J. Math. Soc. Japan
Vol. 41, No. 2, 1989

A note on fundamental dimensions of Whitney continua of graphs

By Hisao KATO

(Received Sept. 28, 1987)

1. Introduction.

By a *continuum*, we mean a compact connected metric space. Let X be a continuum with metric d. By the *hyperspace* of X, we mean

$$C(X)=\{A \mid A \text{ is a nonempty subcontinuum of } X\}$$

with the *Hausdorff metric* d_H, i.e., $d_H(A, B)=\inf\{\varepsilon>0 \mid U(A, \varepsilon)\supset B$ and $U(B, \varepsilon)\supset A\}$, where $U(A, \varepsilon)=\{x\in X \mid d(x, A)<\varepsilon\}$. In [22], Whitney showed that for any continuum X there exists a map $\omega: C(X)\to[0, \omega(X)]$ satisfying

(1) $\omega(\{x\})=0$ for every $x\in X$, and

(2) if $A, B\in C(X)$, $A\subset B$ and $A\neq B$, then $\omega(A)<\omega(B)$.

Any such map ω is called a *Whitney map*. We may think of the map ω as measuring the size of a continuum. It is well-known that every Whitney map ω is monotone, i.e., $\omega^{-1}(t)$ is a continuum for each $0<t<\omega(X)$. The continuum $\omega^{-1}(t)$ $(0\leqq t<\omega(X))$ is called a *Whitney continuum*. Note that $\omega^{-1}(0)$ is homeomorphic to X and $\omega^{-1}(\omega(X))=\{X\}$. Naturally, we are interested in the structure of $\omega^{-1}(t)$ $(0<t<\omega(X))$. Let X be a continuum. Then the *fundamental dimension* $\mathrm{Fd}(X)$ of X is defined as follows (see [1] or [16]): $\mathrm{Fd}(X)=\min\{\dim Z \mid Z$ is a continuum such that Z has the same shape as $X\}$. In particular, if P is a compact connected polyhedron, then $\mathrm{Fd}\,P=\min\{\dim Z \mid Z$ is a compact connected polyhedron such that Z has the same homotopy type as $P\}$.

In [11] and [2], Kelley and Duda investigated the dimension of $C(G)$ for a graph G. In particular, Duda described and analyzed polyhedral models for hyperspaces of graphs (see [2] and [3]). In [5, (2.4)], we showed that $\omega^{-1}(t)$ is a polyhedron for any graph G, any Whitney map ω for $C(G)$ and $t\in[0, \omega(G)]$ (cf. [15]).

In [5, (2.9)], we defined an index $n(G)$ for a graph G and showed that if ω is any Whitney map for $C(G)$, then $\mathrm{Fd}\,\omega^{-1}(t)\leqq n(G)-1$ for each t. Also, we showed that Whitney continua of graphs admit all homotopy types of compact connected ANR's ([7]).

In this paper, we give a sharper result than [**5**, (2.9)]. We define a more precise index $I(G)$ ($\leqq n(G)$) for a graph G and show that $\mathrm{Fd}\,\omega^{-1}(t)\leqq I(G)-1$ for any Whitney map ω for $C(G)$ and $0\leqq t\leqq\omega(G)$. In general, $I(G)$ is not equal to $n(G)$.

We refer readers to [**11**] and [**17**] for hyperspace theory, and to [**1**] and [**16**] for shape theory.

The author wishes to thank the referee for his helpful comments, in particular, the proof of (3.4).

