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On the semisimplicity of Hecke algebras

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0. Let (W, S) be a Coxeter system [**2**], t an indeterminate, $q=t^2$, and $H(W, t)$ a free $\boldsymbol{C}[t]$-module with a basis $\{T(w)\}_{w\in W}$ parametrized by the elements of W. Here $\boldsymbol{C}$ denotes the field of complex numbers. Then $H(W, t)$ has an associative $\boldsymbol{C}[t]$-algebra structure characterized by the conditions

$$(T(s)+1)(T(s)-q)=0, \qquad \text{if } s\in S,$$

and

$$T(w)T(w')=T(ww'), \qquad \text{if } l(w)+l(w')=l(ww'),$$

where l is the length function [**2**]. See [**2**; Chap. 4, §2, Ex. 23] for the algebra structure of $H(W, t)$. See [**5**] for the significance of $H(W, t)$ in the representation theory. Let α be a complex number, $\varphi_\alpha: \boldsymbol{C}[t]\to\boldsymbol{C}$ the $\boldsymbol{C}$-algebra homomorphism defined by $\varphi_\alpha(t)=\alpha$, and $H(W, \alpha)=H(W, t)\otimes_{\boldsymbol{C}[t]}(\boldsymbol{C}, \varphi_\alpha)$.

From now on, we assume that W is finite, and (except in the final remark) not of type $A_1\times\cdots\times A_1$. Let w_0 be the longest element of W, $N=l(w_0)$, and $G(q)=q^N\sum_{w\in W}q^{l(w)}$.

The purpose of this note is to prove the following theorem.

THEOREM. *The $\boldsymbol{C}$-algebra $H(W, \alpha)$ is semisimple if and only if $G(\alpha^2)\neq 0$.*

1. Let

$$R_i: H(W, t)\longrightarrow M_{n_i\times n_i}(\boldsymbol{C}[t]) \qquad (i=1, 2)$$

be $\boldsymbol{C}[t]$-algebra homomorphisms. Here $M_{m\times n}$ denotes the set of $m\times n$-matrices. Let $T\hat{}(w)=q^{N-l(w)}T(w^{-1})$, $A\in M_{n_1\times n_2}(\boldsymbol{C}[t])$ and

$$B=\sum_{w\in W}R_1(T(w))AR_2(T\hat{}(w)).$$

LEMMA. *For $x\in W$, $R_1(T(x))B=BR_2(T(x))$.*

PROOF. We may assume that $x=s\in S$. Let $X=\{w\in W\,|\,l(sw)>l(w)\}$. Since W is a disjoint union of X and sX, it is enough to prove that

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$$R_1(T(s))\{R_1(T(w))AR_2(T\hat{\ }(w))+R_1(T(sw))AR_2(T\hat{\ }(sw))\}$$
$$=\{R_1(T(w))AR_2(T\hat{\ }(w))+R_1(T(sw)AR_2(T\hat{\ }(sw))\}R_2(T(s))$$

holds for all $w\in X$. The verification of this equality is easy and omitted.

2. Let $K=\boldsymbol{C}(t)$. Assume that $R_i\otimes K\ (i=1, 2)$ are irreducible representations.

If $R_1\otimes K$ is not isomorphic to $R_2\otimes K$, then as a consequence of the above lemma, we have

$$\sum_w R_1(T(w))_{ij}R_2(T\hat{\ }(w))_{kl}=0, \qquad \text{for all } i, j, k, l.$$

Here $(\ )_{ij}$ etc. mean matrix components. Hence if we put $\chi_i=\operatorname{trace} R_i$,

$$\sum_w \chi_1(T(w))\chi_2(T\hat{\ }(w))=0.$$

3. Assume that $R_1\otimes K=R_2\otimes K=R\otimes K$, and is irreducible. Let $\chi_1=\chi_2=\chi$ and $d_\chi=d_\chi(q)$ be the generic degree of R: The generic degree is characterized by the conditions

$$d_\chi(1)=\chi(1),$$

and

$$d_\chi(1)/d_\chi(q)=G(q)^{-1}\sum_{w\in W}\chi(T(w))\chi(T\hat{\ }(w)).$$

4. We now show the following well known result. However the proof given here seems to be simpler than known ones.

LEMMA. *The K-algebra $H(W, t)\otimes K$ is semisimple.*

PROOF. Assume that there is a non-zero element h of the Jacobson radical of $H(W, t)\otimes K$. By multiplying an element of $\boldsymbol{C}[t]$, we may assume that $h\in H(W, t)$. Furthermore, dividing by a power of $t-1$, we may assume that h is not contained in $(t-1)H(W, t)$. Let $\varphi: H(W, t)\to\boldsymbol{C}W$ be the $\boldsymbol{C}$-algebra homomorphism characterized by $\varphi(t)=1$ and $\varphi(T(w))=w$. Then for any $(c_{xy})\in\boldsymbol{C}^{W\times W}$,

