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On the existence of periodic solutions to nonlinear abstract parabolic equations

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Introduction.

This paper concerns the nonlinear parabolic equation in a real Hilbert space H, which is of the form

(E) $$\frac{d}{dt}u(t)+\partial\varphi(u(t))\ni f(t),$$

where $f\in L_{\mathrm{loc}}^{2}(\boldsymbol{R};H)$, φ is a proper l.s.c. (lower semi-continuous) convex functional on H and $\partial\varphi$ is the subdifferential of φ.

The existence of periodic solutions to (E) has been studied by many authors under some assumptions on $\partial\varphi$ and f (see [4], [7], [8], [12]).

The purpose of this paper is to show the existence of anti-periodic solutions to (E) under some condition different from coerciveness. This is motivated by the fact that generally elliptic operators defined on unbounded domains of $\boldsymbol{R}^{n}$ are not coercive. We show the existence of anti-periodic solutions in case $\partial\varphi$ is *odd* (Theorem 1.1). Next we apply this result to a nonlinear heat equation defined on an exterior domain of $\boldsymbol{R}^{n}$ (Section 3). Finally we give examples to see that the conditions assumed in Theorem 1.1 are essential for the existence of a periodic solution to (E) (see Propositions 1.1 and 1.2).

1. Results.

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\ \|$. We consider the existence of periodic solutions to the equation;

(E; φ, f) $$\frac{d}{dt}u(t)+\partial\varphi(u(t))\ni f(t).$$

Here φ is a proper l.s.c. convex functional on H and $\partial\varphi$ is the subdifferential of φ and $f\in L_{\mathrm{loc}}^{2}(\boldsymbol{R};H)$.

Let g be a locally square-integrable function on $\boldsymbol{R}$ with values in H. Then

g is said to be $2T$-periodic [T-anti-periodic] if $g(t+2T)=g(t)$ [$g(t+T)=-g(t)$] a.e. $t\in\boldsymbol{R}$.

Our main result is the following:

THEOREM 1.1. *Suppose;*

(1.1) *φ is even (i.e. $\varphi(-x)=\varphi(x)$, $x\in H$).*

(1.2) *f is T-anti-periodic.*

Then there is a unique T-anti-periodic solution to (E).

COROLLARY 1.1. *Under the conditions* (1.1) *and* (1.2), *there is a $2T$-periodic solution to* (E).

REMARK 1.1. It is known that the periodic solution to (E) is unique if φ is strictly convex.

Condition (1.1) of Theorem 1.1 differs from the topological condition given in [7]. Hence Theorem 1.1 is more useful for the case of nonlinear heat equations defined on unbounded domains of $\boldsymbol{R}^n$ (see Section 3).

Now we give some remarks on the conditions (1.1), (1.2).

The following condition often appears in considering asymptotic behavior of solutions to (E; φ, 0) (cf. [6], [10]):

(1.3) *There is a constant $c>0$ such that $\varphi(-cx)\leqq\varphi(x)$ holds for each $x\in H$.*

We claim that Corollary 1.1 does not hold under the assumptions (1.3), (1.2). In fact we have:

PROPOSITION 1.1. *There are a l.s.c. convex functional φ_1 and $f_1\in L^2_{loc}(\boldsymbol{R}; H)$ such that;*

(i) *φ_1, f_1 satisfies* (1.3), (1.2), *respectively.*

(ii) *There is no periodic solution to* (E; φ_1, f_1).

(See Section 4.)

We next consider the condition (1.2). We know;

PROPOSITION A (Haraux [7]). *Suppose that $f(\cdot)$ is $2T$-periodic and that* (E; φ, f) *has a $2T$-periodic solution. Then*

$$(2T)^{-1}\int_0^{2T} f(t)dt \in \mathrm{Cl}[\mathfrak{R}(\partial\varphi)]. \tag{1.4}$$

One gets (1.4) directly under the assumptions (1.1) and (1.2). In fact, (1.1) yields that

(1.5) $$0 \in \partial\varphi(0) \subset \mathfrak{R}(\partial\varphi).$$

On the other hand, by (1.2) one has

(1.6) $$\int_0^{2T} f(t)dt = 0.$$

(1.5) and (1.6) together yield (1.4).