2. The index $I(G)$.

Let G be a graph (=1-dimensional connected finite polyhedron with a triangulation T). For each edge $e=\langle V, W\rangle$ of G, let

$$\mathcal{A}(e)=\{A \mid A \text{ is an arc in } G \text{ joining } V \text{ and } W\}.$$

For each $A_0\in\mathcal{A}(e)$, let

$$\Phi(A_0)=\{\mathcal{A} \mid \mathcal{A}\subset\mathcal{A}(e) \text{ such that } \bigcup\{A \mid A\in\mathcal{A},\ A\neq A_0\} \text{ does not contain } A_0\}$$

and let

$$\alpha(A_0)=\max\{|\mathcal{A}| \mid \mathcal{A}\in\Phi(A_0)\},$$

where $|\mathcal{A}|$ denotes the cardinal number of $\mathcal{A}$. Then define

$$I(e)=\min\{\alpha(A_0) \mid A_0\in\mathcal{A}(e)\}.$$

The index $I(G)$ is finally defined as

$$I(G)=\max\{I(e) \mid e \text{ is an edge of } G\}.$$

Clearly, the index $I(G)$ is topological invariant. Now, recall the index $n(G)$ (see [**5**]): $n(G)=\max\{|\mathcal{A}(e)| \mid e \text{ is an edge of } G\}$. Clearly, $I(G)\leqq n(G)$. We shall give examples to show that in general $I(G)$ is not equal to $n(G)$.

Let G be a graph and $e=\langle V, W\rangle$ be an edge of G. For each edge e' contained in $\bigcup\{A \mid A\in\mathcal{A}(e)\}$, set $\mathcal{B}_e(e')=|\{A\in\mathcal{A}(e) \mid A\subset \mathrm{Cl}(G-e')\}|+1$.

The next proposition is convenient for calculating the index $I(G)$.

(2.1) PROPOSITION. *If $A_0\in\mathcal{A}(e)$, then $\alpha(A_0)=\max\{\mathcal{B}_e(e') \mid e'$ is an edge contained in $A_0\}$.*

Next, we give some examples in order to clarify the difference between $I(G)$ and $n(G)$.

(2.2) EXAMPLE. Let Δ be a 3-simplex with vertices a_i ($i=0, 1, 2, 3$) and let G be the 1-skeleton of Δ, i.e., $G=\Delta^1$. Consider the following arcs: $A_1=\langle a_0, a_1\rangle$, $A_2=\langle a_0, a_2\rangle\cup\langle a_2, a_1\rangle$, $A_3=\langle a_0, a_3\rangle\cup\langle a_3, a_1\rangle$, $A_4=\langle a_0, a_2\rangle\cup\langle a_2, a_3\rangle\cup\langle a_3, a_1\rangle$ and $A_5=\langle a_0, a_3\rangle\cup\langle a_3, a_2\rangle\cup\langle a_2, a_1\rangle$ (see Figure 1).

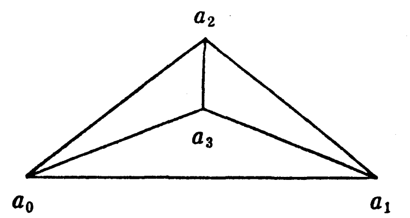

Figure 1.

Then $\alpha(A_1)=5$, $\alpha(A_j)=4$ $(j=2, 3, 4, 5)$. Thus $I(\langle a_0, a_1\rangle)=4$, hence $I(G)=4$. Clearly, $n(G)=5$. By the main result (3.1) of this paper, we see that $\operatorname{Fd}\omega^{-1}(t)\leqq 4-1=3$ for any Whitney map ω for $C(G)$ and $0\leqq t\leqq\omega(G)$. In fact, for any Whitney map ω for $C(G)$, $\operatorname{Fd}\omega^{-1}(t)=3$ for some t (see (3.8) Example).

(2.3) EXAMPLE. Let G be a graph as below (Figure 2).

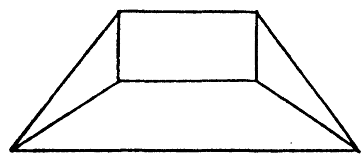

Figure 2.

