J. Math. Soc. Japan
Vol. 40, No. 4, 1988

On a problem of Yamamoto concerning biquadratic Gauss sums, II

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(Received April 20, 1987)

§ 1. Introduction.

For a prime number $p\equiv 5 \pmod 8$ take positive integers a and b such that $p=a^2+4b^2$ and put $\omega=\omega_p=a+2bi$. Consider the Gauss sum

$$\tau_p=\sum_{m=1}^{p-1}\left(\frac{m}{\omega}\right)_4 e^{2\pi i m/p},$$

where $\left(\frac{m}{\omega}\right)_4$ is the biquadratic residue symbol in Gauss' number field $\boldsymbol{Q}(i)$. We write

$$\tau_p=\varepsilon_p\omega^{1/2}p^{1/4} \quad \text{with} \quad 0<\arg(\omega^{1/2})<\frac{\pi}{4}.$$

It is known that $\varepsilon_p^4=1$. Furthermore we put

$$C_p=\sum_{m=1}^{(p-1)/2}\left(\frac{m}{\omega}\right)_4.$$

For a complex number z we denote by $\bar{z}$ the complex conjugate of z and put $\mathrm{Re}(z)=(z+\bar{z})/2$ and $\mathrm{Im}(z)=(z-\bar{z})/2i$. Yamamoto [**10**] observed that the inequality

$$\mathrm{Im}(\varepsilon_p\overline{C_p})>0 \tag{1}$$

holds for $p<4{,}000$ and proposed the question whether this is always true. In the previous paper [**7**], the author reported a counter-example for (1). At the same time, it was also mentioned that there is only one counter-example for (1) up to 1,000,000. The purpose of this paper is to explain the tendency of the inequality (1) to be satisfied. We shall prove the following theorem.

THEOREM 1. *The limit*

$$\lim_{x\to\infty}\frac{\#\{p\,;\,p\leqq x,\ p\equiv 5 \pmod 8,\ \text{the inequality (1) holds for } p\}}{\#\{p\,;\,p\leqq x,\ p\equiv 5 \pmod 8\}},$$

where p denotes rational prime numbers, exists and lies between 0.9997 *and* 0.9998.

For an element μ of $\mathcal{O}:=\boldsymbol{Z}[i]$ prime to 2, denote by χ_μ the Dirichlet character modulo $2m$ induced from $\left(\frac{\cdot}{\mu}\right)_4$ where m is the smallest positive integer contained in the ideal $\mu\mathcal{O}$. Then

$$L(1, \chi_\omega) := \sum_{n=1}^{\infty} \chi_\omega(n) n^{-1} = -\frac{\pi i}{2p} \tau_p \overline{C_p}$$

because

$$C_p = -\left(2 - \left(\frac{2}{\omega}\right)_4^{-1}\right) B_p \qquad \text{if} \quad B_p := \frac{1}{p} \sum_{m=1}^{p-1} \left(\frac{m}{\omega}\right)_4 m\,,$$

cf. Barkan [**1**]. It follows that the inequality (1) is equivalent to

$$\mathrm{Re}\,(L(1, \chi_\omega)/\omega^{1/2}) > 0\,. \tag{2}$$

We normalize the argument $\arg(z)$ of a complex number $z \neq 0$ by $-\pi \leqq \arg(z) < \pi$ and define the square root $z^{1/2}$ by $z^{1/2} = e^{(\log|z| + i \arg(z))/2}$. Note that there is the obvious one to one correspondence between the sets

$$\{\omega_p\,;\ p \text{ is a rational prime number, } p \equiv 5 \,(\mathrm{mod}\,8)\}$$

and

$$\left\{\omega\,;\ \omega \text{ is a prime number of } \boldsymbol{Q}(i),\ 0 < \arg(\omega) < \frac{\pi}{2},\ \omega \equiv 1 \,(\mathrm{mod}\,2),\ \left(\frac{-1}{\omega}\right)_4 = -1\right\}. \tag{3}$$

In the following the letter ω will always denote an element of the set (3). For every real number x greater than 5, we use the notation

$$\nu_x(\omega\,; \cdots) = \frac{\#\{\omega\,; N\omega \leqq x,\ \omega \text{ satisfies the property } \cdots\}}{\#\{\omega\,; N\omega \leqq x\}}\,.$$

Here $N\omega = \omega\bar{\omega}$. Then Theorem 1 is equivalent to the same assertion on the limit

$$\lim_{x\to\infty} \nu_x(\omega\,; \mathrm{Re}\,(L(1, \chi_\omega)/\omega^{1/2}) > 0)\,. \tag{4}$$

To prove this assertion we first formulate the distribution of $\arg(L(1, \chi_\omega))$ by the method developed in Elliott [**5**] and [**6**]:

THEOREM 2. *Let, for each rational integer k,*

$$\phi(q, k) = \frac{1}{2} + \frac{1}{4}\left(\frac{1+iq^{-1}}{|1+iq^{-1}|}\right)^k + \frac{1}{4}\left(\frac{1-iq^{-1}}{|1-iq^{-1}|}\right)^k. \tag{5}$$