Therefore one might expect that Corollary 1.1 hold if (1.6) is assumed instead of (1.2). But we have;

PROPOSITION 1.2. *There are a l.s.c. convex functional* φ_2 *and* $f_2 \in L^2_{loc}(\boldsymbol{R}; H)$ *such that;*

(i) φ_2, f_2 *satisfies* (1.1), (1.6), *respectively.*

(ii) *There is no periodic solution to* (E; φ_2, f_2).

(See Section 5.)

Finally we note that Corollary 1.1 does not hold in case of considering the equation

(E)′ $$\frac{d}{dt}u(t) + Au(t) \ni f(t),$$

where A is the infinitesimal generator of a unitary group in H. In fact we have the following example;

$$H = \boldsymbol{R}^2, \quad A = \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix}, \quad f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

(Then A is odd, f is T-anti-periodic and (E)′ has no periodic solution.)

2. Proof of Theorem 1.1.

For each $a \in \mathrm{Cl}[\mathfrak{D}(\varphi)]$ there is a unique solution $u_a \in W^{1,1}_{loc}((0, \infty); H) \cap C^0([0, \infty); H)$ to (E) with $u(0) = a$. We define a single-valued mapping S by $Sa = -u_a(T)$ for $a \in \mathrm{Cl}[\mathfrak{D}(\varphi)]$.

To show that S has a fixed point in $\mathrm{Cl}[\mathfrak{D}(\varphi)]$ we use the following fixed point theorem;

THEOREM A (Browder and Petryshyn [**5**]). *Let* S *be a nonexpansive self-mapping of a nonempty closed convex set* C *of* H. *Then* S *has a fixed point in* C *if and only if for any* $x_0 \in C$ *the sequence of Picard iterates* $\{x_n\}$ *starting at* x_0 (*i.e.* $x_{n+1} = Sx_n$) *is bounded in* H.

Let u be the solution to (E) with arbitrary initial-value $u_0 \in \mathrm{Cl}[\mathfrak{D}(\varphi)]$. Then the definition of $\{u_n\}$ means that $u_n = (-1)^n u(nT)$, $n \in \boldsymbol{N}$. Hence it is sufficient to show that the set $\{u(t);\ t\geqq 0\}$ is bounded in H.

In what follows we show the boundedness of $\{u(t);\ t\geqq 0\}$. By (1.1) the relation $\partial\varphi(-x)=-\partial\varphi(x)$ holds for each $x\in\mathfrak{D}(\partial\varphi)$. Hence

$$u'(t)-f(t)\in -\partial\varphi(u(t))=\partial\varphi(-u(t))$$

holds for a.e. $t\geqq 0$, where $u'(t)=(d/dt)u(t)$. Therefore, by (1.2) and the monotonicity of $\partial\varphi$, we have

$$\begin{aligned}\frac{d}{dt}\|u(t+T)+u(t)\|^2 &= 2(u'(t+T)+u'(t),\ u(t+T)+u(t))\\ &= 2(u'(t+T)-f(t+T)+u'(t)-f(t),\ u(t+T)-(-u(t)))\\ &\leqq 0, \qquad \text{a.e. } t\geqq 0,\end{aligned}$$

or

$$\|u(t+T)+u(t)\|\leqq\|u(T)+u(0)\|\ (=c_1), \qquad t\geqq 0. \tag{2.1}$$

On the other hand Condition (1.1) also yields that $0\in\partial\varphi(0)$. Hence

$$\begin{aligned}\frac{d}{dt}\|u(t)\| &= \|u(t)\|^{-1}(u'(t),\ u(t))\\ &= \|u(t)\|^{-1}\{(\partial\varphi(u(t))-\partial\varphi(0),\ u(t)-0)+(f(t),\ u(t))\}\\ &\leqq \|u(t)\|^{-1}\{0+\|f(t)\|\|u(t)\|\}=\|f(t)\|, \qquad \text{a.e. } t\geqq 0.\end{aligned} \tag{2.2}$$