By using (2.1), we can calculate $I(G)$ as follows:

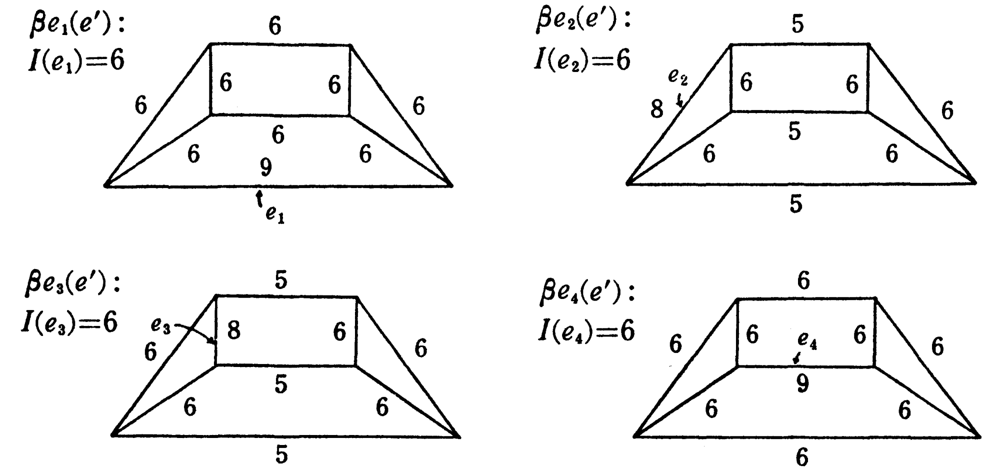

Figure 3.

Then $I(G)=6$ and $n(G)=9$. Hence $\operatorname{Fd}\omega^{-1}(t)\leqq 6-1=5$ for any Whitney map ω for $C(G)$ and $0\leqq t\leqq\omega(G)$.

(2.4) REMARK. Note that for any graph G, $n(G)=k$ if and only if $I(G)=k$, for $k=1, 2, 3$.

3. The main theorem.

In this section, we prove the following main theorem (3.1). The proof is similar to one of [**5**, (2.9)], but we use more precise information.

(3.1) THEOREM. *Let G be a graph and let ω be any Whitney map for $C(G)$. Then* $\mathrm{Fd}\,\omega^{-1}(t)\leqq I(G)-1$ *for each* $0\leqq t\leqq\omega(X)$.

To prove (3.1), we shall need the following:

(3.2) (M. Lynch [**14**]). *Let X be a continuum and let ω be any Whitney map for $C(X)$. If $A\in C(X)$ and $\omega(A)\leqq t\leqq\omega(X)$, then the set*

$$C_A(X, \omega, t) = \{B\in\omega^{-1}(t) \mid A\subset B\}$$

is a nonempty AR.

(3.3) (S. Nowak [**18**]). *Let X, Y be compacta. Then*

$$\mathrm{Fd}(X\cup Y) \leqq \max\{\mathrm{Fd}(X), \mathrm{Fd}(Y), \mathrm{Fd}(X\cap Y)+1\}.$$

(3.4) LEMMA. *Let G be a graph and let V and W be two vertices of G. Let $\mathcal{A}(V, W)=\{A\mid A$ is an arc in G with end points V and $W\}$. Suppose that $A_0\in\mathcal{A}(V, W)$. If $A_1, A_2, \cdots, A_m\in\mathcal{A}(V, W)$, $\bigcup_{i=1}^m A_i$ does not contain A_0 and $\bigcup_{i=0}^m A_i\neq\bigcup\{A\mid A\in\mathcal{A}(V, W)\}$, then there is some $A_{m+1}\in\mathcal{A}(V, W)$ such that $A_{m+1}\neq A_i$ $(i=1, 2, \cdots, m)$ and $\bigcup_{i=1}^{m+1}A_i$ does not contain A_0.*

