prove (1.d). Let $(X, B, \mathcal{I})$ be symmetric. Since $p \nmid n$, it follows from (1) that

$$p \nmid r \quad \text{or} \quad p \nmid v\,. \tag{22}$$

Since the order of dual design $(B, X, \bar{\mathcal{I}})$ is equal to the one of block design $(X, B, \mathcal{I})$, so

$$p \nmid \text{the order of } (B, X, \bar{\mathcal{I}})\,. \tag{23}$$

(22) and (23) show that both the design $(X, B, \mathcal{I})$ and its dual design $(B, X, \bar{\mathcal{I}})$ satisfy the assumption of (1.a) or the one of (1.c). This implies that

$$|X^P/G| = |B^P/G|.$$

The proof of (1.d) is completed.

Now we will prove (2). Since $X^P=x_1G\cup\cdots\cup x_mG$, it is easy to show that

$$(X\times X)^P/G = \bigcup_{i=1}^{m}(x_iG\times X^P)/G\,, \tag{24}$$

$$(X\times B)^P/G = \bigcup_{i=1}^{m}(x_iG\times B^P)/G\,. \tag{25}$$

From Lemma 2 we obtain for all $i\in\{1, 2, \cdots, m\}$

$$|(x_iG\times X^P)/G| = |X^P/G_{x_i}|, \tag{26}$$
$$|(x_iG\times B^P)/G| = |B^P/G_{x_i}|. \tag{27}$$

Since $B^P=\beta_1G\cup\cdots\cup\beta_lG$, we have

$$(X\times B)^P/G = \bigcup_{j=1}^{l}(X^P\times\beta_jG)/G, \tag{28}$$
$$(B\times B)^P/G = \bigcup_{j=1}^{l}(B^P\times\beta_jG)/G. \tag{29}$$

From Lemma 2, it follows that for all $j\in\{1, 2, \cdots, l\}$

$$|(X^P\times\beta_jG)/G| = |X^P/G_{\beta_j}|, \tag{30}$$
$$|(B^P\times\beta_jG)/G| = |B^P/G_{\beta_j}|. \tag{31}$$

Now we will prove (2. c). So we assume that $p\nmid r$ or $p\nmid b$, by using (1. a) or (1. c) we see that

$$|X^P/G_{x_i}| \leqq |B^P/G_{x_i}| \qquad (1\leqq i\leqq m). \tag{32}$$

From (24), (25), (26), (27) and (32) it follows that

$$|(X\times X)^P/G| \leqq |(X\times B)^P/G|.$$

We have proved (2. c).

Next we will prove (2. a). Since $\{\beta_j\}$ is a G_{β_j}-orbit on B of length 1, it follows from Proposition 1 that

$$|X^P/G_{\beta_j}| \leqq |B^P/G_{\beta_j}| \qquad \text{for } 1\leqq j\leqq l. \tag{33}$$

(2. a) follows from (28), (29), (30), (31) and (33). Thus our proof of (2. a) is done.

We will prove (2. d). Let $(X, B, \mathcal{I})$ be symmetric. As in the proof of (1. d), we see that both the design $(X, B, \mathcal{I})$ and its dual design $(B, X, \bar{\mathcal{I}})$ satisfy the assumption of (2. c) and that of (2. a). Thus (2. d) follows from (2. a) and (2. c). This completes the proof of (2. d).

We will prove (2. b). If $p\nmid r$ or $p\nmid b$, then by (1. a) or (1. c) we have

$$|X^P/G_{x_j}| \leqq |B^P/G_{x_j}| \qquad \text{for } 1\leqq j\leqq m. \tag{34}$$

From (24), (25), (26), (27) and (34) it follows that

$$|(X\times X)^P/G| \leqq |(X\times B)^P/G|.$$

Thus in this case (2. b) is proved. Assume that $p\mid r$ and $p\mid b$. Let us recall that $X^P=x_1G\cup\cdots\cup x_mG$ and

$$\text{rank } A_{11}B_{11} = m-1 \leqq \text{rank } A_{11}. \tag{35}$$

We have the following lemma.

LEMMA 7. *Let* $M=\{G_{x_1}, G_{x_2}, \cdots, G_{x_m}\}$. *Then all groups in* M *except one have an orbit on* B *whose length is relatively prime to* p.