Then, the infinite product

$$\varphi(k) = \prod_q \phi(q, k)\,, \tag{6}$$

where q runs over all rational primes greater than 2, and the infinite series

$$G(z) = \frac{1}{2} + z - \frac{1}{2\pi i} \sum_{k \in \boldsymbol{Z} - \{0\}} \frac{\varphi(k)}{k} e^{-2\pi i k z} \qquad (z \in \boldsymbol{R}) \tag{7}$$

both converge absolutely, and we have

$$\lim_{x\to\infty} \nu_x\left(\omega\,; \theta_1 \leqq \frac{1}{2\pi} \arg(\omega) \leqq \theta_2,\ -\frac{1}{2} \leqq \frac{1}{2\pi} \arg(L(1, \chi_\omega)) \leqq z\right) = 4(\theta_2 - \theta_1) G(z)$$

for every z with $-1/2 \leqq z < 1/2$ and every θ_1, θ_2 such that $0 \leqq \theta_1 \leqq \theta_2 \leqq 1/4$. The function $G(z)$ is infinitely differentiable on $\boldsymbol{R}$ and satisfies $G(-1/2)=0$.

The next theorem follows easily from the above.

THEOREM 3. *For every real number z such that $-1/2 \leqq z \leqq 3/8$, we have*

$$\lim_{x\to\infty} \nu_x\Big(\omega\,;\ -\frac{1}{2} \leqq \frac{1}{2\pi}\arg(L(1, \chi_\omega)/\omega^{1/2}) \leqq z\Big) = 8\int_0^{1/8}\Big(G(z+\theta) - G\Big(-\frac{1}{2}+\theta\Big)\Big)d\theta\,.$$

It follows that the limit (4) is equal to

$$8\int_0^{1/8}\Big(G\Big(\frac{1}{4}+\theta\Big) - G\Big(-\frac{1}{4}+\theta\Big)\Big)d\theta \tag{8}$$

and we get Theorem 1 by the numerical calculation of this quantity. Theorem 2 will be proved in Section 2. Theorem 1 and Theorem 3 will be proved in Section 3. The possibility that the inequality (1) can be reduced to an assertion like (2) has already been remarked by Heath-Brown and Patterson in Berndt and Evans [2].

§ 2. The distribution of arg$(L(1, \chi_\omega))$.

1. The letter q will always denote an odd rational prime number. For a real number $h>0$ and a Dirichlet character χ, put

$$D_h = 16\prod_{q\leqq h} q \qquad \text{and} \qquad L_h(1, \chi) = \prod_{q\leqq h}(1-\chi(q)q^{-1})^{-1}\,.$$

Because of the reciprocity law of the biquadratic residue symbol, the value $L_h(1, \chi_\omega)$ depends only on the residue class of ω modulo D_h. We define the map $Y_h : (\mathcal{O}/D_h\mathcal{O})^\times \to T := \boldsymbol{R}/\boldsymbol{Z}$ by

$$Y_h(\mu \bmod D_h) = \frac{1}{2\pi}\arg(L_h(1, \chi_\mu))\,.$$

Usually, T will be identified with the interval $[-1/2, 1/2)$. Denote by Ω_h the subset of $(\mathcal{O}/D_h\mathcal{O})^\times$ represented by integers $\mu \in \mathcal{O}$ such that $\mu \equiv 1 \pmod 2$ and $\left(\frac{-1}{\mu}\right)_4 = -1$. Let, for $z \in [-1/2, 1/2)$,

$$G_h(z) = \frac{1}{\#\Omega_h}\#\Big\{c \in \Omega_h\,;\ -\frac{1}{2} \leqq Y_h(c) \leqq z\Big\}\,.$$

Then it is easy to see that

$$\lim_{x\to\infty} \nu_x\Big(\omega\,;\ \theta_1 \leqq \frac{1}{2\pi}\arg(\omega) \leqq \theta_2,\ -\frac{1}{2} \leqq \frac{1}{2\pi}\arg(L_h(1, \chi_\omega)) \leqq z\Big) = 4(\theta_1-\theta_2)G_h(z)$$

if $0 \leqq \theta_1 \leqq \theta_2 \leqq 1/4$.

In order to consider the limit of $G_h(z)$ when h tends to the infinity, we quote a result from Fourier analysis. A real valued function $F(z)$ on $[-1/2, 1/2)$ will be called a distribution function if it satisfies the following three conditions:

(i) $F(z)$ increases in the wide sense with z,

(ii) it is continuous on the right,

(iii) $F\left(-\frac{1}{2}\right) \geqq 0$ and $\lim_{z\to 1/2-0} F(z)=1$.