PROOF. There is some $A\in\mathcal{A}(V, W)$ such that $\bigcup_{i=0}^m A_i$ does not contain A. Choose two points S and T of $A\cap(\bigcup_{i=0}^m A_i)$ such that $A(S, T)\cap(\bigcup_{i=0}^m A_i)=\{S, T\}$, where $A(S, T)$ denotes the arc from S to T in A. The set $\mathcal{K}(S)=\{k\mid S\in A_k\}$ and $\mathcal{K}(T)=\{k\mid T\in A_k\}$ are not empty. Suppose $k\in\mathcal{K}(S)\cap\mathcal{K}(T)$. Then let A_{m+1} be formed from A_k by replacing the arc $A_k(S, T)$ in A_k from S to T with the arc $A(S, T)$. Obviously, $\bigcup_{i=0}^{m+1}A_i\supset A(S, T)$ and $A_0-(\bigcup_{i=1}^{m+1}A_i)\neq\emptyset$. Hence A_{m+1} satisfies the desired conditions. Suppose next that $\mathcal{K}(S)\cap\mathcal{K}(T)=\emptyset$ and let $k(S)$ and $k(T)$ be in $\mathcal{K}(S)$ and $\mathcal{K}(T)$ respectively. Let $A_{k(S)}(S_1, S_2)$ be the closure of the component of $A_{k(S)}-A_{k(T)}$ which contains S; and let $A_{k(T)}(T_1, T_2)$ be the closure of the component of $A_{k(T)}-A_{k(S)}$ which contains T. We may assume that $k(T)\neq 0$. In $A_{k(S)}$, we have the set $\{V, S_1, S, S_2, T_1, T_2, W\}$ which has the possible orderings:

(i) $V\leqq T_1 < T_2\leqq S_1 < S < S_2\leqq W$,
(ii) $V\leqq T_1\leqq S_1 < S < S_2\leqq T_2\leqq W$,
(iii) $V\leqq S_1 < S < S_2\leqq T_1 < T_2\leqq W$,

and three others with T_1 and T_2 interchanged. For the cases (i) and (ii), let $A_{m+1}=A_{k(S)}(V, T_1)\cup A_{k(T)}(T_1, T)\cup A(T, S)\cup A_{k(S)}(S, W)$; and for the case (iii), let $A_{m+1}=A_{k(S)}(V, S)\cup A(S, T)\cup A_{k(T)}(T, T_1)\cup A_{k(S)}(T_1, W)$. In all three cases, one can easily verify that $A(S, T)\subset\bigcup_{i=0}^{m+1}A_i$ and $A_0-\bigcup_{i=1}^{m+1}A_i\neq\emptyset$. Hence A_{m+1} satisfies the desired conditions. The remaining three cases are completed by interchanging T_1 and T_2.

The next lemma is easily proved. We omit the proof.

(3.5) LEMMA. *If L and G are graphs and $L\subset G$, then $I(L)\leqq I(G)$.*

(3.6) LEMMA. *Suppose that V and W are two vertices of a graph G and $|\mathcal{A}(V, W)|\geqq 2$. For $A_0\in\mathcal{A}(V, W)$, let*

$$\alpha(A_0)=\max\{|\mathcal{A}|\,|\,\mathcal{A}\subset\mathcal{A}(V, W) \text{ such that } \bigcup\{A|\ A\in\mathcal{A},\ A\neq A_0\} \text{ does not contain } A_0\}.$$

If ω is a Whitney map for $C(G)$ and $C_V(G, \omega, t)\cap C_W(G, \omega, t)\neq\emptyset$ for some $t\in(0, \omega(X)]$, then

$$\mathrm{Fd}\,(C_V(G, \omega, t)\cap C_W(G, \omega, t))\leqq\alpha(A_0)-2, \quad \text{for each } A_0\in\mathcal{A}(V, W).$$

