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Coisotropic calculus and Poisson groupoids

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Introduction.

Lagrangian submanifolds play a special role in the geometry of symplectic manifolds. From the point of view of quantization theory, or simply a categorical approach to symplectic geometry [**Gu-S2**], [**W3**], lagrangian submanifolds are the "elements" of symplectic manifolds. Since the canonical transformations between symplectic manifolds P_1 and P_2 are those whose graphs are lagrangian in $P_2\times P_1^-$ (the "$^-$" indicating that the symplectic structure on P_1 has been multiplied by -1), one calls arbitrary lagrangian submanifolds of a product $P_2\times P_1^-$ *canonical relations*. It turns out that, under a transversality or clean intersection assumption, the composition of canonical relations is again canonical. Thus the canonical relations can be taken as morphisms in a symplectic "category"; the quotation marks, which are present because of the difficulties raised by the transversality condition, can be removed if we restrict attention to symplectic vector spaces and linear canonical relations.

The purpose of this paper is to extend the lagrangian calculus from symplectic to Poisson manifolds, i. e., manifolds foliated by symplectic manifolds of varying dimensions. The notion of lagrangian submanifold becomes less useful in this case (it is not even so clear how to define it when the dimension of the symplectic leaves jumps), and in fact it is the *coisotropic* submanifolds which will play the essential role. A closed submanifold C of a Poisson manifold P is coisotropic if, for every function $f\in C^\infty(P)$ vanishing on C, the hamiltonian vector field X_f is tangent to C; equivalently, the set $I_C=\{f\in C^\infty(P)|f_{|C}\equiv 0\}$, an ideal with respect to multiplication, is required to be a subalgebra for the Poisson bracket. The latter definition is purely algebraic and shows that the notion of coisotropic extends as far as Poisson *algebras*.

Our main results are as follows.

1) The graph of $f\colon P_1\to P_2$ is coisotropic in $P_2\times P_1^-$ if and only if f is a Poisson map.

2) Under suitable clean intersection assumptions, the composition of coiso-

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tropic relations is coisotropic.

3) If $\phi: P \to Q$ is a submersion, P a Poisson manifold, then $\phi^*(C^\infty(Q))$ is a Poisson subalgebra if and only if the equivalence relation $\phi^{-1} \circ \phi$ is coisotropic.

4) If $C \subset P$ is coisotropic, the normal bundle $T_C P/TC$ carries a natural Poisson structure "dual" to a Lie algebroid structure on the conormal bundle $TC^\perp \subset T_C^* P$; the latter is inherited in a natural way from the Lie algebroid structure on T^*P associated to the Poisson structure.

As an application of the coisotropic calculus, we derive the basic properties of Poisson groupoids. A Poisson groupoid is a Lie groupoid G with a Poisson structure for which the graph of multiplication is coisotropic in $G \times G^- \times G^-$. The special cases where the groupoid is a group or the Poisson manifold is symplectic correspond precisely to the Poisson groups of Drinfel'd [**D**] and the symplectic groupoids of Karasev [**K**] and the author [**Ct-D-W**], [**W5**].

We give a new proof and generalization of the reduction theorem of Semenov-Tian-Shansky [**S**]. In addition, extending Drinfel'd's work in the case of Poisson groups, we begin a study of the duality of Poisson groupoids, a concept which reveals new connections between familiar examples.

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1. Composition of Poisson relations.

(**1.1**) **Poisson structures.** We will be using Poisson structures on various levels, so we recall the definitions here.

(1.1.1) DEFINITION. (a) A *Poisson structure on a commutative algebra* A is a Lie algebra structure $\{\ ,\ \}$ on A such that, for each element $h \in A$, the operator $X_h: f \mapsto \{f, h\}$ is a derivation of the (commutative) multiplication.

(b) A *Poisson structure on a vector space* V is a skew-symmetric bilinear form on V^*.

(c) A *Poisson structure on a (smooth) vector bundle* E is a (smooth) field of Poisson structures on the fibres of E.

(d) A *Poisson structure on a manifold* P is a Poisson structure on the commutative algebra $C^\infty(P)$.

(1.1.2) DEFINITION. A commutative algebra [manifold, vector space, vector bundle] with a Poisson structure is called a Poisson algebra [manifold, vector space, vector bundle].

There are various well known connections between the objects defined in (1.1.1). A Poisson structure on the manifold P defines a Poisson structure π on the vector bundle TP by

$$\pi(df, dg) = \{f, g\}. \tag{1.1.3}$$

Conversely, a Poisson structure on TP defines a bracket operation on $C^\infty(P)$ by (1.1.3) which is a Poisson structure on P if and only if the Schouten bracket $[\pi, \pi]$ of π with itself vanishes. A Poisson structure on the vector space V defines a ("constant") Poisson structure on V considered as a manifold.

(1.2) **Coisotropic subobjects.** The most useful subobjects of Poisson structures turn out to be those which we call coisotropic.

(1.2.1) DEFINITION. (a) A *coisotrope* in a Poisson algebra A is a subset I which is an ideal for the commutative algebra structure and a subalgebra for the Lie algebra structure.

(b) A subspace W of a Poisson vector space V is *coisotropic* if the annihilator $W^\perp \subset V^*$ is isotropic for the Poisson structure, i.e., $\pi(\omega_1, \omega_2)=0$ whenever $\omega_1, \omega_2 \in W^\perp$.

(c) A subbundle F of a Poisson vector bundle E is *coisotropic* if $F^\perp$ is isotropic in E^*.

(d) A submanifold M of a Poisson P is *coisotropic* if TM is coisotropic in the restricted tangent bundle T_MP, i.e., if each tangent space T_mM is coisotropic in T_mP.

(1.2.2) PROPOSITION. *Let M be a closed submanifold of a Poisson manifold P. Then the following conditions are equivalent.*

(a) *M is coisotropic.*

(b) *The ideal $I_M \subset C^\infty(P)$ consisting of functions which are zero on M is a coisotrope.*

(c) *For every $h \in I_M$, the hamiltonian vector field X_h is tangent to M.*

PROOF. If f and g belong to I_M, then the restriction to M of df and dg are sections of $TM^\perp \subset T_M^*P$. Using (1.1.3), we find that $\{f, g\} \in I_M$ if and only if $\pi(df_{|M}, dg_{|M})=0$. Since all sections of $TM^\perp$ are realizable in this way, we have the equivalence of (a) and (b).

Noting that a vector field on P is tangent to M if and only if it annihilates I_M, we conclude immediately that (b) and (c) are equivalent. □

(1.2.3) REMARK. The notion of coisotropic makes sense for submanifolds which are locally closed or even immersed. In the locally closed case, (1.2.2) is still valid since M is dense in the set $\bar{M}$ where all the elements of I_M vanish.

On the other hand, for an immersed submanifold such as a dense leaf of a foliation, (1.2.2) fails because I_M is too small to detect any properties of M.

We note that a subspace $W \subset V$ is coisotropic if and only if it is coisotropic as a submanifold. In this setting as well as that of vector bundles, we have the following easily verified characterization of coisotropicity.

(1.2.4) PROPOSITION. *Let E be a Poisson vector bundle and $\tilde{\pi}: E^* \to E$ the operator inducing the Poisson structure, defined by $\langle \omega_1, \tilde{\pi}\omega_2 \rangle = \pi(\omega_1, \omega_2)$. Then the subbundle $F \subset E$ is coisotropic if and only if $\tilde{\pi}(F^\perp) \subset F$.*

It is also possible to study coisotropic objects by reducing to the symplectic case, in the following way. If V is a Poisson vector space, then $\tilde{\pi}(V^*)$ inherits a nondegenerate Poisson structure. In fact, π passes naturally to a form Ω on $V^*/\operatorname{Ker}\tilde{\pi} \cong \tilde{\pi}(V^*)$. Since Ω is nondegenerate, i.e., Ω is a *symplectic structure* on $\tilde{\pi}(V^*)$, the inverse $\tilde{\Omega}^{-1}$ defines a nondegenerate Poisson structure π_s on $\tilde{\pi}(V^*)$.

