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Existence of maximal hypersurfaces in an asymptotically anti-de Sitter spacetime satisfying a global barrier condition

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§ 1. Introduction.

Maximal hypersurfaces are spacelike hypersurfaces of a Lorentzian manifold which are critical points of the induced area functional. The (universal) anti-de Sitter spacetime is a geodesically complete spacetime of constant negative curvature, which is a useful model for spatially noncompact spacetime as the Minkowski spacetime. The purpose of this paper is to prove the existence of entire maximal hypersurfaces in an asymptotically anti-de Sitter spacetime satisfying a global barrier condition (see Section 4).

Maximal hypersurfaces play a very important role in the study of Lorentzian geometry. In fact, Gerhardt [**10**] and Galloway [**9**] proved splitting theorems with respect to time and space in a spatially closed globally hyperbolic Lorentzian manifold satisfying the timelike convergence condition (cf. [**4**]). In [**14**], Schoen-Yau proved the positive mass conjecture under the assumption of the existence of an asymptotically flat maximal hypersurface in an asymptotically flat spacetime.

In a spatially closed Lorentzian manifold, many general results for the existence of compact maximal hypersurfaces were shown in Bartnik [**1**] and [**10**]. In a spatially noncompact Lorentzian manifold, several difficulties arise when considering the existence of noncompact maximal hypersurfaces. However, in [**1**] it was proved that there exist entire maximal hypersurfaces in an asymptotically flat spacetime satisfying a uniform interior condition (cf. [**7**]). It should be remarked that a complete maximal hypersurface in the Minkowski space is totally geodesic (cf. [**5**], [**6**]).

Although several useful gradient estimates for spacelike hypersurfaces have been known, for example in [**1**], [**2**], [**6**], [**10**], [**15**], to prove our main result we need more general gradient estimates since the lapse functions of asymptotically anti-de Sitter spacetimes are unbounded. In Section 3, modifying the technique in [**1**], we shall prove the gradient estimates. The gradient estimates nevertheless depend on the a priori decay of height functions of spacelike hypersurfaces. For one sufficient condition to control the decay of height functions of maximal hypersurfaces, we impose a global barrier condition upon asymptotically anti-de Sitter spacetimes. It should be pointed out that, although the anti-de Sitter spacetime satisfies the global barrier condition, the norm of gradient of the barrier function defined by the condition is uniformly bounded and converges uniformly to zero at spacelike infinity (see Section 5).

§ 2. Notation and formulas.

First we set up our terminology and notation.

Let $(\mathcal{V}, g)$ be a spacetime (i.e., a time oriented Lorentzian 4-manifold, cf. [**12**]) with Lorentzian metric g of signature $(-, +, +, +)$. Let ∇ denote the Levi-Civita connection of $(\mathcal{V}, g)$. We shall use the summation convention with Roman indices in the range $1 \leqq i, j, \cdots \leqq 3$ and Greek indices in $0 \leqq \lambda, \mu, \cdots \leqq 3$.

A function $t \in C^\infty(\mathcal{V})$ is said to be a time function (cf. [**8**]) if ∇t ($=\operatorname{grad} t$) is a nonzero timelike vector field. The lapse function $\alpha \in C^\infty(\mathcal{V})$ of t is defined by

$$\alpha^{-2} = -\langle \nabla t, \nabla t \rangle . \tag{2.1}$$

The future-directed unit normal vector T on the time slice $\mathcal{S}_t = \{p \in \mathcal{V}\,;\, t(p)=t\}$ is given by

$$T = -\alpha \nabla t . \tag{2.2}$$

Let M be a spacelike hypersurface in $\mathcal{V}$. We choose a local field of Lorentz orthonormal frames $\{V, e_1, e_2, e_3\}$ in $\mathcal{V}$ such that, restricted to M, the vectors $\{e_1, e_2, e_3\}$ are tangent to M and the vector V is future-directed. Then the second fundamental form A and the mean curvature H are given by

$$A(e_i, e_j) = -\langle e_i, \nabla_{e_j} V \rangle , \tag{2.3}$$

$$H = \sum_{i=1}^{3} A(e_i, e_j) = -\operatorname{div}_M V . \tag{2.4}$$

The following calculations are due to Bartnik [**1**]. For completeness we review them briefly.