PROOF. Note that $C_V(G, \omega, t)\cap C_W(G, \omega, t)=\bigcup\{C_A(G, \omega, t)\,|\,A\in A(V, W)\}$. Consider the following polyhedron P with a triangulation K, i.e., $|K|=P$: The vertices of K are elements of $\mathcal{A}(V, W)$, and $\langle A_1, A_2, \cdots, A_m\rangle\in K$ if and only if $\omega(\bigcup_{i=1}^{m}A_i)\leqq t$. By (3.2), we can see that there is a map $f: P\to C_V(G, \omega, t)\cap C_W(G, \omega, t)$ such that $f(st(A; \mathrm{Sd}\,K))\subset C_A(G, \omega, t)$ for each vertex A of K, where $\mathrm{Sd}\,K$ is the barycentric subdivision of K and $st(A; \mathrm{Sd}\,K)$ denotes the closed star of the vertex $A\in K$. Then f is a homotopy equivalence (cf. [**5**, **7**]). Now, we shall show that $\mathrm{Fd}\,P\leqq\alpha(A_0)-2$ for each $A_0\in\mathcal{A}(V, W)$. Let $A_0\in\mathcal{A}(V, W)$. Suppose that A_0 is not a vertex of K (i.e., $\omega(A_0)>t$). For each simplex $\langle A_1, A_2, \cdots, A_m\rangle$ of K, $\omega(\bigcup_{i=1}^{m}A_i)\leqq t$, hence $\bigcup_{i=1}^{m}A_i$ does not contain A_0. Then $m\leqq\alpha(A_0)-1$, hence $\mathrm{Fd}\,P\leqq\dim P\leqq\alpha(A_0)-2$. Now, we assume that A_0 is a vertex of K. Consider the following set.

$$\nabla(A_0)=\{\mathcal{A}|\ \mathcal{A} \text{ is a family of vertices of } K \text{ such that } \bigcup_{i=1}^{m}A_i\supset A_0,\ A_0 \text{ is not an element of } \mathcal{A} \text{ and } \langle A_1, A_2, \cdots, A_m\rangle\in K, \text{ where } \mathcal{A}=\{A_1, A_2, \cdots, A_m\}\}.$$

First, we assume that $\nabla(A_0)=\emptyset$. If $\langle A_1, A_2, \cdots, A_m\rangle$ is a simplex of K and $A_i\neq A_0$ $(i=1, 2, \cdots, m)$, then $\bigcup_{i=1}^{m}A_i$ does not contain A_0, hence $\dim\langle A_1, A_2, \cdots, A_m\rangle\leqq\alpha(A_0)-2$. If there is an $(\alpha(A_0)-1)$-simplex $\langle A_0, A_1, \cdots, A_{\alpha(A_0)-1}\rangle$ in K, by (3.4) we have $\bigcup_{i=0}^{\alpha(A_0)-1}A_i=\bigcup\{A|\ A\in\mathcal{A}(V, W)\}$. Hence $\omega(\bigcup\{A|\ A\in\mathcal{A}(V, W)\})\leqq t$, which implies that $\mathrm{Fd}\,P=0$. Thus we can conclude that $\mathrm{Fd}\,P\leqq\alpha(A_0)-2$.

Next, we assume that $\nabla(A_0)\neq\emptyset$. Let $s=\max\{|\mathcal{A}|\,|\,\mathcal{A}\in\nabla(A_0)\}$ and $s'=\min\{|\mathcal{A}|\,|\,\mathcal{A}\in\nabla(A_0)\}\geqq 2$. If $\mathcal{A}\in\nabla(A_0)$ and $|\mathcal{A}|=s$, then $\langle\mathcal{A}\rangle=\langle A_1, A_2, \cdots, A_s\rangle$ is a free face of $\langle A_0, \mathcal{A}\rangle=\langle A_0, A_1, A_2, \cdots, A_s\rangle$ in P, i.e., $\langle A_0, \mathcal{A}\rangle$ is the unique