Therefore

$$\|u(t+T)\|-\|u(t)\|\leqq\int_t^{t+T}\|f(s)\|ds=\int_0^T\|f(s)\|ds\ (=c_2), \qquad t\geqq 0. \tag{2.3}$$

Now we assume that the set $\{u(t);\ t\geqq 0\}$ is unbounded. Then there is the sequence $\{t_n\}$ in $[0,\infty)$ defined by

$$t_n=\inf\{t\geqq 0;\ \|u(t)\|\geqq n\}, \qquad n\geqq N,$$

where N is a large integer. Note by definition that

$$\|u(s)\|\leqq\|u(t_n)\|=n, \qquad 0\leqq s\leqq t_n, \quad n\geqq N. \tag{2.4}$$

Moreover by (2.2) and (1.2) one has $t_n\uparrow\infty$ as $n\to\infty$.

Fix an arbitrary $n\geqq N$ with $t_n\geqq T$. Let $v(t)$ $(t\in[t_n-T,\infty))$ be the solution of the initial-value problem

$$\begin{cases}\dfrac{d}{dt}v(t)+\partial\varphi(v(t))\ni 0, & t\geqq t_n-T,\\ v(t_n-T)=u(t_n-T).\end{cases}$$

Then one has the estimates

$$(2.5)\qquad \|v(t_n)-u(t_n)\| \leqq \int_{t_n-T}^{t_n} \|f(t)\|dt = \int_0^T \|f(t)\|dt \ (=c_2)\,; \text{ and}$$

$$(2.6)\qquad \varphi(v(t_n)) \leqq \varphi(v(t)), \qquad t\in[t_n-T,\, t_n).$$

(1.1) and (2.6) together yield that

$$(2.7)\qquad (-v(t_n),\, v'(s)) \leqq -(v(s),\, v'(s)), \qquad \text{a.e. } s\in[t_n-T,\, t_n),$$

since the definition of subdifferential yields that

$$(-v(t_n)-v(s),\, -v'(s)) \leqq \varphi(-v(t_n))-\varphi(v(s)).$$

By (2.7) and (2.4) we have

$$\begin{aligned}(2.8)\quad (v(t_n),\, v(t_n)-v(t_n-T)) &= \int_{t_n-T}^{t_n} (v(t_n),\, v'(s))ds \\ &\leqq \int_{t_n-T}^{t_n} (-v(s),\, v'(s))ds = 2^{-1}\{\|v(t_n-T)\|^2-\|v(t_n)\|^2\} \\ &\leqq 2^{-1}\|v(t_n-T)\|^2 = 2^{-1}\|u(t_n-T)\|^2 \leqq 2^{-1}n^2.\end{aligned}$$

Put $y=v(t_n)-u(t_n)$ and $z=v(t_n-T)+u(t_n)$ $(=u(t_n-T)+u(t_n))$. Then estimates (2.1) and (2.5) mean that $\|y\|\leqq c_2$ and $\|z\|\leqq c_1$, respectively. Hence

$$\begin{aligned}(2.9)\qquad (v(t_n),\, v(t_n)-v(t_n-T)) &= (u(t_n)+y,\; u(t_n)+y+u(t_n)-z) \\ &\geqq 2\|u(t_n)\|^2-(c_1+c_2)\|u(t_n)\|-c_2(c_2+c_1) \\ &= 2n^2-(c_1+c_2)n-c_2(c_2+c_1).\end{aligned}$$

(2.8) and (2.9) together yield

$$2n^2-(c_1+c_2)n-c_2(c_2+c_1) \leqq 2^{-1}n^2.$$

Since c_1 and c_2 are independent of n, this estimate is a contraction. Therefore the set $\{u(t)\,;\ t\geqq 0\}$ is bounded.