$$\sum_{x,y}c_{xy}x\varphi(h)y=\varphi(\sum_{x,y}c_{xy}T(x)hT(y))$$

is a nilpotent element. Hence $\varphi(h)$ is contained in the Jacobson radical of $\boldsymbol{C}W$. Hence $\varphi(h)=0$ and $h\in\ker(\varphi)=(t-1)H(W, t)$, which contradicts our assumption. Hence the Jacobson radical of $H(W, t)\otimes K$ is zero.

5. For two linear functionals φ_1, φ_2 of $H(W, t)\otimes K$, let

$$\langle\varphi_1, \varphi_2\rangle=G(q)^{-1}\sum_{w\in W}\varphi_1(T(w))\varphi_2(T\hat{\ }(w)).$$

THEOREM. *Let $\chi_1, \cdots, \chi_n$ be the irreducible characters of $H(W, t)\otimes K$. Then*

$$\langle \chi_i, \chi_j \rangle = \begin{cases} d_{\chi_i}(1)/d_{\chi_i}(q), & \text{if } i=j, \\ 0, & \text{otherwise.} \end{cases}$$

6. (i) The above theorem can be considered as a q-analogue of the first orthogonality relation of the character values of a finite group. Note that in the case where W is a Weyl group, this formula was obtained in [**3**; (2.4)].

(ii) It is known that K is a splitting field for $H(W, t)\otimes K$. See [**6**], its references, and [**1**].

(iii) The generic degrees d_χ are calculated explicitly. See [**1**] and its references. From these calculations, we can see that $d_\chi(q)$ is always a polynomial in q. If W is a Weyl group, this phenomenon can be explained by the following fact: If q_0 is a prime power, then $d_\chi(q_0)$ is a degree of an irreducible representation of a finite Chevalley group, and, is an integer. But no unified explanation of this phenomenon (including the cases of type H_3, H_4 and $I_2(p)$) seems to be known.

7. Define a linear functional δ on $H(W, t)\otimes K$ by

$$\delta(T(x)) = \begin{cases} \sum_w q^{l(w)}, & \text{if } x=1, \\ 0, & \text{if } x\neq 1. \end{cases}$$

The following equality can be proved easily.

$$\delta(T(x)T\check{\ }(y)) = \begin{cases} G(q), & \text{if } x=y, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\delta(hh')=\delta(h'h)$ for $h, h'\in H(W, t)\otimes K$. Since $H(W, t)\otimes K$ is semisimple and since K is a splitting field for it, δ can be expressed as a linear combination

$$\delta = \sum_{i=1}^{n} c_i \chi_i \qquad (c_i\in K).$$

By the orthogonality relation, we have

$$\langle \delta, \chi_i \rangle = c_i d_{\chi_i}(1)/d_{\chi_i}(q).$$

On the other hand

$$\langle \delta, \chi_i \rangle = G(q)^{-1} \sum_{w\in W} \delta(T(w))\chi_i(T\check{\ }(w)) = \chi_i(1) = d_{\chi_i}(1).$$

Hence

(7.1)
$$\delta = \sum_{i=1}^{n} d_{\chi_i}(q)\chi_i.$$

8. Let $\delta_\alpha=\delta|_{t\to\alpha}$ and $\chi_{i,\alpha}=\chi_i|_{t\to\alpha}$. Here $|_{t\to\alpha}$ means the specialization $t\to\alpha$, which is possible since the values of δ and χ_i on $H(W, t)$ are polynomials in t. (This fact can be proved by a standard argument on representations over quotient fields of principal ideal domains, and by (7.1). Furthermore, using the notion of W-graphs, the first author [3] proved that all the values of χ_i at $T(w)$ are polynomials in t whose coefficients are algebraic integers.) We can also show that $\chi_{i,\alpha}$ is a trace of some representation of $H(W, \alpha)$. By (7.1), we get

$$\delta_\alpha = \sum_{i=1}^{n} d_{\chi_i}(\alpha^2)\chi_{i,\alpha}. \tag{8.1}$$

Let $\mathrm{rad}\, H(W, \alpha)$ be the Jacobson radical of $H(W, \alpha)$. Since $\mathrm{rad}\, H(W, \alpha)$ is nilpotent, $\chi_{i,\alpha}(\mathrm{rad}\, H(W, \alpha))=0$. Hence by (8.1), $\delta_\alpha(\mathrm{rad}\, H(W, \alpha))=0$.