The domain of a distribution function $F(z)$ will be often extended to $\boldsymbol{R}$ by defining $F(z+n)=F(z)+n$ for every integer n. By the characteristic function of a distribution function $F(z)$ we understand the function $\psi: \boldsymbol{Z}\to\boldsymbol{C}$ defined by

$$\psi(k)=\int_a^{a+1} e^{2\pi i k z} dF(z)=e^{2\pi i k a}-2\pi i k\int_a^{a+1} e^{2\pi i k z} F(z) dz .$$

Here a is an arbitrary real number at which F is continuous. Let F_n $(n=1, 2, \cdots)$ and F be distribution functions and $\psi_n(k)$ and $\psi(k)$ be their respective characteristic functions. Assume $F(-1/2)=0$ and

$$\lim_{n\leftarrow\infty} \psi_n(k)=\psi(k)$$

for every integer k. Then it is known that

$$\lim_{n\to\infty} F_n(z)=F(z)$$

holds for every z (cf. Elliott [4]).

LEMMA 1. i) *The characteristic function* $\varphi_h(k)$ *of the distribution function* $G_h(z)$ *is equal to* $\prod_{q\leq h}\phi(q, k)$ *where* $\phi(q, k)$ *is defined by* (5).

ii) *The estimate*

$$\phi(q, k)=1+O(q^{-2}) \qquad (q\to\infty)$$

holds for every integer k *and we can define* $\varphi(k)$ *by* (6).

iii) *There is a positive constant* c *such that, if* $|k|$ *is sufficiently large,*

$$\varphi(k) \leqq \exp\left(-\frac{c|k|}{\log|k|}\right).$$

Hence it is possible to define $G(z)$ *by* (7) *and then* $G(z)$ *is infinitely differentiable on* $\boldsymbol{R}$.

iv) *We have*

$$\lim_{h\to\infty} G_h(z)=G(z) .$$

REMARK. We understand that the infinite product in (6) is zero if there exists q such that $\phi(q, k)=0$.

PROOF. i) It suffices to prove the assertion in case h is a positive rational integer and we can consider inductively. If $h<3$, then $L_h(1, \chi)=1$ for every character χ and it follows that

$$G_h(z)=\begin{cases}0 & \text{if } -\frac{1}{2}\leqq z<0\\ 1 & \text{if } 0\leqq z<\frac{1}{2},\end{cases}\qquad \varphi_h(k)=1 \qquad (k\in\boldsymbol{Z}).$$

Hence the assertion is true in this case. Assume that the assertion holds when h is a positive integer n. If $n+1$ is not a prime number there is nothing to prove because $\varphi_n(k)=\varphi_{n+1}(k)$. Consider the case $n+1$ is a prime number p. In general the formula

$$\varphi_h(k)=\sum_{z\in T} m_h(z)e^{2\pi ikz}$$

holds with

$$m_h(z)=\frac{1}{\#\Omega_h}\#\{c\in\Omega_h\,;\,Y_h(c)=z\}.$$

We have, for every $\zeta\in\{1,-1,i,-i\}$ and every $\alpha\in\mathcal{O}$ prime to D_n,

$$\frac{\#\{\mu \bmod D_{n+1}\in\Omega_{n+1}\,;\,\mu\equiv\alpha\ (\bmod D_n),\ \chi_\mu(p)=\zeta\}}{\#\{\mu \bmod D_{n+1}\in\Omega_{n+1}\,;\,\mu\equiv\alpha\ (\bmod D_n)\}}=\frac{1}{4}.$$

Hence

$$m_{n+1}(z)=\frac{1}{2}m_n(z)+\frac{1}{4}m_n(z+z_p)+\frac{1}{4}m_n(z-z_p)$$

if we put

$$z_p=-\frac{1}{2\pi}\arg\left(1-\frac{i}{p}\right).$$

Therefore

$$\varphi_{n+1}(k)=\phi(p,k)\sum_{z\in T} m_n(z)e^{2\pi ikz}=\prod_{q\leqq n+1}\phi(q,k)$$

by the assumption.

ii) Because

$$\arg(1+iq^{-1})=q^{-1}+O(q^{-2})$$

we see

$$\left(\frac{1+iq^{-1}}{|1+iq^{-1}|}\right)^k=\exp(ik\arg(1+iq^{-1}))=1+ikq^{-1}+O(q^{-2}),$$

and similarly

$$\left(\frac{1-iq^{-1}}{|1-iq^{-1}|}\right)^k=1-ikq^{-1}+O(q^{-2}).$$

Hence the assertion follows.

iii) To use the estimate later we give slightly sharper one than is needed. Let $|k|\geqq 100$ and $|k/q|\leqq 1/4$. Then, because $\binom{k}{r}\leqq k^r$,

$$\frac{1}{2}\{(1+iq^{-1})^k+(1-iq^{-1})^k\}=\sum_{\substack{r=0\\2|r}}^{k}(-1)^{r/2}\binom{k}{r}q^{-r}$$

$$\leqq 1-\frac{k(k-1)}{2}q^{-2}+\left(\frac{k}{q}\right)^4+\left(\frac{k}{q}\right)^8+\cdots$$

$$\leqq 1-\frac{99}{200}\left(\frac{k}{q}\right)^2+\frac{256}{255}\left(\frac{k}{q}\right)^4 \leqq 1-\frac{1}{5}\left(\frac{k}{q}\right)^2 \leqq \exp\left(-\frac{1}{5}\left(\frac{k}{q}\right)^2\right).$$