(1.2.5) PROPOSITION. *Let W be a subspace of the Poisson vector space V. Then the following conditions are equivalent:*

(a) *W is coisotropic in V.*

(b) *$W \cap \tilde{\pi}(V^*)$ is coisotropic in V.*

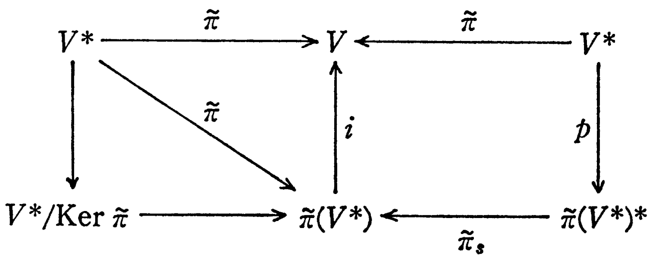
(c) *$W \cap \tilde{\pi}(V^*)$ is coisotropic in $\tilde{\pi}(V^*)$ with respect to π_s.*

PROOF. (b) implies (a) because any subspace containing a coisotropic one is coisotropic. Next, we note that

$$\tilde{\pi}([W \cap \tilde{\pi}(V^*)]^\perp) = \tilde{\pi}(W^\perp + \tilde{\pi}(V^*)^\perp) = \tilde{\pi}(W^\perp + \operatorname{Ker}\tilde{\pi}) = \tilde{\pi}(W^\perp).$$

If W is coisotropic, $\tilde{\pi}(W^\perp) \subset W$. Since $\tilde{\pi}(W^\perp)$ is obviously contained in $\tilde{\pi}(V^*)$ as well, we conclude that (a) and (b) are equivalent.

Now let $i: \tilde{\pi}(V^*) \to V$ be the inclusion and denote its dual by $p: V^* \to \tilde{\pi}(V^*)^*$. For a subspace $U \subset \tilde{\pi}(V^*)$, its annihilators $U^\perp$ and U^0 in V^* and $\tilde{\pi}(V^*)^*$ respectively are related by $U^\perp = p^{-1}(U^0)$. From the commutativity of the diagram,



we conclude that $\tilde{\pi}(U^\perp) = \tilde{\pi}(p^{-1}U^0) = i \circ \tilde{\pi}_s(U^0) = \tilde{\pi}_s(U^0)$. So $\tilde{\pi}(U^\perp) \subset U$ if and only if $\tilde{\pi}_s(U^0) \subset U$, from which we have the equivalence of (b) and (c). $\square$

Applying Proposition 1.2.5 pointwise, we have:

(1.2.6) COROLLARY. *Let M be a submanifold of the Poisson manifold P which has clean intersection with each symplectic leaf of P. Then M is coisotropic if and only if its intersection with each symplectic leaf is coisotropic in P, or, equivalently, in the symplectic leaf.*

(**1.3**) **Poisson relations.** Let P_1 and P_2 be any sets; the graph of a transformation $f: P_2 \to P_1$ will be defined (contrary to the most common convention) as $\{(f(y), y) | y \in P_2\}$. Accordingly, if $R \subset P_1 \times P_2$ and $S \subset P_2 \times P_3$ are subsets, considered as relations $R: P_2 \to P_1$ and $S: P_3 \to P_2$, the composite relation $R \circ S: P_3 \to P_1$ has as graph

$$\{(x, z) \in P_1 \times P_3 |\ (x, y) \in R \text{ and } (y, z) \in S \text{ for some } y \in P_2\}.$$

(1.3.1) DEFINITION. If P_1 and P_2 are Poisson manifolds, a *Poisson relation* $R: P_2 \to P_1$ is a coisotropic submanifold of the product $P_1 \times P_2^-$. Poisson relations between Poisson vector spaces and bundles are defined analogously.

We will see in the next subsection that the graph of a mapping is coisotropic if and only if the mapping is Poisson. For now, we go right ahead to consider the composition of Poisson relations. Following the pattern set by Guillemin [**Gu**] in the symplectic/lagrangian case, we begin with the linear case.

(1.3.2) PROPOSITION. *Let W_1 and W_2 be Poisson vector spaces, R and C coisotropic subspaces of $W_1 \times W_2$ and W_2 respectively. Then*

$$R(C) = \{x \in W_1 | (x, y) \in R \quad \textit{for some} \quad y \in C\}$$

is coisotropic in W_1.

PROOF. Let $\tilde{\pi}$ denote the product Poisson structure on $W = W_1 \times W_2$. Then $\tilde{\pi}(W^*) = \tilde{\pi}_1(W_1^*) \times \tilde{\pi}_2(W_2^*)$. Let $R' = R \cap \tilde{\pi}(W^*)$ and $C' = C \cap \tilde{\pi}_2(W_2^*)$. By Proposition 1.2.5, R' and C' are coisotropic in the nondegenerate Poisson vector spaces $\tilde{\pi}_1(V_1^*) \times \tilde{\pi}_2(V_2^*)$ and $\tilde{\pi}_2(V_2^*)$ respectively. Since $R(C)$ contains $R'(C')$, it suffices to show that $R'(C')$ is coisotropic in $\tilde{\pi}_1(V_1^*)$; i.e., we are reduced to the symplectic case.

In a symplectic vector space, a subspace is coisotropic if and only if it contains a lagrangian subspace, so we are reduced to the case where V_1 and V_2 are symplectic and R and C are lagrangian. But this case is just Lemma 1, p. 26, in [**Gu**]. □

(1.3.3) REMARK. A direct proof in the Poisson case can be given based on the following lemma from linear algebra: Let $R^* = \{(\omega_1, \omega_2) \in W_1^* \times W_2^* | \langle \omega_1, x_1 \rangle = \langle \omega_2, x_2 \rangle$ for all $(x_1, x_2) \in R\}$; then $R(C)^\perp = R^*(C^\perp)$.

We now have:

(1.3.4) THEOREM. *Let V_1, V_2 and V_3 be Poisson vector spaces, $S: V_2 \to V_1$ and $T: V_3 \to V_2$ linear Poisson relations. Then $S \circ T: V_3 \to V_1$ is a Poisson relation.*

PROOF. Let $W_1=V_1\times V_3^-\times V_2^-\times V_3$, $W_2=V_2^-\times V_3$, $C=T$ and $R=\{(x_1, x_3, x_2, x_3)|(x_1, x_2)\in S\}$. C is coisotropic because T is (changing the sign of the Poisson structure in $V_2\times V_3^-$ does not cause any problem); R is, up to reordering of factors, the product of $S\subset V_1\times V_2^-$ and the diagonal $\Delta_{V_3}\subset V_3^-\times V_3$, which is seen to be coisotropic by a direct calculation.

Applying Proposition 1.3.2, we find that $R(C)$ is coisotropic. But,

$$\begin{aligned} R(C) &= \{(x_1, x_3)\in V_1\times V_3 |\ (x_1, x_3, x_2, x_3)\in R \text{ and } (x_2, x_3)\in T \text{ for some } x_2\in V_2\} \\ &= \{(x_1, x_3)\in V_1\times V_3 |\ (x_1, x_2)\in S \text{ and } (x_2, x_3)\in T \text{ for some } x_2\in V_2\} \\ &= S\circ T. \qquad \square \end{aligned}$$

Theorem 1.3.4, together with the obvious statements about identities and inverses, implies that the Poisson vector spaces and Poisson relations form a category containing that of symplectic vector spaces and canonical relations (see [**Gu-S2**], [**W3**]).

To extend Theorem 1.3.4 to vector bundles, we need to make an assumption to insure that the composition is smooth.

(1.3.5) COROLLARY. *Let E_1, E_2, E_3 be Poisson vector bundles, $S: E_2\to E_1$ and $T: E_3\to E_2$ linear Poisson relations (i.e., coisotropic subbundles of $E_1\times E_2^-$ and $E_2\times E_3^-$ respectively). If $S\circ T\subset E_1\times E_3^-$ is a subbundle, it is a Poisson relation.*

(1.3.6) REMARKS. (a) A sufficient condition for $S\circ T$ to be a subbundle is that $S\times T\cap E_1\times\Delta_{E_2}\times E_3$ have constant dimension in $E_1\times E_2\times E_2\times E_3$.

(b) The condition that $S\circ T$ must be a subbundle prevents the Poisson vector bundles and relations from forming a category.