Now applying Theorem A we conclude that there is a T-anti-periodic solution to (E).

The uniqueness of the anti-periodic solution to (E) is obtained by the following:

PROPOSITION B (Baillon-Haraux [2]). *The difference of any two $2T$-periodic solutions to* (E) *is a constant vector of H.*

3. An application to a generalized Lin's equation.

Since Condition (1.1) differs from coerciveness, Theorem 1.1 seems to be more useful in case of nonlinear heat equations defined on unbounded domains of $\boldsymbol{R}^n$.

In this section, we show the existence of a solution to the equation;

$$
\begin{cases} \dfrac{\partial v}{\partial t}(x, t)-\Delta v(x, t)=0, & (x, t)\in\Omega\times\boldsymbol{R}, \\ \dfrac{\partial v}{\partial n}(x, t)+g[v(x, t)-h(x, t)]=0, & (x, t)\in\Gamma\times\boldsymbol{R}, \end{cases} \tag{3.1}
$$

with

$$
-v(x, t+T)=v(x, t), \qquad (x, t)\in\Omega\times\boldsymbol{R}. \tag{3.2}
$$

Here Ω is an exterior domain of $\boldsymbol{R}^n$ with smooth compact boundary Γ and n denotes the outer normal vector on Γ.

The equation (3.1) with $n=1$ ($\Omega=[0, \infty)$) is discussed in [**1**; Section 6.2]. According to [**1**] the function g with argument $v(x, t)-h(x, t)$ has the form $c_1[v(x, t)-c_2\sin t]^3$ in Lin's problem and is also a power function in radiation problems. In most physical situation g and h are continuous and $h(t)$ is periodic, representing a pulsating energy source.

Our result is the following:

THEOREM 3.1. *Suppose;*

(g1) *g is a nondegenerate measurable function on $\boldsymbol{R}$,*

(g2) *g is odd (i.e. $g(-r)=-g(r)$, $r\in\boldsymbol{R}$),*

(h1) *$h(\cdot, t)\in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}; C^2(\Gamma))$,*

(h2) *$h(\cdot, t+T)=-h(\cdot, t)$, $t\in\boldsymbol{R}$.*

Then there is a unique solution $v\in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}; L^2(\Omega))$ to (3.1) *and* (3.2).

To show this we express the equation (3.1) in the subdifferential form

$$
\frac{d}{dt}u(t)+\partial\varphi(u(t))\ni f(t), \qquad t\in\boldsymbol{R}, \tag{3.3}
$$

which is defined in the space $L^2(\Omega)$, as follows:

Extend the function h on $\bar{\Omega}\times\boldsymbol{R}$ satisfying $h(\cdot, t)\in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}; L^2(\Omega))\cap L^2_{\mathrm{loc}}(\boldsymbol{R}; H^2(\Omega))$, $h(\cdot, t+T)=-h(\cdot, t)$, $t\in\boldsymbol{R}$, and $(\partial/\partial n)h(x, t)=0$, $(x, t)\in\Gamma\times\boldsymbol{R}$. Put

$$
u(x, t)=v(x, t)-h(x, t), \quad f(x, t)=\frac{\partial h}{\partial t}(x, t)-\Delta h(x, t).
$$

Then we have the following;

(3.4) $\quad f(\cdot, t)\in L^2_{\mathrm{loc}}(\boldsymbol{R}; L^2(\Omega))$ with $f(\cdot, t+T)=-f(t)$, $t\in\boldsymbol{R}$.