Note that, for all q and k,

$$0 \leqq \phi(q, k) \leqq 1 .$$

It follows that

$$\varphi(k) \leqq \sum_{q \geqq 4|k|} \phi(q, k) \leqq \exp\left(-\frac{k^2}{5}\sum_{q \geqq 4|k|} q^{-2}\right) \tag{9}$$

if $|k| \geqq 100$. By the prime number theorem,

$$\sum_{q \geqq x} q^{-2} \sim \frac{1}{x \log x} \qquad (x \to \infty).$$

This completes the proof.

iv) By the fact stated before Lemma 1, it suffices to show that $G(z)$ increases in the wide sense and $G(-1/2)=0$. The latter follows easily from the fact $\varphi(-k)=\varphi(k)$. Because of the property (i) of distribution functions the sequence $\varphi_h(k)$ $(k \in \boldsymbol{Z})$ is positive definite (i.e., $\sum_{j,k=1}^{n} \varphi_h(j-k)\xi_j\bar{\xi}_k \geqq 0$ for an arbitrary n and arbitrary complex numbers $\xi_1, \cdots, \xi_n$) for every h and hence the sequence $\varphi(k)=\lim_{h\to\infty}\varphi_h(k)$ $(k \in \boldsymbol{Z})$ also has this property, from which follows that $G(z)$ increases in the wide sense.

LEMMA 2. *Let* $h(x)=\log\log x$ *and* $0 \leqq \theta_1 \leqq \theta_2 \leqq 1/4$. *Then*

$$\lim_{x\to\infty} \nu_x\left(\omega\,;\, \theta_1 \leqq \frac{1}{2\pi}\arg(\omega) \leqq \theta_2,\ -\frac{1}{2} \leqq \frac{1}{2\pi}\arg(L_{h(x)}(1, \chi_\omega)) \leqq z\right) = 4(\theta_2-\theta_1)G(z).$$

PROOF. We need a theorem proved by Mitsui [8]: For an integral ideal $\mathfrak{a}$ of $\boldsymbol{Q}(i)$, an integer $\mu \in \mathcal{O}$ such that $(\mathfrak{a}, \mu)=1$ and real numbers x, θ with $x>0$, $-1/2 \leqq \theta < 1/2$, we denote by $\pi(\mathfrak{a}, \mu\,;\, x, \theta)$ the number of prime numbers ρ of $\boldsymbol{Q}(i)$ satisfying the following conditions;

$$\rho \equiv \mu \pmod{\mathfrak{a}}, \qquad N\rho \leqq x, \qquad -\frac{1}{2} \leqq \frac{1}{2\pi}\arg(\rho) \leqq \theta .$$

Then, for any positive constant A, there exists a positive number c, depending only on A, such that if $\mathfrak{a}$ satisfies

$$N\mathfrak{a} \leqq (\log x)^A ,$$

the estimate

$$\pi(\mathfrak{a}, \mu\,;\, x, \theta) = \frac{4\left(\theta+\frac{1}{2}\right)}{\#(\mathcal{O}/\mathfrak{a})^\times}\int_2^x \frac{dt}{\log t} + O(xe^{-c\sqrt{\log x}})$$

holds uniformly in $\mathfrak{a}$, μ and θ. Because $D_{h(x)}$ satisfies

$$D_{h(x)}=\prod_{q\leqq h(x)} q \ll (\log x)^2,$$

the above theorem shows

$$\nu_x\left(\omega\,;\,\theta_1\leqq\frac{1}{2\pi}\arg(\omega)\leqq\theta_2,\ -\frac{1}{2}\leqq\frac{1}{2\pi}\arg(L_{h(x)}(1,\chi_\omega))\leqq z\right)$$

$$\sim\nu_x\left(\omega\,;\,\theta_1\leqq\frac{1}{2\pi}\arg(\omega)\leqq\theta_2,\ -\frac{1}{2}\leqq Y_{h(x)}(\omega \bmod D_{h(x)})\leqq z,\ (\omega,\,D_{h(x)})=1\right)$$

$$=4(\theta_2-\theta_1)G_{h(x)}(z)+O(e^{-c\sqrt{\log x}})$$

with a positive constant c. By Lemma 1 this converges to $4(\theta_2-\theta_1)G(z)$ when x tends to the infinity.

LEMMA 3. *There exist functions $f(x)$ and $g(x)$, which are defined for sufficiently large real numbers and satisfy $f(x)\geqq 0$, $g(x)\geqq 0$ and $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}g(x)=0$, so that the estimate*

$$\nu_x(\omega\,;\,|\arg(L(1,\chi_\omega))-\arg(L_{h(x)}(1,\chi_\omega))|>f(x))\ll g(x) \tag{10}$$

holds.