Going on to manifolds, we need to make the usual assumptions to insure that composition behaves nicely.

(1.3.7) DEFINITION. Let $S: P_2\to P_1$ and $T: P_3\to P_2$ be relations.

(a) We say that S and T form a clean pair if

(i) the submanifolds $Q=S\times T$ and $D=P_1\times\Delta_{P_2}\times P_3$ intersect cleanly in $P_1\times P_2\times P_2\times P_3$; i.e., $Q\cap D$ is a submanifold with $T(Q\cap D)=T_{Q\cap D}Q\cap T_{Q\cap D}D$;

(ii) the restriction to $Q\cap D$ of the projection $pr_{13}\ P_1\times P_2\times P_2\times P_3\to P_1\times P$ has constant rank.

(b) If (i) and (ii) are satisfied, we call the pair (S, T) *very clean* if:

(iii) $S\circ T$ is a submanifold of $P_1\times P_3$; and

(iv) the map pr_{13} from $Q\cap D$ onto $S\circ T$ is a submersion.

(1.3.8) REMARK. In the symplectic/lagrangian case, (i) implies (ii) in Definition 1.3.7, but this is not so in general.

We can now state the main result of this section.

(1.3.9) THEOREM. *Let $S: P_2 \to P_1$ and $T: P_3 \to P_2$ be a very clean pair of Poisson relations. Then $S \circ T: P_3 \to P_1$ is a Poisson relation.*

PROOF. The assumption that (S, T) is a clean pair implies that, for each $(p_1, p_2) \in S$ and $(p_2, p_3) \in T$, the tangent space of $S \circ T$ at (p_1, p_3) equals $T_{(p_1, p_2)}S \circ T_{(p_2, p_3)}T$. Now Theorem 1.3.4 and Definition 1.2.1(d) imply that $S \circ T$ is coisotropic. □

(1.3.10) REMARK. Although the class of Poisson relations is closed under inversionand very clean composition, the difficulties connected with arbitrary composition prevent the Poisson relations from being the mappings of a category.

2. Examples of Poisson relations and composition.

(2.1) **Applying Poisson relations to coisotropic submanifolds.** Let O be a Poisson manifold with just one point (and the zero Poisson structure). Then $P \times O^- \cong P$ for any Poisson manifold P, and so the coisotropic submanifolds $C \subset P$ are just the Poisson relations from O to P.

(2.1.1) DEFINITION. Let $R: P_2 \to P_1$ be a relation and $C \subset P_2$ a submanifold. We say that C is in [very] clean position for R if $R: P_2 \to P_1$ and $C: O \to P_2$ are a [very] clean pair of relations.

(2.1.2) PROPOSITION. *If a coisotropic submanifold $C \subset P_2$ is in very clean position for a Poisson relation $R: P_2 \to P_1$, then $R(C)$ is a coisotropic submanifold of P_1.*

PROOF. There are two easy proofs available to us at this point. The first is to notice that $R(C) = R \circ C$, where C is considered as a Poisson relation $O \to P_2$, and then to apply Theorem 1.3.9. The second is to apply Proposition 1.3.2 pointwise. □

(2.2) **Poisson maps.**

(2.2.1) PROPOSITION. *Let P_1 and P_2 be Poisson manifolds, $i: M \to P_2$ the inclusion of any submanifold, and $\phi: M \to P_1$ a C^∞-mapping. Then the graph $R = (\mathrm{id}_{P_1} \times i)(\phi) = \{(\phi(x), x) \mid x \in M\} \subset P_1 \times P_2^-$ is a Poisson relation if and only if*

(2.2.2) *for any C^∞-functions f_1, h_1 on P_1 and f_2, h_2 on P_2 such that $\phi^* f_1 = i^* f_2$, $\phi^* h_1 = i^* h_2$, we have $\phi^*\{f_1, h_1\} = i^*\{f_2, h_2\}$.*

When these conditions hold, M is necessarily coisotropic.

PROOF. The ideal $I_M \subset C^\infty(P_1 \times P_2)$ of functions vanishing on R is generated by functions of the type $(f_1 - f_2)(x, y) = f_1(x) - f_2(y)$ for $f_1 \in C^\infty(P_1)$ and $f_2 \in C^\infty(P_2)$ such that $\phi^* f_1 = i^* f_2$.

On $P_1 \times P_2^-$, $\{f_1-f_2, h_1-h_2\}=\{f_1, h_1\}-\{f_2, h_2\}$, so $\{I_R, I_R\} \subset I_R$ if and only if condition (2.2.2) holds.

Finally, if R is coisotropic, so is $R^{-1}(P_1)=M$ by Proposition 2.1.2. In fact, the submanifolds $R^{-1} \times P_1=\{(x, \phi(x), y) | x \in M, y \in P_1\}$ and $P_2 \times \Delta_{P_1}=\{(z, w, w) | z \in P_2, w \in P_1\}$ intersect transversely and thus cleanly along $K=\{(x, \phi(x), \phi(x)) | x \in M\}$ in $P_2 \times P_1 \times P_1$, and the projection of K into P_2 is a diffeomorphism with M. □

(2.2.3) COROLLARY. *Let $\phi: P_2 \to P_1$ be a map between Poisson manifolds. Then ϕ is a Poisson map if and only if its graph is a Poisson relation.*

(2.2.4) REMARK. Another proof of Proposition 2.2.1 can be given by beginning with the linear version and applying it to the derivative $T\phi$. With our present approach, one may derive the linear case from the general one.

(2.2.5) COROLLARY. *Let $\phi: P_1 \to P_2$ be a Poisson map.*

(a) *If C is a coisotropic submanifold of P_2 such that $\phi^{-1}(C)$ is a submanifold with $T(\phi^{-1}C)=(T\phi)^{-1}(TC)$ (e.g. if ϕ is transverse to C), then $\phi^{-1}(C)$ is coisotropic.*

(b) *If D is a coisotropic submanifold of P_1 such that $\phi|_{P_1}$ is of constant rank and $\phi(D)$ is a submanifold of P_2, then $\phi(D)$ is coisotropic.*

PROOF. In each case, Proposition 2.1.2 applies, with $R=\phi^{-1}: P_2 \to P_1$ for (a) and $R=\phi$ for (b). □

(2.3) Equivalence relations and reduction. If $\phi: P \to Q$ is a (set theoretic) mapping, then $\phi^{-1} \circ \phi: P \to P$ is the equivalence relation defined by $x \sim y$ if and only if $\phi(x)=\phi(y)$.

(2.3.1) PROPOSITION. *Let P be a Poisson manifold, ϕ a submersion from P to the manifold Q. Then Q has a (unique) Poisson structure making P into a Poisson map if and only if $\phi^{-1} \circ \phi$ is a Poisson relation.*

PROOF. We note first that $\phi^{-1}: Q \to P$ and $\phi: P \to Q$ form a very clean pair of relations. In fact, $\phi^{-1} \times \phi=\{(x, \phi(x), \phi(y), y) | x, y \in P\}$ is transverse to $P \times \Delta_Q \times P$ because ϕ is a submersion. The intersection $\{(x, \phi(x), \phi(y), y) | \phi(x)=\phi(y)\}$ is then embedded in $P \times P$ by the projection. It follows from Theorem 1.3.9 and Corollary 2.2.3 that $\phi^{-1} \circ \phi$ is coisotropic if ϕ is a Poisson map.

The proof of the converse is similar to that of Proposition 2.2.1. We must show that $\phi^*(C^\infty(Q))$ is a subalgebra of $C^\infty(P)$ with respect to the Poisson bracket. For f_1 and f_2 in $C^\infty(Q)$, define $g_i(x, y)=f_i(\phi(x))-f_i(\phi(y))$ on $P \times P^-$. Both g_1 and g_2 vanish on the submanifold $\phi^{-1} \circ \phi \subset P \times P^-$; since $\phi^{-1} \circ \phi$ is coisotropic, so does their bracket

$$\{g_1, g_2\}(x, y) = \{f_1\circ\phi, f_2\circ\phi\}(x) - \{f_1\circ\phi, f_2\circ\phi\}(y).$$

Thus $\{f_1\circ\phi, f_2\circ\phi\}$ is constant on fibres of ϕ and so, since ϕ is a submersion, it lies in $\phi^*(C^\infty(Q))$. □

(2.3.2) REMARK. If instead of a fibration of the Poisson manifold P we merely have a foliation, then a local application of the proposition shows that the sheaf of germs of functions constant on leaves is closed under Poisson bracket if and only if the equivalence relation defined by the leaves of the foliation is a (possibly immersed) coisotropic submanifold of $P\times P^-$. In the symplectic case, this gives a new characterization of symplectically complete foliations [**L**].