(3.5) $\quad v(\cdot, t)\in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}; L^2(\Omega))$ if and only if $u(\cdot, t)\in W^{1,2}_{\mathrm{loc}}(\boldsymbol{R}; L^2(\Omega))$.

(3.6) $\quad v$ satisfies (3.1) if and only if u satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t}(x, t)-\Delta u(x, t) = f(x, t), & (x, t)\in\Omega\times\boldsymbol{R}, \\ \dfrac{\partial u}{\partial n}(x, t)+g[u(x, t)] = 0, & (x, t)\in\Gamma\times\boldsymbol{R}. \end{cases}$$

Put

$$\varphi(u) = \begin{cases} 2^{-1}\int_{\Omega}|\nabla u(x)|^2dx+\int_{\Gamma}G[u(s)]ds, \\ \qquad \text{if } u\in H(\Omega) \text{ and the second term is finite,} \\ +\infty, \quad \text{otherwise,} \end{cases} \tag{3.7}$$

where G is the function defined by $G(r)=\int_0^r g(s)ds$, $r\in\boldsymbol{R}$. Since g is nonnegative by (g1), G is a convex function on $\boldsymbol{R}$. Hence φ is a l.s.c. convex functional on $L^2(\Omega)$. By definition,

$$\begin{aligned} \mathfrak{D}(\partial\varphi) &= \left\{u\in H^2(\Omega)\,;\ \frac{\partial u}{\partial n}(s)+g[u(s)]=0 \ \text{ on } \Gamma\right\} \\ \partial\varphi(u) &= \{-\Delta u\} \qquad \text{for } u\in\mathfrak{D}(\partial\varphi). \end{aligned} \tag{3.8}$$

By (3.5), (3.6) and (3.8), we have;

LEMMA 3.1. *$v\in W^{1,2}_{loc}(\boldsymbol{R}\,; L^2(\Omega))$ is a solution to* (3.1) *if and only if u is a solution to the equation* (3.3) *with φ defined by* (3.7).

((3.5), (3.6) and (3.8), hence also Lemma 3.1, are obtained under the assumptions (g1) and (h1).) Next, by (g2) and (h2), we have;

LEMMA 3.2. (i) *φ is even, and* (ii) $f(\cdot, t+T)=-f(\cdot, t)$, $t\in\boldsymbol{R}$.

Now, applying Theorem 1.1, we get the existence and the uniqueness of the solution to (3.1) and (3.2). Hence we proved Theorem 3.1.

4. Proof of Proposition 1.1.

To prove Proposition 1.1 we constract a l.s.c. convex functional φ_1 and $f_1\in L^2_{loc}(\boldsymbol{R}\,;H)$ with the following property;

(a) f_1 is T-anti-periodic.

(b) There are a l.s.c. convex functional ϕ on H and $c\in(0, 1]$ such that

(4.1) $\partial\phi$ is linear, and

$$c\{\varphi_1(x)-\varphi_1(0)\} \leqq \phi(x)-\phi(0) \leqq \varphi_1(x)-\varphi_1(0), \qquad x\in H. \tag{4.2}$$

(c) There is no periodic solution of (E; φ_1, f_1).

REMARK 4.1. Property (b) yields (1.3). In fact, (4.1) yields that ϕ is even. Hence by (4.2)

$$\begin{aligned}&\varphi_1(-cx)-\varphi_1(0) \leqq c^{-1}\{\phi(-cx)-\phi(0)\} = c^{-1}\{\phi(cx)-\phi(0)\}\\&= c^{-1}\{\phi(cx+(1-c)0)-\phi(0)\} \leqq c^{-1}\{c\phi(x)+(1-c)\phi(0)-\phi(0)\}\\&= \phi(x)-\phi(0) \leqq \varphi_1(x)-\varphi_1(0), \qquad x\in H.\end{aligned}$$

This estimate means that (1.3) holds.