A function $u \in C^\infty(M)$ is said to be the height function of M if u is the restriction of the time function to M, that is, $u = t|_M$. Then we have

$$\nabla^M u = \alpha^{-1}(\nu V - T) , \tag{2.5}$$

$$|\nabla^M u|^2 = \alpha^{-2}(\nu^2 - 1) , \tag{2.6}$$

where $\nabla^M u = \operatorname{grad}_M u$ and $\nu = -\langle T, V \rangle$. Hence $\nu \geqq 1$. From (2.4) and (2.5) we obtain

(2.7) $$H\nu = -\mathrm{div}_M(\alpha\nabla^M u) - \mathrm{div}_M T\,.$$

We carry out local calculations which are of use in Section 4. Let $(t, x^i) = (x^\lambda)$ be local coordinates (t is still the time function) of $\mathcal{V}$ so that the metric g can be written as

(2.8) $$g_{\lambda\mu}dx^\lambda dx^\mu = -(\alpha^2-\beta^2)dt^2 + 2\beta_i dt dx^i + g_{ij}dx^i dx^j\,,$$

where β is the shift vector $g^{ij}\beta_i\partial/\partial x^j$ and $(g^{ij})=(g_{ij})^{-1}$. Note that, since the vector fields $\{\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3\}$ are tangent to $\mathcal{S}_t$, $D\phi = g^{ij}\partial\phi/\partial x^j\cdot\partial/\partial x^i$ for each $\phi\in C^\infty(\mathcal{V})$ is also tangent to $\mathcal{S}_t$. The future-directed unit normal vector T can be expressed as

(2.9) $$T = \alpha^{-1}(\partial/\partial t - \beta)\,,$$

and the second fundamental form A^0 and the mean curvature H^0 of $\mathcal{S}_t$ are given by

(2.10) $$A^0_{ij} = -\langle\partial/\partial x^i, \nabla_{\partial/\partial x^j}T\rangle = -\frac{1}{2}\alpha^{-1}\partial g_{ij}/\partial t + \frac{1}{2}\alpha^{-1}\mathcal{L}_\beta g_{ij}\,,$$

(2.11) $$H^0 = g^{ij}A^0_{ij} = -\frac{1}{2}\alpha^{-1}g^{ij}\partial g_{ij}/\partial t + \alpha^{-1}\mathrm{div}^0\beta\,,$$

where $\mathcal{L}_\beta$ is the Lie derivative with respect to β and div^0 is the divergence on $\mathcal{S}_t$.

The height function u of M can be extended to one on $\mathcal{V}$ satisfying $\partial u/\partial t = 0$. Since M is a level set of $u-t$, the future-directed unit vector V can be expressed as

(2.12) $$V = \nu(U+T)\,,$$

where $U = \alpha Du(1+\langle\beta, Du\rangle)^{-1}$, and hence $\nu = (1-|U|^2)^{-1/2}$. Let $\{\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$, where $\bar{e}_0 = T$ and $\bar{e}_1 = |Du|^{-1}Du$ whenever $Du\neq 0$, be a local field of Lorentz orthonormal frames in $\mathcal{V}$. Choosing $\{V, e_1, e_2, e_3\}$ so that $e_1 = |\nabla^M u|^{-1}\nabla^M u$ whenever $Du\neq 0$, we obtain

(2.13) $$\nabla^M u = \alpha^{-1}\nu^2(U+|U|^2T)\,, \qquad e_1 = \nu(\bar{e}_1 + |U|T)\,,$$

(2.14) $$\begin{aligned} H &= -\nu\,\mathrm{div}_M(U+T) \\ &= -\nu\,\mathrm{div}^0 U + \nu H^0 - \nu^3|U|^2\bar{e}_1(|U|) - \nu|U|\langle\bar{e}_1, \nabla_T T\rangle - \frac{1}{2}\nu^3 T(|U|^2) \\ &= -\mathrm{div}^0[U/(1-|U|^2)^{1/2}] + \nu H^0 - \nu\langle U, \nabla_T T\rangle - \frac{1}{2}\nu^3 T(|U|^2)\,, \end{aligned}$$

(2.15) $$\begin{aligned} \mathrm{div}_M T &= -H^0 + \nu^2 U^i U^j A^0_{ij} + \nu^2\langle U, \nabla_T T\rangle \\ &= -H^0 + (\nu^2-1)(A^0_{11} - \alpha^{-1}T(\alpha)) + \langle\nabla u, \nabla^M\alpha\rangle\,. \end{aligned}$$

§ 3. Gradient estimates.

In this section we give main estimates for ν.

Keeping our notation in Section 2, for each nonnegative integer n we define a positive definite norm $\|\cdot\|_n$ on the space $\mathcal{T}(\mathcal{V})$ of tensor fields on $\mathcal{V}$. The norm $\|\cdot\|_n$ is defined by

$$\|X\|_0=\|X\|=\sup_{\mathcal{V}}\Big(\sum_{\lambda,\cdots,\mu,\cdots=0}^{3}|X(\bar{e}_\lambda,\cdots,\bar{\omega}^\mu,\cdots)|^2\Big)^{1/2},$$

$$\|X\|_n=\sum_{j=0}^{n}\|\nabla^j X\|$$

for $X\in\mathcal{T}(\mathcal{V})$, where $\{\bar{\omega}^0, \bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3\}$ is the frame dual to $\{\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3\}$.