This will be proved in the next subsection. Here we prove Theorem 2 admitting this lemma. By the continuity of $G(z)$ and Dini's theorem the convergence proved in Lemma 2 is uniform with respect to z. Therefore we see

$$\overline{\lim_{x\to\infty}}\,\nu_x\left(\omega\,;\,\theta_1\leqq\frac{1}{2\pi}\arg(\omega)\leqq\theta_2,\ -\frac{1}{2}\leqq\frac{1}{2\pi}\arg(L(1,\chi_\omega))\leqq z\right)$$

$$\leqq\lim_{x\to\infty}\left\{\nu_x\left(\omega\,;\,\theta_1\leqq\frac{1}{2\pi}\arg(\omega)\leqq\theta_2,\ -\frac{1}{2}\leqq\frac{1}{2\pi}\arg(L_{h(x)}(1,\chi_\omega))\leqq z+f(x)\right)+O(g(x))\right\}$$

$$=4(\theta_2-\theta_1)G(z).$$

Similarly the inferior limit is not smaller than $4(\theta_2-\theta_1)G(z)$. This proves Theorem 2.

2. Let us prove Lemma 3. The letter $\mathfrak{p}$ will always denote a prime ideal of $K:=\boldsymbol{Q}(i)$ which is prime to 2 and $\chi_{\mathfrak{p}}$ will denote the Dirichlet character modulo $2p$ induced from $\left(\frac{\cdot}{\mathfrak{p}}\right)_4$ where p is the rational prime divided by $\mathfrak{p}$. For each real number $x>5$, put

$$\nu_x(\mathfrak{p}\,;\,\cdots)=\frac{\#\{\mathfrak{p}\,;\,N\mathfrak{p}\leqq x,\ \mathfrak{p}\text{ satisfies the condition }\cdots\}}{\pi(x\,;\,K)}.$$

with

$$\pi(x\,;\,K):=\#\{\mathfrak{p}\,;\,N\mathfrak{p}\leqq x\}.$$

Note that

$$4\#\{\omega ; N\omega \leqq x\} \sim \pi(x ; K) \qquad (x \to \infty).$$

Then to prove Lemma 3 it is sufficient to show the similar assertion in which the left hand side of (10) is replaced by

$$\nu_x(\mathfrak{p} ; |\arg(L(1, \chi_{\mathfrak{p}})) - \arg(L_{h(x)}(1, \chi_{\mathfrak{p}}))| > f(x)).$$

Furthermore this reduces to the existence of $f(x)$ and $g(x)$ such that

$$\nu_x\left(\mathfrak{p} ; \left|\frac{L(1, \chi_{\mathfrak{p}})}{L_{h(x)}(1, \chi_{\mathfrak{p}})} - 1\right| > f(x)\right) \ll g(x)$$

or, equivalently,

$$\nu_x(\mathfrak{p} ; |\sum_{q > h(x)} \chi_{\mathfrak{p}}(q) q^{-1}| > f(x)) \ll g(x). \tag{11}$$

We have the following fact (Elliott [6], Proof of Lemma 22.7): Let y and Q be real numbers with $Q \geqq 9$ and $(\log Q)^{20} \leqq y \leqq Q^2$. Then there exist positive constants c_1 and c_2 such that

$$\#\{\chi ; |\sum_{q > y} \chi(q) q^{-1}| > c_1 y^{-1/10}\} < c_2 Q^{7/8}.$$

Here χ denotes primitive Dirichlet characters to moduli not exceeding Q. Putting $H(x) = (\log x)^{20}$ and considering the case $Q = x$ and $y = H(x)$, we get

$$\nu_x(\mathfrak{p} ; |\sum_{q > H(x)} \chi_{\mathfrak{p}}(q) q^{-1}| > c_1 (\log x)^{-2}) \ll x^{-1/8} \log x. \tag{12}$$

To deal with the contribution from prime numbers q such that $h(x) < q \leqq H(x)$ we need a lemma.

LEMMA 4. *Let x and y be positive real numbers satisfying $y \leqq (\log x)^A$ where A is a positive constant. Then there exists $c > 0$ so that the following estimate holds for arbitrary complex numbers $a_1, a_2, \cdots$:*

$$\sum_{N\mathfrak{p} \leqq x} \left| \sum_{m \leqq y} a_m \left(\frac{m}{\mathfrak{p}}\right)_4 \right|^2 \ll \pi(x ; K) \left| \sum_{\substack{mn^{-1} \in (Q(i)^{\times})^4 \\ m, n \leqq y}} a_m \overline{a_n} \right| + x e^{-c\sqrt{\log x}} \Big(\sum_{m \leqq y} |a_m| \Big)^2.$$