We construct φ_1, ϕ and f_1 in the space l^2. Let ε, $\varepsilon_1>0$ and $\{e_i\}_{i\geq 0}$ be the orthogonal basis of l^2. Put

$$\begin{aligned}&z_1 = e_0-\sum_{n=1}^{\infty}\varepsilon^n e_n, \qquad z_2 = e_0+\sum_{n=1}^{\infty}\varepsilon^n e_n,\\&X_1 = \{x\in l^2;\ (z_1, x)>0 \text{ and } (e_0, x)>0\},\\&X_2 = \{x\in l^2;\ (z_2, x)>0 \text{ and } (e_0, x)<0\},\end{aligned}$$

where $(\cdot, \cdot)$ denotes the inner product in l^2. We define the functionals as follows:

$$\phi(x) = 6^{-1}(e_0, x)^2+3^{-1}\sum_{n=1}^{\infty}(2n)^{-1}\varepsilon^n(e_n, x)^2,$$

$$\varphi_1(x) = \begin{cases} 2^{-1}(z_1, x)^2+3\phi(x) & \text{if } x\in \bar{X}_1,\\ 2^{-1}(z_2, x)^2+3\phi(x) & \text{if } x\in \bar{X}_2,\\ 3\phi(x) & \text{otherwise,}\end{cases}$$

$$f(t) = \rho(t)e_0,$$

where

$$\rho(t) = \begin{cases} 1, & t\in[2mT, (2m+1)T-\varepsilon_1),\\ -2\varepsilon_1^{-1}, & t\in[(2m+1)T-\varepsilon_1, (2m+1)T),\\ -1, & t\in[(2m+1)T, (2m+2)T-\varepsilon_1),\\ 2\varepsilon_1^{-1}, & t\in[(2m+2)T-\varepsilon_1, (2m+2)T).\end{cases} \tag{4.3}$$

Then properties (a) and (b) hold with arbitrary ε, $\varepsilon_1\in(0, 1)$.

We claim that (c) holds with sufficiently small ε, $\varepsilon_1>0$. Indeed, let $u\in W^{1,1}_{\text{loc}}([0, \infty); l^2)$ be the solution of the initial-value problem

$$\begin{cases} \dfrac{d}{dt}u(t)+\partial\varphi_1(u(t)) \ni f_1(t), & t>0,\\ u(0) = e_0. \end{cases}$$

To see (c), we have only to show that the set $\{u(2mT);\ m\in N\}$ is unbounded in l^2 with the aid of Theorem A in Section 2. We show this in a few lemmas.

LEMMA 4.1. *Put* $u_k(t)=(e_k, u(t))$, $k=0, 1, 2, \cdots$, *and* $a(t)=\sum_{k=1}^{\infty}\varepsilon^k u_k(t)$. *Then one has*

$$\frac{d}{dt}u_0(t) = \begin{cases} -u_0(t)+a(t)+\rho(t) & \textit{if } u(t)\in X_1,\\ -u_0(t)-a(t)+\rho(t) & \textit{if } u(t)\in X_2,\\ -u_0(t)+\theta(t)a(t)+\rho(t) & \textit{otherwise,}\end{cases} \tag{4.4}$$

with $\theta(t)\in[-1, 1]$, *and, for* $n\geqq 1$,

$$\frac{d}{dt}u_n(t)=\begin{cases}\varepsilon^n\{u_0(t)-a(t)-n^{-1}u_n(t)\}, & \text{if } u(t)\in X_2,\\ \varepsilon^n\{u_0(t)-a(t)-n^{-1}u_n(t)\}, & \text{if } u(t)\in X_2,\\ \varepsilon^n\{-a(t)-n^{-1}u_n(t)\}, & \text{if } u(t)\in \bar{X}_1\cap\bar{X}_2,\\ \varepsilon^n\{-n^{-1}u_n(t)\}, & \text{otherwise}.\end{cases} \tag{4.5}$$