To prove the estimates for ν, we need the following lemma.

LEMMA 1([1]). *Let $(\mathcal{V}, g)$ be a spacetime with a time function t. Let M be a spacelike hypersurface. Then the following holds.*

$$\Delta_M\nu=\nu(|A|^2+\mathrm{Ric}(V, V))+\langle T, \nabla^M H\rangle-T(H_T), \tag{3.1}$$

where $T(H_T)$ is the variation of mean curvature of M under the deformation vector field T. This can be expressed as

$$\begin{aligned}T(H_T)&=-\frac{1}{2}\sum_{i=1}^{3}(\nabla_V\mathcal{L}_T g)(e_i, e_i)+\sum_{i=1}^{3}(\nabla_{e_i}\mathcal{L}_T g)(V, e_i)\\&=-\frac{1}{2}H(\mathcal{L}_T g)(V, V)-\sum_{i,j=1}^{3}(\mathcal{L}_T g)(e_i, e_j)\cdot A(e_i, e_j).\end{aligned}\tag{3.2}$$

PROPOSITION 1. *Let $(\mathcal{V}, g)$ be a spacetime with a time function t. Let M be a compact spacelike hypersurface with the height function u and the mean curvature H. Suppose that there exist constants $\delta\,(>0)$, $C_1, \cdots, C_4$ and k such that*

$$\|\mathrm{Ric}(V, V)\|,\ \|\nabla\mathcal{L}_T g\|,\ \|\alpha^{-1}\nabla\nabla\alpha\|\leqq C_1, \tag{3.3}$$

$$\|A^0\|,\ \|\alpha^{-1}\nabla\alpha\|,\ \|\mathcal{L}_T g\|\leqq C_2, \tag{3.4}$$

$$\|H\|_1\leqq C_3, \tag{3.5}$$

$$\sup_M(|u(x)-k|\alpha(x)^{1+\delta})\leqq C_4. \tag{3.6}$$

(i) *If $\partial M=\emptyset$, then for all $x\in M$*

$$\nu(x)\leqq 2\exp[\min\{m_+-(u(x)-k)\alpha_K(x)^{1+\delta},\ (u(x)-k)\alpha_K(x)^{1+\delta}-m_-\}], \tag{3.7}$$

where $K=K(C_1, \cdots, C_4, \delta)$, $\alpha_K=\max(\alpha, K)$ and $m_+=\sup_M\{(u-k)\alpha_K^{1+\delta}\}$, $m_-=\inf_M\{(u-k)\alpha_K^{1+\delta}\}$.

(ii) *If $\partial M\neq\emptyset$ and there exists a constant C_5 such that ∂M satisfies the conditions*

(3.8) $$\|H_{\partial M}\| = \sup_{\partial M}\Big(\sum_{\lambda=0}^{3} |\langle H_{\partial M}, \bar{e}_\lambda\rangle|^2\Big)^{1/2} \leqq C_5,$$

(3.9) $$u|_{\partial M} = k,$$

where $H_{\partial M}$ is the mean curvature vector of ∂M, then for all $x \in M$

(3.10) $$\nu(x) \leqq 2\exp(m),$$

where $K=K(C_1, \cdots, C_5, \delta)$ and $m=\sup_M(|u-k|\alpha_K^{1+\delta})$.

PROOF. We may assume that $k=0$.

(i) We consider the function

$$\nu(x)\exp(u(x)\alpha_K(x)^{1+\delta}), \qquad x \in M,$$

where K is a large constant to be fixed later. $\nu\exp(u\alpha_K^{1+\delta})$ attains a maximum at some point $x_0 \in M$.

If $\alpha(x_0) \leqq K$, the function $\nu\exp(K^{1+\delta}u)$, $x \in M$ also attains a maximum at x_0. By the arguments in [**1**, Theorem 3.1, (i)], there exists a constant $C'=C'(C_1, C_2, C_3)$ such that

$$\nu(x_0)^2 \leqq K^{2+2\delta}\alpha(x_0)^{-2}\{K^{2+2\delta}\alpha(x_0)^{-2} - C'(K^{1+\delta}\alpha(x_0)^{-1}+1)\}^{-1}.$$

Then we can choose a large constant $K=K(C', \delta)$ so that $\nu(x_0) \leqq 2$. Hence

(3.11) $$\nu(x) \leqq 2\exp(m_+ - u(x)\alpha_K(x)^{1+\delta})$$

for all $x \in M$.