PROOF. This kind of formula has been considerded in Elliott [3]. The left hand side is equal to

$$\sum_{m, n \leqq y} a_m \overline{a_n} \sum_{N\mathfrak{p} \leqq x} \left(\frac{m}{\mathfrak{p}}\right)_4 \overline{\left(\frac{n}{\mathfrak{p}}\right)_4}$$

$$= \Big(\sum_{\substack{mn^{-1} \in (Q(i)^{\times})^4 \\ m, n \leqq y}} a_m \overline{a_n} \Big) \cdot \pi(x ; K) + \sum_{\substack{mn^{-1} \notin (Q(i)^{\times})^4 \\ m, n \leqq y}} a_m \overline{a_n} \sum_{N\mathfrak{p} \leqq x} \left(\frac{mn^3}{\mathfrak{p}}\right)_4.$$

The map $\mathfrak{p} \mapsto \left(\frac{mn^3}{\mathfrak{p}}\right)_4$ gives an ideal class character whose conductor divides

$16mn$. Hence, by the result of Mitsui quoted in the proof of Lemma 2, there is a positive constant c so that

$$\sum_{N\mathfrak{p}\leq x}\left(\frac{mn^3}{\mathfrak{p}}\right)_4 = O(xe^{-c\sqrt{\log x}})$$

holds uniformly for all $m, n \leqq y$ with $mn^{-1} \notin (\boldsymbol{Q}(i)^{\times})^4$. Furthermore we have

$$\left|\sum_{\substack{mn^{-1}\notin(\boldsymbol{Q}(i)^{\times})^4\\ m,n\leqq y}} a_m \overline{a_n}\right| \leqq \sum_{m,n\leqq y} |a_m a_n| = \Big(\sum_{m\leqq y}|a_m|\Big)^2 .$$

This completes the proof.

For positive integers m and all sufficiently large real numbers x we inductively define functions $\log_m x$ by

$$\log_m x = \log(\log_{m-1} x), \qquad m \geqq 2$$

with

$$\log_1 x = \log x, \qquad x > 0 .$$

LEMMA 5. $$\nu_x\left(\mathfrak{p}; \left|\sum_{h(x)<q\leqq H(x)} \chi_{\mathfrak{p}}(q) q^{-1}\right| > \frac{1}{\log_3 x}\right) \ll \frac{\log_3 x}{\log_2 x}.$$

PROOF. Use Lemma 4 with $y = H(x)$ and

$$a_n = \begin{cases} \dfrac{1}{q}, & n = q \text{ is an odd prime greater than } h(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{N\mathfrak{p}\leq x}\left|\sum_{h(x)<q\leqq H(x)} \chi_{\mathfrak{p}}(q) q^{-1}\right|^2$$

$$\ll \pi(x;K)\sum_{h(x)<q\leqq H(x)} q^{-2} + xe^{-c\sqrt{\log x}}\Big(\sum_{h(x)<q\leqq H(x)} q^{-1}\Big)^2 \ll \frac{\pi(x;K)}{\log_2 x \log_3 x}$$

because

$$\sum_{h(x)<q} q^{-2} \sim \frac{1}{h(x)\log h(x)}, \qquad \sum_{q\leqq H(x)} q^{-1} \ll H(x) .$$

Therefore, denoting by M the left hand side of the formula to be proved, we have

$$\pi(x;K)M(\log_3 x)^{-2} \ll \frac{\pi(x;K)}{\log_2 x \log_3 x} \qquad \text{and} \qquad M \ll \frac{\log_3 x}{\log_2 x} .$$

Combining this lemma with (12), we see that (11) holds with

$$f(x) = \frac{1}{\log_3 x}, \qquad g(x) = \frac{\log_3 x}{\log_2 x} .$$

Lemma 3 is proved.

§3. The distribution of $\arg(L(1, \chi_\omega)/\omega^{1/2})$.

We first prove Theorem 3. Let $-1/2 \leqq z \leqq 3/8$. For an arbitrary positive integer N,

$$(13)\qquad \nu_x\Big(\omega;\, -\frac{1}{2} \leqq \frac{1}{2\pi}\arg(L(1, \chi_\omega)/\omega^{1/2}) \leqq z\Big)$$

$$\leqq \sum_{n=1}^{N} \nu_x\Big(\omega;\, \frac{n-1}{4N} \leqq \frac{1}{2\pi}\arg(\omega) \leqq \frac{n}{4N},\ -\frac{1}{2}+\frac{n-1}{8N} \leqq \frac{1}{2\pi}\arg(L(1, \chi_\omega)) \leqq z+\frac{n}{8N}\Big).$$

Hence, by Theorem 2, the superior limit of (13) when x tends to the infinity does not exceed

$$\frac{1}{N}\sum_{n=1}^{N}\Big(G\Big(z+\frac{n}{8N}\Big)-G\Big(-\frac{1}{2}+\frac{n-1}{8N}\Big)\Big).$$

Similarly, the inferior limit is not smaller than

$$\frac{1}{N}\sum_{n=1}^{N}\Big(G\Big(z+\frac{n-1}{8N}\Big)-G\Big(-\frac{1}{2}+\frac{n}{8N}\Big)\Big).$$