A proof similar to that of Proposition 2.3.1, using the full strength of Proposition 2.2.1, leads to the following more general result.

(2.3.3) PROPOSITION. *Let $i: M\to P$ be the inclusion of a submanifold in the Poisson manifold P and let $\phi: P\to Q$ be a submersion. Then $(i\times i)(\phi^{-1}\circ\phi) = \{(x, y)\mid x, y\in M \text{ and } \phi(x)=\phi(y)\}$ is a Poisson relation from P to P if and only if M is coisotropic and*

(2.3.4) *there is a (unique) Poisson structure on Q such that, for any C^∞-functions f_1, f_2 on Q and h_1, h_2 on P such that $\phi^* f_i = i^* h_i$ we have $\phi^*\{f_1, f_2\} = i^*\{h_1, h_2\}$.*

In the language of [**Ma-R**], condition (2.3.4) says that the triple $(P, M, \operatorname{Ker} T\phi)$ is Poisson reducible; i.e., Poisson brackets on Q can be defined by pullback to M and extension to P, followed by restriction to M and push-forward to Q. Note that $\operatorname{Ker} T\phi$ must contain the characteristic "distribution" $\tilde{\pi}(TM^\perp)$ of the coisotropic submanifold M generated by the hamiltonian vector fields of I_M, but it could be larger. This is important because, except in the symplectic case, $\tilde{\pi}(TM^\perp)$ might not be a smooth subbundle of TM even if $\tilde{\pi}$ has constant rank.

3. Normal structures.

(3.1) Lie algebroid structure on the conormal bundle. The space $\Omega^1(P)$ of 1-forms on a Poisson manifold P is a Lie algebra with respect to the bracket [**Ct-D-W**], [**K**], [**Ko**]

(3.1.1) $$\{\omega_1, \omega_2\} = d[\pi(\omega_1, \omega_2)] - \tilde{\pi}\omega_1 \lrcorner\, d\omega_2 + \tilde{\pi}\omega_2 \lrcorner\, d\omega_1.$$

In fact, this bracket and the map $-\tilde{\pi}: T^*P\to TP$ make T^*P into a Lie algebroid [**Ct-D-W**], [**M**]; i.e. the map induced by $\tilde{\pi}$ on sections is a Lie algebra homomorphism to the vector fields and, for $f\in C^\infty(P)$, (3.1.2) $$\{\omega_1, f\omega_2\} = f\{\omega_1, \omega_2\} + (-\tilde{\pi}\omega_1 \rfloor df)\omega_2.$$

It turns out that this Lie algebroid structure can be restricted to the conormal bundle of any coisotropic submanifold.

(3.1.3) PROPOSITION. *Let $i: C \to P$ be the inclusion mapping of a coisotropic manifold. Then:*

(a) $\operatorname{Ker} i^*$ *is a subalgebra of the Lie algebra* $\Omega^1(P)$;

(b) $\{\omega \in \Omega^1(P) \mid \omega_{|C} = 0\}$ *is an ideal in* $\operatorname{Ker} i^*$.

PROOF. For any tangent vector $X \in TP$,

(3.1.4) $$\begin{aligned}\{\omega_1, \omega_2\}(X) &= d[\pi(\omega_1, \omega_2)](X) - (\tilde{\pi}\omega_1 \rfloor d\omega_2)(X) + (\tilde{\pi}\omega_2 \rfloor d\omega_1)(X) \\ &= d[\pi(\omega_1, \omega_2)](X) - d\omega_2(\tilde{\pi}\omega_1, X) + d\omega_1(\tilde{\pi}\omega_2, X).\end{aligned}$$

If $i^*\omega_1 = i^*\omega_2 = 0$, then $\pi(\omega_1, \omega_2) = 0$ on the coisotropic submanifold C, $i^*d\omega_1 = i^*d\omega_2 = 0$, and $\tilde{\pi}\omega_1$ and $\tilde{\pi}\omega_2$ are tangent to C. Then, for X tangent to C, all three terms on the right-hand side of (3.1.4) are zero; thus $i^*\{\omega_1, \omega_2\} = 0$, which proves (a).

To prove (b), we suppose that $\omega_{2|C} = 0$, but that X is merely in $T_C P$. Once again, we must show that $\{\omega_1, \omega_2\}(X) = 0$. This time, only the third term on the right-hand side of (3.1.4) is necessarily zero. Then we have

$$\begin{aligned}\{\omega_1, \omega_2\}(X) &= d[\pi(\omega_1, \omega_2)](X) - d\omega_2(\tilde{\pi}\omega_1, X) \\ &= (-d(\tilde{\pi}\omega_1 \rfloor \omega_2) - \tilde{\pi}\omega_1 \rfloor d\omega_2)(X) \\ &= -(\mathfrak{L}_{\tilde{\pi}\omega_1}\omega_2)(X).\end{aligned}$$

But, since $\tilde{\pi}\omega_1$ is tangent to C and $\omega_{2|C} = 0$, it follows that $\mathfrak{L}_{\tilde{\pi}\omega_1}\omega_{2|C} = 0$, so $\{\omega_1, \omega_2\}(X) = 0$. □

(3.1.5) COROLLARY. *The conormal bundle $N^*(C) = (T_C P)^\perp$ is a Lie algebroid with the Lie algebra structure given through Proposition* 3.1.3 *and the "anchor" $\rho: N^*(C) \to TC$ given by the restriction of $-\tilde{\pi}$.*

PROOF. The ideal $\{\omega \in \Omega^1(P) \mid \omega_{|C} = 0\}$ is the kernel of the restriction homomorphism from $\operatorname{Ker} i^*$ to *Sect* $N^*(C)$, so *Sect* $N^*(C)$ inherits a Lie algebra structure from that on $\Omega^1(P)$. The identity $\{\theta_1, f\theta_2\} = f\{\theta_1, \theta_2\} + (\rho\theta_1 \rfloor df)\theta_2$ for $\theta_1, \theta_2 \in$ *Sect* $N^*(C)$ and $f \in C^\infty(C)$ follows directly from (3.1.2). □

(3.1.6) EXAMPLES. (a) C is a Poisson submanifold if and only if $\rho = 0$; in this case $N^*(C)$ is just a Lie algebra bundle. When C is a single symplectic leaf, the fibres of $N^*(C)$ are the transverse Lie algebras [**W4**]; for a general Poisson submanifold C, the fibres of $N^*(C)$ are subalgebras of the transverse algebras.

(b) If P is a symplectic manifold, then $N^*(C)$ is naturally isomorphic to the characteristic distribution $\tilde{\pi}(TC^{\perp})$, $\rho: \tilde{\pi}(TC^{\perp}) \to TC$ is the inclusion, and the Lie algebra structure on $\tilde{\pi}(TC^{\perp})$ is the usual bracket of vector fields. We note that in this case $N^*(C)$ is the Lie algebroid of the holonomy groupoid [**E**] of the characteristic foliation of C (which may be a non-Hausdorff manifold; see [**Wi**]).

(3.2) **Linearized Poisson structure on the normal bundle.** Since the dual bundle A^* to a Lie algebroid A always carries a Poisson structure for which the inclusion $Sect(A) \subset C^{\infty}(A^*)$ is a Lie algebra homomorphism [**Ct-D-W**], [**Cu**], Corollary 3.1.5 has the following immediate consequence.

(3.2.1) PROPOSITION. *Let $C \subset P$ be a coisotropic submanifold. Then the normal bundle $N(C) = T_C P/TC$ has a natural Poisson structure which is linear in the sense that the functions $N(C) \to \boldsymbol{R}$ which are linear on fibres form a Lie subalgebra of $C^{\infty}(N(C))$.*

Since, as a manifold, $N(C)$ represents a tubular neighborhood of C in P, it is natural to try to compare the linear structure on $N(C)$ with that on P near C.