We now assume that $\alpha(x_0) > K$. Then the function $\nu(x)\exp(u(x)\alpha(x)^{1+\delta})$, $x \in M$ also attains a maximum at x_0. Hence at x_0

(3.12) $$0 = \nu\nabla^M(u\alpha^{1+\delta}) + \nabla^M\nu,$$
$$0 \geqq \Delta_M(u\alpha^{1+\delta}) - |\nabla^M(u\alpha^{1+\delta})|^2 + \nu^{-1}\Delta_M\nu.$$

On the other hand, from (3.1)-(3.5) we obtain

(3.13) $$\Delta_M\nu \geqq \nu|A|^2 - C(\nu^3+\nu^2|A|) \geqq (1-\varepsilon)\nu|A|^2 - C(\varepsilon^{-1})\nu^3$$

for any $\varepsilon > 0$. Using the inequalities

$$|A|^2 \geqq \Big(1+\frac{1}{3}\Big)\lambda^2 - H^2, \qquad |\nabla^M\nu|^2 \leqq (1+\varepsilon)\nu^2\lambda^2 + C(\varepsilon^{-1})\nu^4.$$

We obtain

(3.14) $$|A|^2 \geqq \Big(1+\frac{1}{3}\Big)\{(1+\varepsilon^{-1})\nu^{-2}|\nabla^M\nu|^2 - C(\varepsilon^{-1})\nu^2\},$$

where λ is the maximum of the absolute values of the eigenvalues of A. Then, from (3.12), at x_0

$$(3.15)\qquad |A|^2 \geqq (1-\varepsilon)^{-1}\left(1+\frac{1}{4}\right)|\nabla^M(u\alpha^{1+\delta})|^2 - C\nu^2.$$

Substituting (3.13) and (3.15) in (3.12), then at x_0

$$(3.16)\qquad 0 \geqq \Delta_M(u\alpha^{1+\delta}) + \frac{1}{4}|\nabla^M(u\alpha^{1+\delta})|^2 - C\nu^2.$$

From (2.6), (2.7), (2.15) and (3.3)-(3.5) we also have

$$(3.17)\qquad \Delta_M u \geqq -C\alpha^{-1}\nu^2, \qquad \alpha^{-1}|\nabla^M\alpha| \leqq C\nu, \qquad \alpha^{-1}|\Delta_M\alpha| \leqq C\nu^2.$$

It follows from (2.6), (3.6), (3.16) and (3.17) that at x_0

$$(3.18)\qquad \begin{aligned} 0 &\geqq \alpha^{1+\delta}\Delta_M u + 2(1+\delta)\alpha^\delta\langle\nabla^M u, \nabla^M\alpha\rangle + \delta(1+\delta)u\alpha^{\delta-1}|\nabla^M\alpha|^2 + (1+\delta)u\alpha^\delta\Delta_M\alpha \\ &\quad + \frac{1}{4}(\alpha^{2+2\delta}|\nabla^M u|^2 + 2(1+\delta)u\alpha^{1+2\delta}\langle\nabla^M u, \nabla^M\alpha\rangle + (1+\delta)^2u^2\alpha^{2\delta}|\nabla^M\alpha|^2) - C\nu^2 \\ &\geqq \frac{1}{4}\alpha^{2\delta}(\nu^2-1) - C\alpha^\delta\nu^2 - C\nu^2. \end{aligned}$$

Now choosing $K=K(C_1, \cdots, C_4, \delta)$ large enough, then from (3.18) $\nu(x_0)\leqq 2$ and hence we obtain the same estimate (3.11).

Applying the same argument to $\nu\exp(-u\alpha_K^{1+\delta})$, we obtain the estimate (3.7).

(ii) Applying an argument similar to that in [**1**, Theorem 3.1, (iii)] to $\nu\exp(u\alpha_K^{1+\delta})$ and $\nu\exp(-u\alpha_K^{1+\delta})$, we can show the estimate (3.10).

REMARK. This result does not need the estimate of $\|\alpha\|$ under the a priori decay (3.6).

§4. Asymptotically anti-de Sitter spacetimes.

First, we review the (universal) anti-de Sitter spacetime (H_1^4, h) (of constant curvature -1). (H_1^4, h) equals $\boldsymbol{R}^4$ as a set and the metric h is given by (cf. [**3**], [**12**])

$$h_{\lambda\mu}dx^\lambda dx^\mu = -(1+r^2)dt^2 + (1+r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $(t\,(=x^0), x^i)$ is the canonical global coordinate system of $\boldsymbol{R}^4$ and (r, θ, ϕ) is the standard polar coordinate representation of (x^i).