PROOF. By definition, one has

$$\partial\varphi_1(x)=\begin{cases}(x_0-\alpha(x))e_0+\sum_{n=1}^{\infty}\varepsilon^n(-x_0+\alpha(x)+n^{-1}x_n)e_n, & \text{if } x\in X_1,\\ (x_0+\alpha(x))e_0+\sum_{n=1}^{\infty}\varepsilon^n(x_0+\alpha(x)+n^{-1}x_n)e_n, & \text{if } x\in X_2,\\ \left\{\theta\alpha(x)e_0+\sum_{n=1}^{\infty}\varepsilon^n(\alpha(x)+n^{-1}x_n)e_n;\ \theta\in[-1, 1]\right\}, & \text{if } x\in\bar{X}_1\cap\bar{X}_2,\\ x_0e_0+\sum_{n=1}^{\infty}\varepsilon^n n^{-1}x_ne_n, & \text{otherwise},\end{cases}$$

where $x_k=(e_k, x)$, $k=0, 1, 2, \cdots$, and $\alpha(x)=\sum_{k=1}^{\infty}\varepsilon^k x_k$. In fact, for example, if $x\in X_1$ then one has

$$\begin{aligned}\partial\varphi_1(x)&=(z_1, x)z_1+\sum_{n=1}^{\infty}n^{-1}\varepsilon^n(e_n, x)e_n\\ &=(x_0-\alpha(x))\left\{e_0-\sum_{n=1}^{\infty}n^{-1}\varepsilon^n e_n\right\}+\sum_{n=1}^{\infty}n^{-1}\varepsilon^n x_n e_n.\end{aligned}$$

Noting that $\alpha(u(t))=a(t)$ and that $x_0=0$ on $\bar{X}_1\cap\bar{X}_2$, we get both (4.4) and (4.5).

LEMMA 4.2. *Let* $\delta>0$ *be fixed. Let* ε, $\varepsilon_1>0$ *be such that*

$$(1+2\delta)\varepsilon(1-\varepsilon)^{-2}<10^{-1}\delta, \tag{4.6}$$

$$(1+2\delta)\varepsilon_1<\frac{2}{3}\delta\left(1-\frac{1}{T+1}\right). \tag{4.7}$$

Then for each $t\geqq 0$ *one has*

$$|a(t)|\leqq 3^{-1}\delta, \tag{4.8}$$

$$\begin{cases}|u_0(t)-1|\leqq\delta & \text{if } t\in[2mT, (2m+1)T-\varepsilon_1),\\ |u_0(t)+1|\leqq\delta & \text{if } t\in[(2m+1)T, (2m+2)T-\varepsilon_1),\\ |u_0(t)|\leqq 1+\delta & \text{otherwise}.\end{cases} \tag{4.9}$$

PROOF. Put $I=\{t>0; |a(t)|\leqq 3^{-1}\delta\}$. Since $u(0)=e_0$, we see by (4.5) that there is a positive number t_0 satisfying $[0, t_0)\subset I$. We first show that (4.9) holds for $t\in[0, t_0)$. By (4.4)

$$|(d/dt)u_0(t)+u_0(t)-\rho(t)|\leqq\delta/3 \quad \text{for } t\in[0, t_0]. \tag{4.10}$$

Suppose that a nonnegative integer m satisfies

$$(4.11)\qquad 2mT \leqq t_0 \quad \text{and} \quad |u_0(2mT)-1| \leqq \delta .$$

By the definition of ρ, if $t\in[2mT, (2m+1)T-\varepsilon_1)\cap[0, t_0]$ then

$$(d/dt)u_0(t) \leqq -r \qquad \text{if } u_0-1\geqq 3^{-1}\delta+r,$$
$$(d/dt)u_0(t) \leqq r \qquad \text{if } u_0-1\geqq -(3^{-1}\delta+r),$$

where $r=2\delta/\{3(T-\varepsilon_1+1)\}$ $(<(2/3)\delta)$. Hence

$$|u_0(t)-1| \leqq \delta, \qquad \text{if } t\in[2mT, (2m+1)T-\varepsilon_1)\cap[0, t_0],$$
$$|u_0((2m+1)T-\varepsilon_1)-1| \leqq 3^{-1}\delta+r, \qquad \text{if } (2m+1)T-\varepsilon_1\leqq t_0.$$