9. LEMMA. *Assume that $G(\alpha^2)\neq 0$. Let h be an element of $H(W, \alpha)$. If $\delta_\alpha(hT^\wedge(x))=0$ for any x in W, then $h=0$.*

PROOF. Let $h=\sum_{x\in W} c(x)T(x)$ with $c(x)\in \boldsymbol{C}$. Then

$$0 = \delta_\alpha(hT^\wedge(x)) = c(x)G(\alpha^2).$$

Hence $c(x)=0$ for any x in W. Hence $h=0$.

10. PROOF OF THEOREM ("if part"). Assume that $G(\alpha^2)\neq 0$ and $h\in \mathrm{rad}\, H(W, \alpha)$. Then for any $x\in W$, we have $hT^\wedge(x)\in \mathrm{rad}\, H(W, \alpha)$. Hence

$$\delta_\alpha(hT^\wedge(x)) = 0, \qquad \text{for any } x\in W,$$

and $h=0$. Hence $\mathrm{rad}\, H(W, \alpha)=0$, i.e., $H(W, \alpha)$ is semisimple.

11. PROOF OF THEOREM ("only if part"). First, let us consider the case where $\sum_w \alpha^{2l(w)}=0$. Assume that $H(W, \alpha)$ is semisimple. Define a linear function ind on $H(W, \alpha)$ by ind $T(w)=\alpha^{2l(w)}$. Then as is easily seen, ind is a linear character of $H(W, \alpha)$. Let E be the primitive idempotent corresponding to ind. This E satisfies

$$T(s)E = \alpha^2 E \qquad \text{for } s\in S.$$

Hence

$$E = c\sum_{w\in W} T(w)$$

with a non-zero constant $c\ (\in \boldsymbol{C})$. But then we get the equality

$$E = E^2 = c\sum_{w\in W} T(w)E = c\sum_{w\in W} \alpha^{2l(w)}E = 0,$$

which is absurd.

To consider the remaining case, we assume $\alpha=0$. Arrange the elements of W in a sequence $w_1, w_2, \cdots$ so that $l(w_1)\geqq l(w_2)\geqq \cdots$. For any $s\in S$, let $\{a_{ij}\}_{ij}$

be complex numbers such that $T(s)T(w_j)=\sum_i T(w_i)a_{ij}$. Then (a_{ij}) is an upper triangular matrix. In fact

$$T(s)T(w)=\begin{cases} T(sw), & \text{if } l(sw)>l(w), \\ -T(w), & \text{if } l(sw)<l(w). \end{cases}$$

Hence every irreducible representation of $H(W, 0)$ is one dimensional. Since we are assuming that W is not of type $A_1\times\cdots\times A_1$, there are two elements s, s' of S such that $ss'\neq s's$. Then $T(s)T(s')-T(s')T(s)$ $(\neq 0)$ is contained in the Jacobson radical of $H(W, 0)$. Hence $H(W, 0)$ is not semisimple.

12. REMARK. Let us consider the excluded case where W is of type $A_1\times\cdots\times A_1$ (l factors).

Since $H(W, \alpha)$ is commutative, it is semisimple if and only if it has 2^l $(=\dim H(W, \alpha))$ linear characters. Note that every linear character φ of $H(W, \alpha)$ satisfies $(\varphi(s)+1)(\varphi(s)-\alpha^2)=0$ for $s\in S$. For each subset I of S, a linear functional φ_I of $H(W, \alpha)$ given by

$$\varphi_I(T(s))=\begin{cases} -1, & \text{if } s\in I, \\ \alpha^2, & \text{otherwise,} \end{cases}$$

is in fact a character of $H(W, \alpha)$, and thus $\{\varphi_I \mid I\subset S\}$ is the totality of the linear characters. Hence the following conditions are equivalent:

(1) $H(W, \alpha)$ is semisimple.

(2) $\varphi_I\neq\varphi_J$ if $I\neq J$.

(3) $\alpha^2\neq -1$.

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