DEFINITION. Let N be an oriented 3-manifold which satisfies that there exists a compact subset K of N such that $N\setminus K$ is disjoint union of a finite number of subsets $N_1, \cdots, N_m$ with each N_l being diffeomorphic to $\{(x^i)\in\boldsymbol{R}^3;\ \sum_{i=1}^3(x^i)^2>1\}$. Then this diffeomorphism induces a global coordinate system (x^i) of each N_l. Let $(\mathcal{V}, g)$ be a spacetime. $(\mathcal{V}, g)$ is an asymptotically anti-de Sitter spacetime if the following hold.

(i) There exists a diffeomorphism $\Phi: \mathcal{V}\to\boldsymbol{R}\times N$ such that $\pi\circ\Phi: \mathcal{V}\to\boldsymbol{R}$ is a time function, and hence each $\Phi^{-1}(\boldsymbol{R}\times N_l)$ has a global coordinate system $(t(=\pi\circ\Phi),\ x^i(=(\bar{\pi}\circ\Phi)^i))$, where $\pi: \boldsymbol{R}\times N\to\boldsymbol{R}$ and $\bar{\pi}: \boldsymbol{R}\times N\to N$ denote canonical projections.

(ii) On each $(\Phi^{-1}(\boldsymbol{R}\times N_l), (t, x^i))$ there exist constants $\varepsilon\,(0<\varepsilon<1)$ and C_6 such that for each $B\in SO(3)$ (the 3-dimensional rotation group) the metric g has the form

$$(4.1)\qquad g_{\lambda\mu}dx^\lambda dx^\mu = -(\alpha^2-\beta^2)dt^2+2\beta_r dtdr+2\beta_\theta dtd\theta+2\beta_\phi dtd\phi+g_{rr}dr^2 +g_{\theta\theta}d\theta^2+g_{\phi\phi}d\phi^2+2g_{r\theta}drd\theta+2g_{r\phi}drd\phi+2g_{\theta\phi}d\theta d\phi$$

and behaves asymptotically as

$$\begin{aligned}
g_{tt} &= -(\alpha^2-\beta^2) = -(1+r)^2+r^{-\varepsilon}f_{tt}(t, r, \theta, \phi),\\
g_{tr} &= \beta_r = r^{-3-\varepsilon}f_{tr}(t, r, \theta, \phi),\\
g_{t\theta} &= \beta_\theta = r^{-\varepsilon}f_{t\theta}(t, r, \theta, \phi),\\
g_{t\phi} &= \beta_\phi = r^{-\varepsilon}f_{t\phi}(t, r, \theta, \phi),\\
g_{rr} &= -(1+r^2)^{-1}+r^{-4-\varepsilon}f_{rr}(t, r, \theta, \phi),\\
g_{r\theta} &= r^{-3-\varepsilon}f_{r\theta}(t, r, \theta, \phi),\\
g_{r\phi} &= r^{-3-\varepsilon}f_{r\phi}(t, r, \theta, \phi),\\
g_{\theta\theta} &= r^2+r^{-\varepsilon}f_{\theta\theta}(t, r, \theta, \phi),\\
g_{\theta\phi} &= r^{-\varepsilon}f_{\theta\phi}(t, r, \theta, \phi),\\
g_{\phi\phi} &= r^2\sin^2\theta+r^{-\varepsilon}f_{\phi\phi}(t, r, \theta, \phi),
\end{aligned}$$

$$(4.2)\quad \sum_{b,c}|f_{b,c}(t, r, \theta, \phi)|+\sum_{b,c,d}|\partial_d f_{bc}(t, r, \theta, \phi)|+r\sum_{b,c}|\partial_r f_{bc}(t, r, \theta, \phi)| \leqq C_6,$$

where (r, θ, ϕ) is the standard polar coordinate representation of $((B\circ(x^j))^i)$ and the indices b, c, d are in the range $\{t, r, \theta, \phi\}$.

$(\mathcal{V}, g)$ satisfies the global future (resp. past) barrier condition (with respect to $\mathcal{S}_0$) if there exist an entire spacelike hypersurface $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) and a positive constant C_7 such that

$$(4.3)\qquad H_{\mathcal{S}^+}(x)>0 \quad \text{for all } x\in\mathcal{S}^+ \qquad (\text{resp. } H_{\mathcal{S}^-}(x)<0 \quad \text{for all } x\in\mathcal{S}^-),$$

$$(4.4)\qquad \psi^+(x)>0,\quad \limsup_{r\to\infty} r|\psi^+(x)| \leqq C_7<1 \quad \text{for all } x\in\mathcal{S}^+$$
$$(\text{resp. } \psi^-(x)<0,\quad \limsup_{r\to\infty} r|\psi^-(x)| \leqq C_7<1 \quad \text{for all } x\in\mathcal{S}^-),$$

where $H_{\mathcal{S}^+}$ (resp. $H_{\mathcal{S}^-}$) is the mean curvature of $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) and ψ^+ (resp. ψ^-) is its height function.