(3.2.2) DEFINITION. We call the Poisson structure on $N(C)$ in Proposition 3.2.1 the *linearization* at the coisotropic submanifold C of the structure on P.

(3.2.3) EXAMPLE. If $x \in P$ is a point at which the Poisson structure is zero, then $\{x\}$ is coisotropic, and the linearized structure on $N(\{x\}) = T_x P$ is just the one introduced in [**W4**]. It is known in various situations [**C1**], [**C2**], [**W4**] whether the structure on $T_x P$ near 0 is equivalent to the one on P near x.

In general, the Poisson structure on a neighborhood of C depends on more than the Lie algebroid $N^*(C)$. As a minimum of extra data, one must be given the Poisson structure transverse to the characteristic "distribution" $\rho(N^*C) = \tilde{\pi}(TC^{\perp}) \subset TC$ (cf. [**Cu**]). There is one case where we do not have to worry about a structure on C.

(3.2.4) DEFINITION. A submanifold C of a Poisson manifold P is *lagrangian* if $\tilde{\pi}(TC^{\perp}) = TC$.

Clearly C is lagrangian if and only if it is coisotropic and the Lie algebroid $A = N^*(C)$ is transitive, i.e., $\rho(A) = TC$. Equivalently, for each $x \in C$, the intersection of $T_x C$ with the tangent space $\tilde{\pi}(T_x^* P)$ to the symplectic leaf through x is lagrangian in $\tilde{\pi}(T_x^* P)$. (A slightly different definition may be found in [**Av**].)

(3.2.5) EXAMPLES. (a) If P is symplectic, this definition coincides with the usual one. Here $N^*(C) \cong TC$ and the linear Poisson manifold $N(C)$ is just

T^*C with its standard symplectic structure. In this case, we know that a neighborhood of the 0-section in T^*P is Poisson-isomorphic to a neighborhood of C in P [**W1**]; this last result may therefore be considered as a linearization theorem of the same nature as those cited in Example 3.2.3. (In that example, $\{x\}$ is a lagrangian submanifold.)

(b) If $G\to B\to C$ is a principal bundle, then $A=TB/G$ is a vector bundle over C admitting an exact sequence $0\longrightarrow g(B)\longrightarrow A\xrightarrow{\rho} TC\longrightarrow 0$, where $g(B)$ is the bundle associated to B by the adjoint representation. The sections of A are just the G-invariant vector fields on B (infinitesimal gauge transformations covering diffeomorphisms of C), so they form a Lie algebra for which ρ induces a homomorphism to the vector fields on B. In fact, A is a transitive Lie algebroid because ρ is surjective, and the natural Poisson structure on A^* (which fits into the dual exact sequence $0\to T^*C\to A^*\to g^*(B)\to 0$) is that of the phase space for a classical Yang-Mills particle on C with internal variable in g^*. This is nearly the most general example of a transitive Lie algebroid (see [**A-M**], [**M**] for a discussion of what is missed) and so provides a linearized model for the neighborhood of most lagrangian submanifolds in Poisson manifolds. Examples 3.2.3 and 3.2.5(a) correspond to the extreme cases in which either C or G reduces to a single point. It should be interesting to investigate the linearization problem in the intermediate situations.

At the other extreme to the lagrangian submanifolds lie the Poisson submanifolds. Among coisotropic submanifolds, they are characterized by the condition $\tilde{\pi}(TC^{\perp})=0$, i.e., $\rho=0$, the Lie algebroid $N^*(C)$ is a bundle of Lie algebras, and the Poisson manifold $N(C)$ is a bundle of duals of Lie algebras. In case C is a single symplectic leaf, these Lie algebras are all isomorphic. Once the symplectic structure on C is given, the comparison between the Poisson structures on P and $N(C)$ involves the variation of symplectic structure from leaf to leaf, as well as the linearization problem for the transverse structure at any point of C. (See [**Ml**], [**Mn**] for the case where C is a coadjoint orbit.)

For the linearization problem near coisotropic submanifolds of symplectic manifolds, see [**G**].

4. Poisson groupoids.

(**4.1**) **Lie groupoids.** We begin this section by recalling the definition of a Lie groupoid. (See [**Ct-D-W**] or [**M**] for details.)

(4.1.1) DEFINITION. A *groupoid* over a set G_0, called the *base*, is a set G equipped with:

(i) maps $\alpha, \beta: G \rightarrow G_0$ (*source* and *target*),

(ii) a map m from $G_2 \triangleq \{(x, y) \in G \times G \mid \beta(x)=\alpha(y)\}$ to G (*multiplicction*),

(iii) a map $\varepsilon: G_0 \rightarrow G$ (*identities*),

(iv) a map $\iota: G \rightarrow G$ (*inversion*),

such that, for all x, y, z in G for which the expressions below are defined,

(a) $\alpha(m(x, y)) = \alpha(x)$ and $\beta(m(x, y)) = \beta(y)$,

(b) $m(m(x, y), z) = m(x, m(y, z))$ (*associativity*),

(c) $m(\varepsilon(\alpha(x)), x) = x = m(x, \varepsilon(\beta(x)))$ (*identities*),

(d) $m(x, \iota(x)) = \varepsilon(\alpha(x))$ and $m(\iota(x), x) = \varepsilon(\beta(x))$ (*inversion*).

In short, the elements of G are invertible "mappings" in a category whose "objects" are the points of G_0.

(4.1.2) DEFINITION. A *Lie groupoid* (or *differentiable groupoid*) G over a manifold G_0 is a groupoid with a differentiable structure for which:

(i) α and β are differentiable submersions (implying that G_2 is a submanifold of $G \times G$),

(ii) m, ε, and ι are differentiable maps.

Thus, a Lie group is just a Lie groupoid over a one-point base.

From now on, we will usually write xy for $m(x, y)$ and x^{-1} for $\iota(x)$. We will also identify G_0 with G via the embedding ε so that, for instance, equation (d) in Definition 4.1.1 becomes $xx^{-1}=\alpha(x)$ and $x^{-1}x=\beta(x)$.

For each $x \in G$, the *left translation* $l_x: y \mapsto xy$ is a diffeomorphism from $\alpha^{-1}(\beta(x))$ to $\alpha^{-1}(\alpha(x))$; thus, the left translations interchange α-fibres (inverse images of points under α), and hence the following definition makes sense.

(4.1.3) DEFINITION. A vector field X on G is *left invariant* if $X(G) \subset \mathrm{Ker}(T\alpha)$ and if for each $(x, y) \in G_2$, $X(xy)=Tl_x(X(y))$.

The left invariant vector fields form a Lie algebra and can be identified with the sections of the vector bundle $\mathrm{Ker}(T\alpha)|_{G_0}$ which, since it is transverse to TG_0, is naturally isomorphic to the normal bundle $N(G_0, G)$ of G_0 in G. Equipped with the map $\rho: N(G_0, G) \rightarrow TG_0$ given by the projection from $\mathrm{Ker}(T\alpha)|_{G_0}$ to TG_0 along $\mathrm{Ker}(T\beta)|_{G_0}$, $N(G_0, G)$ becomes a Lie algebroid. (See [**M**].)

The correspondence between local Lie groupoids and Lie algebroids is as close as that between Lie groups and Lie algebras, but the global aspects of this correspondence are more complicated. For instance, not every Lie algebroid can be integrated to a global Lie groupoid [**A-M**], [**M**].

(4.2) **Basic properties of Poisson groupoids.**

(4.2.1) DEFINITION. A Poisson structure on a Lie groupoid G is called *multiplicative* if the graph $\{(z, x, y)|z=xy\}$ of the multiplication map m is a Poisson relation from $G\times G$ to G. A Lie groupoid with a multiplicative Poisson structure is called a *Poisson groupoid.*

(4.2.2) EXAMPLES. (a) A Lie group H with a multiplicative Poisson structure is just a Poisson group in the sense of Drinfel'd [**D**]*. (By Corollary 2.2.3, multiplication is a Poisson map from $H\times H$ to H.)

(b) Any Lie groupoid with the zero Poisson structure is a Poisson groupoid.