PROOF. Suppose false. Then without loss of generality we may assume that there exist G_{x_1} and G_{x_2} such that $p \,\big|\, |\beta G_{x_1}|$ and $p \,\big|\, |\beta G_{x_2}|$ for all $\beta \in B$. Since P is a normal subgroup of G_{x_i}, X^P and B^P are G_{x_i}-invariant for $i=1, 2$. Hence we see that for $i=1, 2$,

$$X^P = S_1(i) \cup S_2(i) \cup \cdots \cup S_{m_i}(i), \qquad B^P = T_1(i) \cup T_2(i) \cup \cdots \cup T_{l_i}(i),$$

where $S_j(i)$ and $T_h(i)$ are the G_{x_i}-orbits on X^P and B^P, respectively. Here we may assume that

$$S_1(i) = \{x_i\} \qquad \text{for } i=1, 2. \tag{36}$$

Since $X-X^P$ and $B-B^P$ are G_{x_i}-invariant for $i=1, 2$, we have the following:

$$X-X^P = S_{m_i+1}(i) \cup \cdots \cup S_{m_i+m_i'}(i),$$
$$B-B^P = T_{l_i+1}(i) \cup \cdots \cup T_{l_i+l_i'}(i),$$

where $S_{m_i+j}(i)$ and $T_{l_i+h}(i)$ are the G_{x_i}-orbits on $X-X^P$ and $B-B^P$, respectively for $i=1, 2$. Let $\mathscr{P}(i)=\{S_1(i), \cdots, S_{m_i+m_i'}(i)\}$ and $Q(i)=\{T_1(i), \cdots, T_{l_i+l_i'}(i)\}$ $(i=1, 2)$. Then for $i=1, 2$ $\mathscr{P}(i)$ and $Q(i)$ are partitions of X and B, respectively. As in the proof of (1.a), we have that for $i=1, 2$

$$\delta_{G_{x_i}}(A) = \begin{pmatrix} A_{11}(i) & 0 \\ A_{21}(i) & A_{22}(i) \end{pmatrix}, \tag{37}$$

where $A_{11}(i)$ has size $m_i \times l_i$,

$$\delta_{G_{x_i}}(A^t) = \begin{pmatrix} B_{11}(i) & 0 \\ B_{21}(i) & B_{22}(i) \end{pmatrix}, \tag{38}$$

where $B_{11}(i)$ has size $l_i \times m_i$. By (37), (38) and Proposition 3, we have for $i=1, 2$

$$\begin{pmatrix} |T_1(i)| & & 0 \\ & \ddots & \\ 0 & & |T_{l_i}(i)| \end{pmatrix} B_{11}(i) = A_{11}^t(i) \begin{pmatrix} |S_1(i)| & & 0 \\ & \ddots & \\ 0 & & |S_{m_i}(i)| \end{pmatrix}. \tag{39}$$

Since $|T_j(i)|=0$ in F_p $(i \leqq j \leqq l_i)$ by the assumption for $i=1, 2$, it follows from (36) and (39) that every component of row x_i of matrix $A_{11}(i)$ is equal to 0 for $i=1, 2$. Since $\beta_l G = \sum_j \beta_j' G_{x_i}$ for all $\beta_l \in B^P$,

$$A_{11}(x_i G, \beta_l G) = \sum_{\beta \in \beta_l G} A(x_i, \beta) = \sum_j A_{11}(i)(x_i, \beta_j' G_{x_i}) = 0,$$

for $i=1, 2$.

These facts show that the matrix A_{11} has two row vectors which have all their components equal to 0. Hence rank $A_{11} \leqq m-2$, which contradicts (35). This completes the proof of Lemma 7.

By using (1.b) and Proposition 1 it follows from Lemma 7 that

$$|X^P/G_{x_1}|+\cdots+|X^P/G_{x_m}|-1 \leqq |B^P/G_{x_1}|+\cdots+|B^P/G_{x_m}|.$$

From (24), (25), (26), (27) and the above inequality it follows that

$$|(X\times X)^P/G|-1 \leqq |(X\times B)^P/G|.$$

In this case we have proved (2. b).

We have completed the proof of Theorem.

ACKNOWLEDGEMENTS. I wish to thank Dr. T. Yoshida for sending me manuscript [**5**] before publication. I also wish to thank the referees for their careful reading and valuable suggestions.

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Tsuyoshi ATSUMI
Department of Mathematics
Faculty of Science
Kagoshima University
Kagoshima 890
Japan