REMARK. (1) From (4.2) we can show that there exists a constant $C_8=$

$C_8(C_6, \varepsilon)$ such that

(4.5) $$r^{3+\varepsilon}|A^0| \leqq C_8.$$

(2) The asymptotic condition (4.2) is weaker than that required in [**13**].

(3) Using the standard spacelike hypersurfaces of constant mean curvature in (H_1^4, h), (H_1^4, h) satisfies the global future and past barrier conditions (see Section 5).

Next, we construct barrier hypersurfaces at spacelike infinity. The radial mean curvature equation in (H_1^4, h) for $w=w(r)$,

(4.6) $$-r^{-2}a[r^2a^2w'(1-a^4w'^2)^{-1/2}]'+a'a^2w'(1-a^4w'^2)^{-1/2} = H^*$$

can be solved with $H^*=(3-\varepsilon)R^{11\varepsilon/10}r^{-\varepsilon}$ in $r \geqq R>1$, where $a=a(r)=(1+r^2)^{1/2}$ and H^* is the mean curvature of the graph w in H_1^4. The equation (4.6) equals the following

(4.7) $$-[r^2a^3w'(1-a^4w'^2)^{-1/2}]' = H^*r^2.$$

Then a solution of (4.6) is given by

(4.8) $$w'(r) = -a(r)^{-2}[1+R^{-11\varepsilon/5}r^{-2+2\varepsilon}a(r)^2]^{-1/2},$$

(4.9) $$w(r) = \int_r^\infty a(s)^{-2}[1+R^{-11\varepsilon/5}s^{-2+2\varepsilon}a(s)^2]^{-1/2}ds.$$

PROPOSITION 2. *Let $(\mathcal{V}, g)$ be an asymptotically anti-de Sitter spacetime satisfying* (4.2). *Then there exist constants $\bar{R}=\bar{R}(C_6, \varepsilon)$ and $C_9(>0)$ such that for each $R \geqq \bar{R}$ the hypersurface defined by* (4.9) *is spacelike and satisfies*

(4.10) $$H(w) \geqq 2r^{-\varepsilon}, \qquad r \geqq R,$$

(4.11) $$\nu(w) \leqq 2R^{\varepsilon/10}, \qquad r \geqq R,$$

(4.12) $$0 < w \leqq C_9r^{-1-\varepsilon}, \qquad r \geqq R,$$

(4.13) $$w(R) \geqq \left(\frac{\pi}{2}-\tan^{-1}R\right)-2R^{-1-\varepsilon/10},$$

where $H(w)$ is the mean curvature of the hypersurface defined by w and $\nu(w)(x)=(1-\alpha^2|Dw|^2(1+\langle\beta, Dw\rangle)^{-2})^{-1/2}|_{(x, w(x))}$.

PROOF. First we note that for $r \geqq R$

$$w(r) = R^{11\varepsilon/10}\cdot O(r^{-1-\varepsilon}),$$
$$w'(r) = R^{11\varepsilon/10}\cdot O(r^{-2-\varepsilon}),$$
$$w''(r) = R^{11\varepsilon/10}\cdot O(r^{-3-\varepsilon}),$$
$$(1-a^4w'^2)^{-1/2} \leqq [1+R^{\varepsilon/5}]^{1/2} < 2R^{\varepsilon/10}$$

and

$$\begin{aligned} w(R) &= \int_R^\infty a(r)^{-2}dr - \int_R^\infty a(r)^{-2}[1-\{1+R^{-11\varepsilon/5}r^{-2+2\varepsilon}a(r)^2\}^{-1/2}]dr \\ &\geqq \int_R^\infty a(r)^{-2}dr - \int_R^\infty [R^{11\varepsilon/5}r^{2-2\varepsilon}+a(r)^2]^{-1}dr \\ &\geqq \int_R^\infty a(r)^{-2}dr - \int_R^\infty (R^{2+\varepsilon/5}+r^2)^{-1}dr \\ &\geqq \left(\frac{\pi}{2}-\tan^{-1}R\right)-2R^{-1-\varepsilon/10}. \end{aligned}$$