(c) Let E be a Lie algebroid over M. The natural Poisson structure on E^* (see [**Ct-D-W**], [**Cu**]) is multiplicative (perhaps we should say *additive*) with respect to the group structure for which $\alpha=\beta$ is the vector bundle projection and m is addition in the fibres. Conversely, for any additive Poisson structure on a vector bundle F, the functions on F which are linear on fibres form a subalgebra of $C^\infty(F)$ under the Poisson bracket. Identifying these linear functions with the sections of F^*, we get a Lie algebroid structure on F^*. (The map $F^*\to TM$ comes from the Poisson bracket of functions constant on fibres with functions linear on fibres.) Thus the category of vector bundles with additive Poisson structures is isomorphic by duality to the category of Lie algebroids. (This extends the well-known identification of linear Poisson structures on vector spaces with duals of Lie algebras.)

If $E=M$ is the trivial groupoid, its only multiplicative Poisson structure is zero. In case $E=TM$ has the Lie algebroid structure given by the commutator bracket, the Poisson structure on T^*M is the standard one. When E is the gauge algebroid of a principal bundle, E^* is the phase space of a classical particle in a Yang-Mills field [**St**], [**W2**].

(d) Any symplectic groupoid in the sense of [**K**], [**W5**] (i.e. the Poisson structure is symplectic and the graph of multiplication is lagrangian) is a Poisson groupoid. Conversely, a simple dimension counting argument using Theorem 4.2.3 below shows that, if a multiplicative Poisson structure on a Lie groupoid is symplectic, then the graph of multiplication is necessarily lagrangian and not just coisotropic.

(e) Let P be any set. The coarse groupoid over P has $G=P\times P$, $G_0=P$, $\alpha(a, b)=a$, $\beta(a, b)=b$, $(a, b)(b, c)=(a, c)$, $\varepsilon(a)=(a, a)$, and $(a, b)^{-1}=(b, a)$. If P is a differentiable manifold, the coarse groupoid is a Lie groupoid. If P has a Poisson structure, then the "difference" structure on $P\times P^-$ is multiplicative for the coarse groupoid. It turns out (Corollary 4.2.8 below), that every multi-

* Drinfel'd calls a multiplicative Poisson structure on a Lie group *grouped*. This already sounds awkward in English, and *groupoided* seems even worse.

plicative Poisson structure on the coarse groupoid is of this form. If P is symplectic, then $P\times P^{-}$ is a symplectic groupoid.

(f) Let P be a *regular* Poisson manifold. Then the holonomy groupoid [E], [W] G of the foliation of P by its symplectic leaves has a multiplicative Poisson structure in which the symplectic leaves are coverings of the coarse groupoids of the leaves of P, with symplectic structures as in (e). This kind of Poisson groupoid is locally, but not globally, a product of a trivial groupoid as in (a) and a coarse groupoid as in (e). We note that the holonomy groupoid might not be Hausdorff even if P is.

Many properties of Poisson groupoids can be derived from the composition of Poisson relations.

(4.2.3) THEOREM. *Let G be a Poisson groupoid.*

(a) *The identity section G_0 is coisotropic in G;*

(b) *The inversion $x\mapsto x^{-1}$ is an anti-Poisson mapping;*

(c) *There is a unique Poisson structure on G_0 for which α is a Possson mapping (and β is an anti-Poisson mapping).*

PROOF. (a) Observing that $u\in G$ belongs to G_0 if and only if $uy=y$ for some $y\in G$, we may write $G_0=R(C)$, where $C\subset G\times G$ is the diagonal and $R=\{(x, y, xy) \mid (x, y)\in G_2\}\subset G\times(G\times G)$ is the graph of multiplication with its entries permuted. Now C is coisotropic in $G\times G^{-}$ by Corollary 2.2.3, and R is coisotropic in $G^{-}\times G^{-}\times G$ and thus a Poisson relation from $G\times G^{-}$ to G^{-}. It will follow from Proposition 2.1.2 that $G_0=R(C)$ is coisotropic in G^{-} and hence in G once we have verified the hypothesis of very clean position.

Verifying clean position for $R(C)$ amounts essentially to showing that if $(\delta x, \delta y, \delta z)\subset TG\times TG\times TG$ is any tangent vector to R such that $\delta z=\delta y$, then δx must be tangent to G_0. But the last assertion follows immediately from the fact that TG is itself a groupoid with multiplication given by $Tm\subset T(G\times G\times G)$ $\cong TG\times TG\times TG$. (See **[Ct-D-W]**.) More directly, let $(x(t), y(t), x(t)y(t))$ be a curve in R tangent at $t=0$ to $(\delta x, \delta y, \delta z)$. Since $\delta z=\delta y$, $x(0)$ must lie in G_0, and $(d/dt)|_{t=0}y(t)=(d/dt)|_{t=0}x(t)y(t)$. Multiplying on the right by $y(t)^{-1}$, we find by the differentiability of multiplication that $(d/dt)|_{t=0}\alpha(y(t))=(d/dt)|_{t=0}$ $x(t)$. Since the left-hand side of the last equation is obviously in TG_0, so is the right-hand side δx.

From now on, we will leave verification of clean position to the reader, who may use either of the methods above.

(b) The graph of inversion is $R(C)$, where $R=m^{-1}=\{(x, y, xy) \mid (x, y)\in G_2\}$, thought of this time as a Poisson relation from G to $G\times G$, and $C=G_0$. The clean position hypothesis is easily verified, so $R(C)$ is coisotropic in $G\times G$ by Proposition 2.1.2. Thus ι is a Poisson map from G to G^{-}, i.e. an anti-Poisson map from G to G.

(c) Since $\alpha(z)=\alpha(x)$ if and only if $z=xy$ for some y (let $y=x^{-1}z$), the equivalence relation $A=\{(z, x) \mid \alpha(z)=\alpha(x)\}$ is $R(C)$, where $R=m=\{(xy, x, y) \mid (x, y) \in G_2\}$ considered as a Poisson relation from G to $G\times G^-$, and $C=G$. Then A is coisotropic by Proposition 2.1.2; it now follows from Proposition 2.3.1 that G_0 carries a unique Poisson structure for which α is a Poisson map. □

(4.2.4) EXAMPLES. The induced Poisson structures on G_0 for Examples 4.2.2 are the following:

(a) for G a Poisson group, the trivial structure on a point;

(b) for G a groupoid with zero Poisson structure, the zero Poisson structure on G_0;

(c) for E^* the dual of a Lie algebroid over M, the zero Poisson structure on M;

(d) for a symplectic groupoid, the Poisson structure on the base, as in [**K**], [**W5**];

(e) for a coarse groupoid $P\times P^-$, the given Poisson structure on P;

(f) for the holonomy groupoid of a regular Poisson manifold P, the given Poisson structure on P.

(4.2.5) REMARK. Examples 4.2.2(e) and (f) show that there may exist several global Poisson groupoids over P even if there is no global symplectic groupoid. For example, let $P=S^2\times \boldsymbol{R}$ with the structure described in [**W5**].

The next result, like Theorem 4.2.3, extends a result already known for symplectic groupoids [**K**], [**W5**].

(4.2.6) PROPOSITION. *$\alpha^*(C^\infty(G_0))$ and $\beta^*(C^\infty(G_0))$ are Poisson commuting subalgebras of $C^\infty(G)$.*

PROOF. Let f and h be in $C^\infty(G_0)$. On $G\times G^-\times G^-$, define $F(z, x, y)=f(\alpha(z))-f(\alpha(x))$ and $H(z, x, y)=h(\beta(z))-h(\beta(y))$. On the relation $m=\{(z, x, y) \mid z=xy\}$ we have $\alpha(z)=\alpha(x)$ and $\beta(z)=\beta(y)$, so F and H vanish on m. Since m is coisotropic, $\{F, H\}(z, x, y)=0$ whenever $z=xy$. But

$$\begin{aligned}\{F, H\}(z, x, y) &= \{f(\alpha(z))-f(\alpha(x)),\ h(\beta(z))-h(\beta(y))\}\\ &= \{f(\alpha(z)),\ h(\beta(z))\} = \{\alpha^*f,\ \beta^*h\}(z).\end{aligned}$$

Since every z occurs as the first component of an element of m ($z=z\beta(z)$), we may conclude that $\{\alpha^*f, \beta^*h\}=0$. □

(4.2.7) COROLLARY. *$(\alpha, \beta): G\to G_0\times G_0^-$ is a Poisson map.*

(4.2.8) COROLLARY. *Every multiplicative Poisson structure on the coarse groupoid $P\times P$ is of the "difference" form in* 4.2.2(e).