s-simplex containing $\langle\mathcal{A}\rangle$, where $\mathcal{A}=\{A_1, A_2, \cdots, A_s\}$. Let $P_s=|K-\bigcup\{\langle A_0, \mathcal{A}\rangle, \langle\mathcal{A}\rangle \mid \mathcal{A}\in\nabla(A_0)$ and $|\mathcal{A}|=s\}|$. Then P_s is a strong deformation retract of P, hence $P_s\cong P$. If $\mathcal{A}\in\nabla(A_0)$ and $|\mathcal{A}|=s-1$, $\langle\mathcal{A}\rangle$ is a free face of $\langle A_0, \mathcal{A}\rangle$ in P_s. Let $P_{s-1}=|K-\bigcup\{\langle A_0, \mathcal{A}\rangle, \langle\mathcal{A}\rangle \mid \mathcal{A}\in\nabla(A_0)$ and $|\mathcal{A}|=s$ or $s-1\}|$. Then $P_{s-1}\cong P_s$. If we continue this process, we have a polyhedron $P_{s'}=|K-\bigcup\{\langle A_0, \mathcal{A}\rangle, \langle\mathcal{A}\rangle \mid A\in\nabla(A_0)$ and $s'\leqq|\mathcal{A}|\leqq s\}|$. Then we see that $P_{s'}\cong P$. As in the case $\nabla(A_0)=\emptyset$, we see that $\mathrm{Fd}\, P=\mathrm{Fd}\, P_{s'}\leqq\alpha(A_0)-2$. This completes the proof.

PROOF OF (3.1). By induction on the number i of edges of G, we shall prove this theorem. The statement is easily seen to be true for the case $i=1$. Assume that it is true for the case $i\leqq k$. Let G be a graph which has $(k+1)$ edges. Choose an edge $e=\langle V, W\rangle$ of G such that $\mathrm{Cl}(G-e)$ is connected. Set $L=\mathrm{Cl}(G-e)$. By (3.5), $I(L)\leqq I(G)$. If $e\cap L=\{V\}$, then $\omega^{-1}(t)=\omega_L^{-1}(t)\cup\omega_e^{-1}(t)\cup C_V(G, \omega, t)$, where $\omega_L=\omega|C(L)$, $\omega_e=\omega|C(e)$. Then we can easily see that if $\omega(L)>t$, then $\omega^{-1}(t)\cong\omega_L^{-1}(t)$. Hence $\mathrm{Fd}\,\omega^{-1}(t)=\mathrm{Fd}\,\omega_L^{-1}(t)\leqq I(L)-1\leqq I(G)-1$. If $\omega(L)<t$, then $\omega^{-1}(t)=\omega_e^{-1}(t)\cup C_V(C, \omega, t)\cong C_V(G, \omega, t)$. Hence $\mathrm{Fd}\,\omega^{-1}(t)=0\leqq I(G)-1$ (see (3.2)). Now, we may assume that $e\cap L=\{V, W\}$. Note that $I(G)\geqq 2$. Then we have

$$\omega^{-1}(t) = \omega_L^{-1}(t)\cup C_V(G, \omega, t)\cup C_W(G, \omega, t)\cup\omega_e^{-1}(t)\,.$$

Consider the following two cases (i) $\omega(e)>t$ and (ii) $\omega(e)\leqq t$.

Case (i): $\omega(e)>t$. If $\omega(L)\leqq t$, then $\omega^{-1}(t)=C_V(G, \omega, t)\cup C_W(G, \omega, t)\cup\omega_e^{-1}(t)$. Note that $C_V(G, \omega, t)\cap C_W(G, \omega, t)=\bigcup\{C_A(G, \omega, t) \mid A$ is an arc from V to W in $L\}$. Then for any subfamily $\mathcal{B}$ of arcs from V to W in L, $\bigcap\{C_A(G, \omega, t) \mid A\in\mathcal{B}\}=C_{\cup\{A\mid A\in\mathcal{B}\}}(G, \omega, t)$ is an AR (see (3.2)). Hence we can conclude that $C_V(G, \omega, t)\cup C_W(G, \omega, t)$ is an AR, which implies that $\omega^{-1}(t)\cong S^1$ (=the 1-sphere). Then $\mathrm{Fd}\,\omega^{-1}(t)=1\leqq I(G)-1$. If $\omega(L)>t$, then $\omega_L^{-1}(t)$ is a strong deformation retract of $\omega_L^{-1}(t)\cup C_V(G, \omega, t)\cup C_W(G, \omega, t)$. Hence

$$\mathrm{Fd}\,(\omega_L^{-1}(t)\cup C_V(G, \omega, t)\cup C_W(G, \omega, t)) = \mathrm{Fd}\,\omega_L^{-1}(t) \leqq I(L)-1 \leqq I(G)-1.$$