Theorem 3 follows because N is arbitrary. We remark here that if $3/8<z<1/2$ the limit of (13) when x tends to the infinity is equal to

$$8\int_0^{1/2-z} G(z+\theta)d\theta+8\int_{-1/2-z}^{-7/8} G(z+\theta)d\theta-8\int_0^{1/8} G\Big(-\frac{1}{2}+\theta\Big)d\theta .$$

Let us turn to the proof of Theorem 1. What we have to do is to calculate the quantity (8). It is easy to see from (7) that

$$8\int_0^{1/8} G(z+\theta)d\theta=\frac{7}{8}+z+\frac{2}{\pi^2}\sum_{k\in Z-\{0\}}\frac{\varphi(k)}{k^2}(1-e^{-\pi ik/4})e^{-2\pi ikz}.$$

Therefore (8) is equal to

$$\frac{1}{2}+\frac{2}{\pi^2}\sum_{k\in Z-\{0\}}\frac{\varphi(k)}{k^2}(1-e^{-\pi ik/4})(e^{-\pi ik/2}-e^{\pi ik/2})$$

$$(14)\qquad =\frac{1}{2}+\frac{4}{\pi^2}\sum_{\substack{k=1\\ 2\nmid k}}^{\infty}\frac{\varphi(k)}{k^2}i^k(e^{-\pi ik/4}-e^{\pi ik/4})=\frac{1}{2}+\frac{4\sqrt{2}}{\pi^2}\sum_{\substack{k=1\\ 2\nmid k}}^{\infty}\frac{\varphi(k)}{k^2}(-1)^{(k^2-1)/8}.$$

LEMMA 6. $$\frac{4\sqrt{2}}{\pi^2}\sum_{k>2500}\frac{\varphi(k)}{k^2}<1.3\times 10^{-5}.$$

PROOF. We first prove that

$$(15)\qquad \varphi(k)<\exp\Big(-\frac{k}{100\log(4k)}\Big)$$

for $k \geqq 2500$. By the estimate (9) it is sufficient to prove that

$$S(N):=\sum_{q>N}q^{-2}>\frac{1}{5N\log N} \tag{16}$$

for integers $N\geqq 10000$. The partial summation formula shows

$$S(N)=\sum_{n>N}\pi(n)(n^{-2}-(n+1)^{-2})-\pi(N)N^{-2},$$

where $\pi(x)$ denotes the total number of rational primes not exceeding x. It is known (Rosser [**9**]) that

$$\frac{x}{\log x+2}<\pi(x)<\frac{x}{\log x-4}\qquad (x\geqq 55)\qquad \text{and}$$

$$\pi(x)<\frac{x}{\log x-2}\qquad (e^2<x\leqq e^{100}).$$

Take positive real numbers ε_1, ε_2, ε_3 and ε_4 such that

$$x^{-2}-(x+1)^{-2}\geqq 2(1-\varepsilon_1)x^{-3},$$

$$\frac{1}{x^2\log x}\geqq(1-\varepsilon_2)\left(\frac{1}{x^2\log x}+\frac{1}{x^2(\log x)^2}\right),$$

$$\frac{x}{\log x+2}\geqq(1-\varepsilon_3)\frac{x}{\log x},$$

$$\frac{x}{\log x-2}\leqq(1+\varepsilon_4)\frac{x}{\log x}$$

for $x\geqq 10000$. Then, if $10000\leqq N\leqq e^{100}$,

$$\begin{aligned}S(N)&>2(1-\varepsilon_1)(1-\varepsilon_3)\sum_{n>N}\frac{1}{x^2\log x}-(1+\varepsilon_4)\frac{1}{N\log N}\\&>2\prod_{i=1}^{3}(1-\varepsilon_i)\sum_{n>N}\left(\frac{-1}{x\log x}\right)'-(1+\varepsilon_4)\frac{1}{N\log N}\\&>\left\{2\prod_{i=1}^{3}(1-\varepsilon_i)-(1+\varepsilon_4)\right\}\frac{1}{N\log N}.\end{aligned}$$

We can put

$$\varepsilon_1=\frac{30002}{2\cdot 10000^2},\qquad \varepsilon_2=\frac{1}{\log 10000+1},$$

$$\varepsilon_3=\frac{2}{\log 10000+2},\qquad \varepsilon_4=\frac{2}{\log 10000-2}.$$

Then

$$2\prod_{i=1}^{3}(1-\varepsilon_i)-(1+\varepsilon_4)>\frac{1}{5}$$

and (16) holds if $10000\leqq N\leqq e^{100}$. The case $N\geqq e^{100}$ can be treated similarly appealing to the estimate

$$\pi(x) < \frac{x}{\log x - 4} \qquad (x \geqq 55).$$

It is seen that

$$S(N) > \frac{0.98}{N \log N} \qquad (N \geqq e^{100}).$$

Hence we have proved the inequalities (16) and (15). Take a positive real number a such that

$$\exp\left(-\frac{x}{400 \log x}\right) < x^{-a} \qquad (x > 10000).$$

Then

$$\sum_{k>2500} \frac{\varphi(k)}{k^2} < \sum_{k>2500} k^{-2} \exp\left(-\frac{k}{100 \log (4k)}\right)$$

$$\leqq \int_{2500}^{\infty} x^{-2} \exp\left(-\frac{x}{100 \log (4x)}\right) dx \leqq 4 \int_{10000}^{\infty} x^{-2-a} dx = \frac{4}{1+a} 10000^{-1-a}.$$

Taking $a=0.29$ we see that the estimate of Lemma 6 holds.