PROOF. The map (α, β) is a diffeomorphism in this case. □

(4.2.9) COROLLARY. *The orbit equivalence relation on G_0, in which $u \sim v$ if and only if there is an $x \in G$ with $\alpha(x)=u$ and $\beta(x)=v$, is a Poisson relation. If $G_0/\sim$ is a manifold, it therefore inherits a natural Poisson structure.*

PROOF. The orbit equivalence relation is $(\alpha, \beta)(G) \subset G_0 \times G_0^-$. □

(4.2.10) EXAMPLE. In case Γ is an α-connected symplectic groupoid over Γ_0, the orbits are the symplectic leaves of Γ (see [**Ct-D-W**], [**W5**]), so $\Gamma_0/\sim$ inherits the zero Poisson structure.

The following consequence of Proposition 4.2.6 will be used in § 4.4.

(4.2.11) COROLLARY. *For each $g \in C^\infty(G_0)$, the hamiltonian vector field $X_{\alpha^* g}$ is tangent to the β-fibres, and vice versa.*

(4.3) **Reduction for Poisson group actions.** As another application of the coisotropic calculus, we prove the following theorem of Semenov-Tian-Shansky [**S**].

(4.3.1) THEOREM. *Let G be a Poisson group, P a Poisson manifold, and $l: G \times P \to P$ a Poisson action (i.e. a group action which is also a Poisson map). If H is any coisotropic subgroup of G for which P/H has a C^∞ structure making the projection $p: P \to P/H$ a submersion, then there is a unique Poisson structure on P/H for which p is a Poisson map.*

PROOF. According to Proposition 2.3.1, we must check that $S = p \circ p^{-1}$ is a Poisson relation. Since $S = \{(a, b) \in P \times P \mid l(g, a) = l(h, b)$ for some g and h in $H\}$, $S = R(C)$ where $R = \{(a, b, g, h, l(g, a), l(h, b) \mid (a, b) \in P \times P, (g, h) \in H \times H\}$ considered as a relation from $G \times G \times P \times P$ to $P \times P$, and $C = H \times H \times \Delta_P$ where Δ_P is the diagonal in $P \times P$. R is coisotropic in $P \times P^- \times G \times G^- \times P^- \times P$ because it is obtained by permutation of components from two copies of l, and C is coisotropic in $G^- \times G \times P \times P^-$ since H is coisotropic in G by assumption. Thus $S = R(C)$ is coisotropic in $P \times P^-$ by Proposition 2.1.2. □

(4.3.2) REMARK. Semenov-Tian-Shansky's proof of Theorem 4.3.1 is already quite simple, but our proof has the advantage of extending immediately to Poisson groupoid actions. (See [**Mi-W**] for the symplectic case.) In fact, applying this extended version to the action of a Poisson groupoid on itself by left translations recovers Theorem 4.2.3(c).

(4.4) **Duality.** Since the identity section G_0 of a Poisson groupoid G is coisotropic by Theorem 4.2.3(a), the conormal bundle $N^*(G_0, G)$ is equipped with a Lie algebroid structure, according to Corollary 3.1.5. Notice that, as a vector bundle, $N^*(G_0, G)$ is dual to the Lie algebroid of G.

(4.4.1) DEFINITION. Let G and G' be Poisson groupoids over G_0 provided with a nondegenerate pairing between the vector bundles $N(G_0, G)$ and $N(G_0, G')$, so that each of these bundles is isomorphic to the dual of the other. We say that G and G' are *dual* to one another if the Lie algebroid structure on $N(G_0, G) \cong N^*(G_0, G')$ is the one induced by the Poisson structure on G^*, and vice versa.

Drinfel'd proved in [**D**] that each Poisson group has a dual group which is unique up to covering, and that dual pairs of Poisson groups are in functorial 1-1 correspondence with infinitesimal objects called *Lie algebroids*. We expect that a similar result holds for Poisson groupoids, though we have not yet been able to prove it. (In general, the dual may exist only as a *local* groupoid.) Nevertheless, the concept of duality is already interesting because, as we will see below, it provides a natural organization among the Poisson groupoids in Example 4.2.2.

(4.4.2) EXAMPLES. (See 4.2.2.) (a) The duality of Poisson groups in our sense is the same as described by Drinfel'd. In particular, each Lie group H with the zero Poisson structure is dual to the additive group h^* with the Lie-Poisson structure.

(b)-(c) Any Lie groupoid G with the zero Poisson structure is dual to the vector bundle $N^*(G_0, G)$ with the additive Poisson structure which it carries as the dual of the Lie algebroid of G.

(d)-(e) A symplectic groupoid Γ over the Poisson manifold P is dual to the coarse groupoid $P\times P^-$ with the "difference" Poisson structure.*

(f) The holonomy groupoid of a regular Poisson manifold P with the Poisson structure of Example 4.2.2(f) is dual to itself. The pairing between the Lie algebroid (which is $\tilde{\pi}(T^*P)\subset TP$) and itself is given by the symplectic structure along the leaves.

Some special cases of the examples in 4.4.2 are worth noting. If Γ is a symplectic groupoid, then the coarse groupoid $\Gamma\times\Gamma^-$ is dual to itself. This is a special case of (d)-(e) as well as of (f). The gauge groupoid of a principal bundle, with the zero Poisson structure, is dual to the phase space of a classical particle in a Yang-Mills field [**St**], [**W2**]; this is a special case of (b)-(c). In particular, any coarse groupoid $X\times X$ with the zero Poisson structure is dual to the cotangent bundle T^*X with the standard Poisson structure; this is also a special case of (d)-(e).

(4.4.3) REMARK. Since not every Lie algebroid comes from a global Lie

* Note that, if Γ_P is given the *zero* Poisson structure, it is dual to TP with the tangent Poisson structure of [**Av**] (also see [**Cu**]).

groupoid [**A-M**], [**M**], and not every Poisson manifold is the base of a symplectic groupoid, examples (b)-(c) and (d)-(e) show that a global dual to a Poisson manifold may not exist.

Examples 4.2.6 and 4.4.2 suggest the following general fact.

(4.4.4) THEOREM. *If G and G' are dual Poisson groupoids over G_0, then the Poisson structures which they induce by Theorem* 4.2.3(c) *are opposite to one another.*

In order to prove the theorem, we will give another description of the induced Poisson structure on the base of a Poisson groupoid.

(4.4.5) PROPOSITION. *Let (G, π) be a Poisson groupoid, π_0 the Poisson structure on G_0 for which α is a Poisson map. Then the map $\tilde{\pi}_0: T^*G_0 \to TG_0$ may be factored through the normal bundle $N(G_0, G)$ as $\tilde{\pi}_0 = -\rho\sigma$, where $\rho: N(G_0, G) \to TG_0$ depends only on the groupoid structure, and $\sigma: T^*G_0 \to N(G_0, G)$ depends only on the Poisson structure. Specifically, ρ is given by the projection from* $\mathrm{Ker}(T\alpha)|_{G_0}$ *to TG_0 along* $\mathrm{Ker}(T\beta)|_{G_0}$, *while σ is dual to the map from $N^*(G_0, G)$ to TG_0 obtained by restricting $\tilde{\pi}$ along the coisotropic submanifold G_0.*

PROOF. Let θ be a cotangent vector to G_0 at the point u. First of all, we have $\sigma(\theta) = \tilde{\pi}(\theta')$ $(\mathrm{mod}\, TG_0)$, where θ' is *any* cotangent vector to G at u whose pullback to T_uG_0 is θ. In particular, we may take θ' to be $(T\beta)^*(\theta)$. Applying Corollary 4.2.11 by thinking of θ as the differential of a function g on G_0, so that $\tilde{\pi}(\theta')$ is the normal component at u of the hamiltonian vector field of β^*g, we conclude that $\tilde{\pi}(\theta')$ lies in $\mathrm{Ker}(T\alpha)$. Thus, to compute $\rho(\sigma(\theta))$, we need only project $\tilde{\pi}(\theta')$ into TG_0 along $\mathrm{Ker}(T\beta)$; i.e. we apply $T\beta$ itself. Thus $\rho(\sigma(\theta)) = T\beta(\tilde{\pi}(\theta')) = T\beta(\tilde{\pi}((T\beta)^*(\theta)))$.