Since $(\omega_L^{-1}(t)\cup C_V(G, \omega, t)\cup C_W(G, \omega, t))\cap\omega_e^{-1}(t)$ consists of two points, by (3.3) we can see that $\mathrm{Fd}\,\omega^{-1}(t)\leqq I(G)-1$.

Case (ii): $\omega(e)\leqq t$. Then $\omega^{-1}(t)=\omega_L^{-1}(t)\cup C_V(G, \omega, t)\cup C_W(G, \omega, t)$. By (3.6), $\mathrm{Fd}\,(C_V(G, \omega, t)\cap C_W(G, \omega, t))\leqq I(e)-2\leqq I(G)-2$. Since $C_V(G, \omega, t)$ and $C_W(G, \omega, t)$ are ARs (see (3.2)), by (3.3) we see that $\mathrm{Fd}\,(C_V(G, \omega, t)\cup C_W(G, \omega, t))\leqq I(G)-1$. If $\omega_L^{-1}(t)\cap(C_V(G, \omega, t)\cap C_W(G, \omega, t))\neq\emptyset$, by (3.6)

$$\mathrm{Fd}\,(\omega_L^{-1}(t)\cap(C_V(G, \omega, t)\cap C_W(G, \omega, t))) \leqq \max\{0, I(G)-3\}.$$

Hence we can see that $\mathrm{Fd}\,(\omega_L^{-1}(t)\cap(C_V(G, \omega, t)\cup C_W(G, \omega, t)))\leqq I(G)-2$ (see (3.3)). By (3.3), we can conclude that

$$\mathrm{Fd}\,\omega^{-1}(t) = \mathrm{Fd}\,(\omega_L^{-1}(t)\cup(C_V(G, \omega, t)\cup C_W(G, \omega, t))) \leqq I(G)-1\,.$$

This completes the proof.

(3.7) REMARK. In [**19**, Proposition 12], Petrus proved that if X is a dendrite, then $\omega^{-1}(t)$ is contractible for any Whitney map ω for $C(X)$ and $t\in[0, \omega(X)]$. Hence, (3.1) for the case $I(G)=1$ follows from the result of Petrus.

(3.8) EXAMPLE. Let G be the graph as in (2.2). Let ω be any Whitney map for $C(G)$. Set $t_0=\max\{\omega(\mathrm{Cl}(G-\langle a_i, a_j\rangle))\mid i\neq j\ (i, j=0, 1, 2, 3)\}$. Let $t_0\leqq t<\omega(G)$. Then we shall show that $\mathrm{Fd}\,\omega^{-1}(t)=I(G)-1=3$. Note that $\omega^{-1}(t)=C_{a_0}(G, \omega, t)\cup C_{a_1}(G, \omega, t)$. Then $C_{a_0}(G, \omega, t)\cap C_{a_1}(G, \omega, t)=\bigcup\{C_{A_i}(G, \omega, t)\mid i=1, 2, 3, 4, 5\}$. Consider the following polyhedron P as in the proof of (3.6): The vertices of P are A_i $(i=1, 2, 3, 4, 5)$ and the simplexes are $\langle A_2, A_3, A_4, A_5\rangle$, $\langle A_1, A_2, A_3\rangle$, $\langle A_1, A_2, A_4\rangle$, $\langle A_1, A_2, A_5\rangle$, $\langle A_1, A_3, A_4\rangle$, $\langle A_1, A_3, A_5\rangle$ and their faces. Then $C_{a_0}(G, \omega, t)\cap C_{a_1}(G, \omega, t)\cong P\cong S^2\vee S^2$, where $S^2\vee S^2$ denotes the one point union of 2-spheres. Hence $\omega^{-1}(t)\cong\sum P\cong S^3\vee S^3$. Hence $\mathrm{Fd}\,\omega^{-1}(t)=3$.