In the form (4.1), from (2.14) we have

$$H(w) = -\mathrm{div}^0[W(1-|W|^2)^{-1/2}]+\nu(w)(H^0-\langle W, \nabla_T T\rangle)-\frac{1}{2}\nu(w)^3 T(|W|^2), \tag{4.14}$$

where $W=\alpha Dw(1+\langle\beta, Dw\rangle)^{-1}$. From (4.2), (4.5), (4.8) and using $\langle W, \nabla_T T\rangle = \alpha^{-1}\langle W, D\alpha\rangle$ we can show that

$$\nu(w) \leqq (1-a^4w'^2)^{-1/2}+CR^{11\varepsilon/10}r^{-1-13\varepsilon/10} < (2+Cr^{-1-3\varepsilon/10})R^{\varepsilon/10}, \tag{4.15}$$

$$\left|\frac{1}{2}\nu(w)^3 T(|W|^2)\right| \leqq CR^{11\varepsilon/10}r^{-3-8\varepsilon/5}, \tag{4.16}$$

$$|\nu(w)H^0| \leqq CR^{\varepsilon/10}r^{-3-\varepsilon}, \tag{4.17}$$

$$|\mathrm{div}^0[W(1-|W|^2)^{-1/2}-r^{-2}a[r^2a^2w'(1-a^4w'^2)^{-1/2}]'| \leqq CR^{11\varepsilon/10}r^{-1-4\varepsilon/5}, \tag{4.18}$$

$$|\nu(w)\langle W, \nabla_T T\rangle-(1-a^4w'^2)^{-1/2}a^2a'w'| \leqq CR^{11\varepsilon/10}r^{-1-6\varepsilon/5}. \tag{4.19}$$

Choosing $\bar{R}$ large enough, from (4.6) and (4.14)-(4.19) we obtain that $H(w)\geqq 2r^{-\varepsilon}$ for $r\geqq R\geqq\bar{R}$.

Let M be a spacelike hypersurface with the height function u. We often consider M as a graph over a domain $\mathcal{D}_M\subset\mathcal{S}_0$.

LEMMA 2 ([**1**]). *Let $(\mathcal{V}, g)$ be an asymptotically anti-de Sitter spacetime satisfying the global future and past barrier conditions* (4.3), (4.4). *Then for each $R>0$ there exists a maximal* ($H=0$) *hypersurface M such that*

$$\mathcal{D}_M = \mathcal{S}_0\cap(\mathcal{V}\setminus\mathcal{V}_R), \qquad \partial M = \mathcal{S}_0\cap\partial(\mathcal{V}\setminus\mathcal{V}_R), \qquad \psi^-(x)\leqq u(x)\leqq\psi^+(x),$$

for all $x\in\mathcal{D}_M$, where $\mathcal{V}_R=\{p\in\Phi^{-1}(\boldsymbol{R}\times(\bigcup_{l=1}^m N_l));\ r(p)\geqq R\}$.

We now prove the following main theorem.

THEOREM. *Let $(\mathcal{V}, g)$ be an asymptotically anti-de Sitter spacetime satisfying* (3.3) *and* (4.2). *Suppose that $(\mathcal{V}, g)$ satisfies the global future and past barrier conditions* (4.3), (4.4). *Then there exists an entire maximal hypersurface M satisfying*

$$|u(x)| \leqq C_9 r^{-1-\varepsilon}$$

for $x\in M\cap\mathcal{V}_{R_0}$, where $R_0=R_0(C_7, \varepsilon)\geqq\bar{R}$.

PROOF. For each $\rho>1$, put $B_\rho=\mathcal{S}_0\cap(\mathcal{V}\setminus\mathcal{V}_\rho)$. It follows from Lemma 2 that there exists a solution u_ρ of the Dirichlet problem

$$\begin{cases} H(u)=0 & \text{in } B_\rho \\ u=0 & \text{on } \partial B_\rho \end{cases} \tag{4.20}$$

and u_ρ satisfies

$$\phi^-(x)\leqq u_\rho(x)\leqq \phi^+(x) \tag{4.21}$$

for all $x\in B_\rho$.

From (4.3), (4.4) and (4.13) we then obtain that for a sufficiently large constant $R_0\geqq\bar{R}$

$$w_{R_0}(R_0)\geqq\left(\frac{\pi}{2}-\tan^{-1}R_0\right)-2R_0^{-1-\varepsilon/10}>C_7R_0^{-1},$$

and hence

$$-w_{R_0}(R_0)<\phi^-(R_0)<\phi^+(R_0)<w_{R_0}(R_0). \tag{4.22}$$

It follows from (4.21) and (4.22) that for each $\rho>R_0$

$$-w_{R_0}(R_0)<u_\rho(x)<w_{R_0}(R_0) \tag{4.23}$$

for all $r(x)=R_0$. Since for each $\tau\in\boldsymbol{R}$ $w_{R_0}+\tau$ has properties similar to that of w_{R_0} in Proposition 2 and $H(w_{R_0}+\tau)>H_M=0$ in $B_\rho\cap\mathcal{V}_{R_0}$, the maximum principle shows that the function $u_\rho-w_{R_0}$ attains a maximum on $\{x\in B_\rho;\ r(x)=R_0\}$ or ∂M. It then follows from (4.23) and $\partial M\subset\mathcal{S}_0$ that for all $x\in B_\rho\cap\mathcal{V}_{R_0}$