Since β is an anti-Poisson map, we have $\tilde{\pi}_0 = -T\beta \circ \tilde{\pi} \circ (T\beta)^*$, from which we conclude that $\rho(\sigma(\theta)) = -\tilde{\pi}_0(\theta)$. □

PROOF OF THEOREM 4.4.4. To apply the previous proposition, we must compare the maps ρ' and σ' arising from the dual groupoid G' with ρ and σ. Since $N(G_0, G') \cong N^*(G_0, G)$, we think of the bundle map ρ' associated with the Lie algebroid of G' as a bundle map $\rho': N^*(G_0, G) \to TG_0$. Since this Lie algebroid structure is just the one determined by the coisotropic submanifold $G_0 \subset G$, we find that $\rho' = -\sigma^*$ (see Corollary 3.1.5). Since duality is a symmetric relation, we also have $\sigma' = -\rho^*$.

Now $\tilde{\pi}'_0 = -\rho'\sigma' = -\sigma^*\rho^* = (-\rho\sigma)^* = \tilde{\pi}_0^* = -\tilde{\pi}_0$, and so $\pi'_0 = -\pi_0$. □

(4.4.6) REMARK. It would be interesting to find further structure in the category of Poisson groupoids over a given Poisson manifold and its opposite. In particular, what is the special role of the (local) symplectic groupoid? (Note

Remark 4.2.7.)

(4.5) **Symplectic double groupoids.** In this section, we outline a symplectic approach to understanding the global duality of Poisson groupoids. The successful application of this approach will depend upon future developments in the theory of symplectic groupoids over Poisson manifolds.

If P is any Poisson manifold, there is always a *local* symplectic groupoid Γ_P over P [**K**], [**W5**]; i.e. $\alpha: \Gamma_P \to P$ is a Poisson map. If P has the zero Poisson structure, we can take T^*P for Γ_P; for this and other reasons, it is useful to think of Γ_P as a "Poisson cotangent bundle" or "phase space" [**K**] for a general Poisson manifold P.

To avoid dealing with the unsolved problem of determining which Poisson manifolds are enlargeable in the sense that Γ_P exists as a global groupoid, we will not distinguish below between local and global groupoids.

If $P=E^*$, where E is the Lie algebra [algebroid] of a Lie group [groupoid] G, then we can take $\Gamma_P=T^*G$; the fundamental theorems of Lie then suggest that operations on general Poisson manifolds might be "lifted" to their symplectic groupoids.

The lifting procedure should be defined in the following way. If C is a coisotropic submanifold of P, then $\alpha^{-1}(C)$ is coisotropic in Γ_P by Corollary 2.2.5. We may apply the "method of characteristics" [**Ct-D-W**], [**Gu-S1**], [**Gu-S2**] to $\alpha^{-1}(C)$ and the lagrangian submanifold $(\Gamma_P)_0$, taking the intersection $\alpha^{-1}(C)\cap(\Gamma_P)_0\cong C$ and flowing out along the characteristics of C to obtain a lagrangian submanifold L_C in Γ_P which turns out to be closed under the groupoid operations.

When P has the zero Poisson structure, $L_C\subset T^*P$ is just the conormal bundle. In general, L_C is only an "immersed submanifold" of Γ_P, i.e. the image of a lagrangian immersion $i: \Lambda_C \to \Gamma_P$ for some manifold Λ_C. If we restrict our attention to a small enough neighborhood of the zero section, then L_C is a well-defined local lagrangian subgroupoid (with base C) whose Lie algebroid is the conormal bundle $N^*(C, P)$ with the bracket of Corollary 3.1.5.

It is easy to see that $\Gamma_{P_1\times P_2}\cong\Gamma_{P_1}\times\Gamma_{P_2}$ and $\Gamma_{P^-}=\Gamma_P^-$ for any Poisson manifolds P_1, P_2 and P. Thus we can lift any Poisson relation $R: P_1\to P_2$ to a canonical relation $L_R: \Gamma_{P_1}\to\Gamma_{P_2}$ which is (at least locally) a "groupoid relation" as well. In particular, the domain of L_R is a subgroupoid of Γ_{P_1}, and L_R if single valued is a homomorphism from its domain to Γ_{P_2}. Furthermore, one may check the functorial property $L_{R_1\circ R_2}=L_{R_1}\circ L_{R_2}$ when R_1 and R_2 form a clean pair. (We do not write Γ_R for L_R in order to avoid confusion in the case where R is a Poisson manifold itself.)

Now if G is a Poisson groupoid over G_0, the Poisson relations $m: G\times G\to G$,

$\iota: G \to G^-$, and $G_0: O \to G$ lift to canonical relations $L_m: \Gamma_G \times \Gamma_G \to \Gamma_G$, $L_\iota: \Gamma_G \to \Gamma_{\bar{G}}$, and $L_{G_0}: O \to \Gamma_G$ which endow Γ_G (at least locally) with a second symplectic groupoid structure, this time with L_{G_0} as its base. Since L_m is a subgroupoid, the two groupoid structures are compatible, making Γ_G into a *double groupoid* (see [**B**] or [**Ct-D-W**] for the precise definition).

The Lie algebroid of LG_0 is naturally isomorphic to the conormal bundle of G_0 in G, i.e. the dual of the Lie algebroid of G, with the bracket of Corollary 3.1.5. Since the roles of the two groupoid structures can be interchanged, we may conclude that G and L_{G_0} are dual to one another. At least locally, then, we see that *a dual pair of Poisson groupoids is exactly the pair of bases of a symplectic double groupoid.*

(4.5.1) EXAMPLES. (See 4.2.2 and 4.4.2.) (a) If G and G^* are Poisson groups, then according to Karasev [**K**], the corresponding symplectic double groupoid Γ_G is $G \times G^*$. (Actually, this is probably true only locally, in general.) We note that Manin and Drinfel'd [**D**] also endow $g \oplus g^*$ with a Lie algebra structure, which suggests that Γ_G might carry a group structure in addition to its two groupoid structures. For instance, if G has the zero Poisson bracket, then $\Gamma_G = T^*G$. In this case, the Manin-Drinfel'd structure on $g \oplus g^*$ integrates to the semidirect product group structure on $T^*G \cong G \times g^*$ (with G acting on g^* by the coadjoint representation). Of course, the (canonical) symplectic structure is not multiplicative for this group structure.

(b)-(c) If G is any Lie group*oid* with the zero Poisson structure, then again $\Gamma_G = T^*G$ with the groupoid structure described in [**Ct-D-W**] along with that of addition in the fibres.

(d)-(e) If G is a *symplectic* groupoid over G_0, then $\Gamma_G = G \times G^-$ with the coarse and product groupoid structures. The two bases are G and $G_0 \times G_0^-$. In particular, if $G = G_0 \times G_0^-$ (with G_0 symplectic), then $\Gamma_G = G_0 \times G_0^- \times G_0 \times G_0^-$ with two coarse groupoid structures related by interchange of the factors. If $G = T^*M$ for a manifold M with the zero Poisson structure, then $\Gamma_G = T^*M \times T^*M^- \cong T^*(M \times M)$ with the coarse and cotangent groupoid structures.

(f) Let G be the holonomy groupoid of a regular Poisson manifold P. We suppose for simplicity that $P = S \times M$ where S is symplectic and M has the zero Poisson structure. Then $G = S \times S^- \times T^*C$ and $\Gamma_G = S \times S^- \times S^- \times S \times T^*C \times T^*C^-$. The first groupoid structure is coarse in the first four factors and again in the second two. The second structure (lifted from that on G) is also coarse in the first four factors (with a change in their order) and is given by fibre addition in the last two.

(4.5.2) QUESTIONS. (i) *As suggested by Example* 4.5.1(a), *is there a natural groupoid structure on* Γ_G *with base* G_0 (*rather than* G *or* G^*) *for every Poisson* *groupoid G?*

(ii) *What are the groupoid structures on Γ_G in the general case of Example* 4.5.1(f)*?*

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