By (3.1) and [**5**, (3.2)], we have

(3.9) COROLLARY. *Let* $\underline{X}=\{G_n, p_{n\,n+1}\}$ *be an inverse sequence of graphs. Suppose that* $X=\mathrm{inv}\lim\underline{X}$ *and* ω *is any Whitney map for* $C(X)$. *If* $I(G_n)\leqq m$ *for each* n, *then* $\mathrm{Fd}\,\omega^{-1}(t)\leqq m-1$ *for each* $t\in[0, \omega(X)]$.

References

[1] K. Borsuk, Theory of shape, Monograf. Mat., **59**, PWN, Warszawa, 1975.

[2] R. Duda, On the hyperspace of subcontinua of a finite graph, I, II, Fund. Math., **62** (1968), 265-286; **63** (1968), 225-255.

[3] ———, Correction to the paper "On the hyperspace of subcontinua of a finite graph, I", Fund. Math., **69** (1970), 207-211.

[4] H. Kato, Shape properties of Whitney maps for hyperspaces, Trans. Amer. Math. Soc., **297** (1986), 529-546.

[5] ———, Whitney continua of curves, Trans. Amer. Math. Soc., **300** (1987), 367-381.

[6] ———, Movability and homotopy, homology pro-groups of Whitney continua, J. Math. Soc. Japan, **39** (1987), 435-446.

[7] ———, Whitney continua of graphs admit all homotopy types of compact connected ANR's, Fund. Math., **129** (1988), 161-166.

[8] ———, Various types of Whitney maps on n-dimensional compact connected polyhedra ($n\geqq 2$), Topology Appl., **28** (1988), 17-21.

[9] ———, Shape equivalences of Whitney continua of curves, Canad. J. Math., **40** (1988), 217-227.

[10] ———, Limiting subcontinua and Whitney maps of tree-like continua, Compositio Math., **66** (1988), 5-14.

[11] J.L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc., **52** (1942), 22-36.

[12] J. Krasinkiewicz, Shape properties of hyperspaces, Fund. Math., **101** (1978), 79-91.

[13] J. Krasinkiewicz and S. B. Nadler, Jr., Whitney properties, Fund. Math., **98** (1978), 165-180.

[14] M. Lynch, Whitney levels in $C_p(X)$ are absolute retracts, Proc. Amer. Math. Soc., **97** (1986), 748-750.

[15] ———, Whitney properties of 1-dimensional continua, Bull. Acad. Polon. Sci., **35** (1987), 473-478.

[16] S. Mardešić and J. Segal, Shape theory, North-Holland Mathematical Library, 1982.

[17] S. B. Nadler, Jr., Hyperspaces of sets, Pure and Appl. Math., **49**, Dekker, New York, 1978.

[18] S. Nowak, Some properties of fundamental dimension, Fund. Math., **85** (1974), 211-227.

[19] A. Petrus, Contractibility of Whitney continua in $C(X)$, General Topology Appl., **9** (1978), 275-288.

[20] J. T. Rogers, Jr., Applications of Vietoris-Begle theorem for multi-valued maps to the cohomology of hyperspaces, Michigan Math. J., **22** (1975), 315-319.

[21] ———, Dimension and Whitney subcontinua of $C(X)$, General Topology Appl., **6** (1976), 91-100.

[22] H. Whitney, Regular families of curves I, Proc. Nat. Acad. Sci. U. S. A., **18** (1932), 275-278.

Hisao KATO

Faculty of Integrated Arts and Sciences
Hiroshima University
Higashisenda-machi, Naka-ku
Hiroshima 730
Japan