$$(u_\rho-w_{R_0})(x)\leqq 0. \tag{4.24}$$

Applying the same argument to $-w_{R_0}$ gives that for all $x\in B_\rho\cap\mathcal{V}_{R_0}$

$$(u_\rho+w_{R_0})(x)\geqq 0. \tag{4.25}$$

It then follows from (3.10) and (4.24), (4.25) that for each $\rho>R_0$, u_ρ satisfies

$$\sup_{B_\rho}\{r^{1+\varepsilon}|u_\rho|+\nu(u_\rho)\}\leqq C, \tag{4.26}$$

where C is independent on ρ. Using the Schauder estimates (cf. [**11**]) we obtain that u_ρ is smooth and its derivatives are estimated by constants, which are independent on ρ, on every compact domain in $\mathcal{S}_0$. Then there exist a subsequence $\{u_{\rho_i}\}_{i\in N}\subset\{u_\rho\}_{\rho>R_0}$ and an entire function $u\in C^\infty(\mathcal{S}_0)$ such that u_{ρ_i} converges uniformly to u on every compact domain in $\mathcal{S}_0$ when $i\to\infty$. From (4.20) and (4.26), u is spacelike and $H(u)=0$. This completes the proof of Theorem.

§ 5. Final remarks.

In this section, we show that (H_1^4, h) satisfies the global future barrier condition, and that entire maximal hypersurfaces of (H_1^4, h) have a gap property.

The radial mean curvature equation in (H_1^4, h) for $v=v(r)$

$$-r^{-2}[r^2a^3v'(1-a^4v'^2)^{-1/2}]' = H^* \tag{5.1}$$

can be solved with $H^*=K(>0)$. Then a solution (5.1) is given by

$$v'(r) = -\frac{K}{3}ra^{-2}\left(a^2+\left(\frac{K}{3}\right)^2r^2\right)^{-1/2} = -a^{-2}\left(1+a^2\left(\frac{3}{K}\right)^2r^{-2}\right)^{-1/2}, \tag{5.2}$$

$$v(r) = \int_r^\infty a^{-2}\left(1+a^2\left(\frac{3}{K}\right)^2s^{-2}\right)^{-1/2}ds. \tag{5.3}$$

From (5.3) we then obtain

$$0 \leqq v(r) \leqq \left(1+\left(\frac{3}{K}\right)^2\right)^{-1/2}\int_r^\infty(1+s^2)^{-1}ds \leqq r^{-1}\left(1+\left(\frac{3}{K}\right)^2\right)^{-1/2}. \tag{5.4}$$

It follows from (5.1) and (5.4) that (H_1^4, h) satisfies the global future barrier condition.

Also using the function v we have the following

PROPOSITION 3. *Let M be an entire maximal hypersurface in (H_1^4, h). Suppose that its height function u satisfies the following decay*

$$r^{1+\lambda}|u| \leqq C, \tag{5.5}$$

where λ is a positive constant. Then, $M=\mathcal{S}_0$.

PROOF. We first note that (H_1^4, h) satisfies the timelike convergence condition (cf. [3], [**12**]), that is, $\mathrm{Ric}(X, X)\geqq 0$ for every timelike vector $X\in TH_1^4$.

For each $K>0$ consider the function v defined by (5.3). From (5.5) there exists a constant R_1 such that $u(x)\leqq v(r(x))$ for all $x\in\{p\in\mathcal{S}_0;\ r(p)\geqq R_1\}$. From (5.1) and the timelike convergence condition we can show easily (e.g. [**10**, Lemma 7.2]) $u(x)\leqq v(r(x))$ for all $x\in\mathcal{S}_0$. Letting $K\to 0$ in (5.4), we obtain that $u(x)\leqq 0$ for all $x\in\mathcal{S}_0$. Applying the same argument to $-v$, we also obtain that $u(x)\geqq 0$ for all $x\in\mathcal{S}_0$. Hence $M=\mathcal{S}_0$.

REMARK. For each $K>0$ the hypersurface defined by (5.3) satisfies

$$1 \leqq \nu(v)(x) \leqq 2^{1/2}$$

for all $r(x)\geqq 1$ and

$$\lim_{r(x)\to\infty} \nu(v)(x) = 1.$$

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