LEMMA 7.
$$\frac{4\sqrt{2}}{\pi^2} \sum_{25 \leqq k \leqq 2500} \frac{\varphi(k)}{k^2} < 0.3 \times 10^{-5}.$$

PROOF. We have, for $h=541$ (the 100-th prime number),

$$\varphi(k) \leqq \varphi_h(k) < 10^{-4} \qquad (25 \leqq k \leqq 2500).$$

Here the second inequality is the result of some computer calculation. The lemma follows easily from this estimate.

LEMMA 8. *Assume that h and k are positive integers with $h \geqq 2k$. Then*

$$0 \leqq \varphi_h(k) - \varphi(k) \leqq \frac{k^2}{h} \varphi_h(k).$$

PROOF. What must be shown is that

$$\prod_{q>h} \phi(q, k) \geqq 1 - \frac{k^2}{h} \qquad (h \geqq 2k).$$

If $k/q \leqq 1/2$, we have

$$\frac{1}{2}\left\{\left(1+\frac{i}{q}\right)^k + \left(1-\frac{i}{q}\right)^k\right\} = \sum_{\substack{r=0 \\ 2|r}}^{k} (-1)^{r/2} \binom{k}{r} q^{-r}$$

$$\geqq 1 - \left(\frac{k}{q}\right)^2 - \left(\frac{k}{q}\right)^6 - \cdots \geqq 1 - \frac{16}{15}\left(\frac{k}{q}\right)^2$$

and

$$|1+iq^{-1}|^{-k} = (\sqrt{1+q^{-2}})^{-k} \geqq \left(1-\frac{1}{2q^2}\right)^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \left(\frac{k}{2q^2}\right)^r$$

$$\geqq 1-\frac{k}{2q^2}-\left(\frac{k}{2q^2}\right)^3-\left(\frac{k}{2q^2}\right)^5-\cdots \geqq 1-\frac{144}{143}\cdot\frac{k}{2q^2} \geqq 1-\frac{72}{143}\left(\frac{k}{q}\right)^2.$$

Hence

$$\phi(q,k) \geqq 1-\frac{1}{2}\left(\frac{16}{15}+\frac{72}{143}\right)\left(\frac{k}{q}\right)^2 \geqq \exp\left(-\left(\frac{k}{q}\right)^2\right) \qquad (q \geqq 2k)$$

and

$$\prod_{q>h}\phi(q,k) \geqq \exp\left(-k^2\sum_{q>h}q^{-2}\right) \geqq \exp\left(-\frac{k^2}{h}\right) \geqq 1-\frac{k^2}{h} \qquad (h \geqq 2k).$$

The above lemma shows

$$\left|\sum_{\substack{k=1\\2\nmid k}}^{23}\frac{\varphi(k)}{k^2}(-1)^{(k^2-1)/8}-\sum_{\substack{k=1\\2\nmid k}}^{23}\frac{\varphi_h(k)}{k^2}(-1)^{(k^2-1)/8}\right| \leqq \frac{1}{h}\sum_{\substack{k=1\\2\nmid k}}^{23}\varphi_h(k) \tag{17}$$

for every integer $h \geqq 46$. We list the values of $\varphi_h(k)$ for $h=224737$ (the 20000-th prime number) which are calculated by the help of an electronic computer:

k	$\varphi_h(k)$	k	$\varphi_h(k)$
1	0.9527050 ⋯	3	0.6400176 ⋯
5	0.2706640 ⋯	7	0.0581728 ⋯
9	0.0020544 ⋯	11	0.0016605 ⋯
13	0.0019098 ⋯	15	0.0001763 ⋯
17	0.0001176 ⋯	19	0.0002221 ⋯
21	0.0000350 ⋯	23	0.0000116 ⋯

We see

$$\frac{1}{2}+\frac{4\sqrt{2}}{\pi^2}\sum_{\substack{k=1\\2\nmid k}}^{23}\frac{\varphi_h(k)}{k^2}(-1)^{(k^2-1)/8}=0.99976\cdots \qquad (h=224737) \tag{18}$$

and

$$\frac{4\sqrt{2}}{\pi^2 h}\sum_{\substack{k=1\\2\nmid k}}^{23}\varphi_h(k) < 10^{-5} \qquad (h=224737). \tag{19}$$

Lemma 6, Lemma 7, (17) and (19) show that the absolute value of the difference of (14) and (18) is smaller than 2.6×10^{-5}. This completes the proof of Theorem 1.

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