If $t\in[(2m+1)T-\varepsilon_1, (2m+1)T]\cap[0, t_0]$, then

$$|u_0(t)| \leqq 1+\delta, \qquad \text{if } t\in[(2m+1)T-\varepsilon_1, (2m+1)T]\cap[0, t_0],$$
$$|u_0((2m+1)T)+1| \leqq \delta, \qquad \text{if } (2m+1)T\leqq t_0.$$

Similarly we can show

$$|u_0(t)+1| \leqq \delta, \qquad \text{if } t\in[(2m+1)T, (2m+2)T-\varepsilon_1)\cap[0, t_0],$$
$$|u_0(t)| \leqq 1+\delta, \qquad \text{if } t\in[(2m+2)T-\varepsilon_1, (2m+2)T]\cap[0, t_0],$$
$$|u_0((2m+2)T)-1| \leqq \delta, \qquad \text{if } (2m+2)T\leqq t_0.$$

Since $u(0)=e_0$, integer 0 satisfies the assumption (4.11). Now it is easy to see that (4.9) holds for each $t\in[0, t_0]$.

Next we show that $I=[0, \infty)$. By (4.8) and (4.5) one has

$$|(d/dt)u_n(t)+\varepsilon^n n^{-1}u_n(t)| \leqq \varepsilon^n(1+2\delta)$$

for $t\leqq t_0$ and $n\geqq 1$. Hence we get

$$(4.12)\qquad |u_n(t)| \leqq n(1+2\delta), \qquad t\leqq t_0,\ n\geqq 1.$$

By (4.12) and (4.6) one has

$$|a(t)| = \left|\sum_{k=1}^{\infty} \varepsilon^k u_k(t)\right| \leqq (1+2\delta)\sum_{k=1}^{\infty} \varepsilon^k k = (1+2\delta)\varepsilon(1-\varepsilon)^{-2} \leqq 10^{-1}\delta$$

for $0\leqq t\leqq t_0$, from which it follows that $I=[0, \infty)$.

Consequently estimates (4.8) and (4.9) hold for each $t\geqq 0$.

LEMMA 4.3. *There is a sequence* $\{t_n\}\subset[0, \infty)$ *satisfying*

$$(4.13)\qquad \|u(t_n)\| \geqq n(1-2\delta), \qquad n\geqq 1.$$

PROOF. We see by (4.8) that

$$u(t) \in X_1 \quad \text{if } |u(t_0)-1| \leqq \delta,$$
$$u(t) \in X_2 \quad \text{if } |u(t_0)+1| \leqq \delta.$$

Hence by (4.5) and (4.9) we see that for each $n \geqq 1$, there is a positive number t_n satisfying $u_n(t_n) \geqq n(1-2\delta)$. Since $\|u(t_n)\| \geqq |u_n(t_n)|$ by the definition of $u_n(t)$, we have (4.13).

5. Proof of Proposition 1.2.

We constract φ_1, f_1 with required properties in the space l^2. Let ε, $\varepsilon_1>0$. Put

$$z_1 = e_0 - \sum_{n=1}^{\infty} \varepsilon^n e_n, \qquad z_2 = e_0 + \sum_{n=1}^{\infty} \varepsilon^n e_n \qquad \text{and} \quad M \geqq 2,$$

where $\{e_n\}_{n\geq 0}$ is the orthogonal basis of l^2. We define the functionals φ_2 and f_2 as follows;

$$\varphi_2(x) = \phi_1(x) + \phi_2(x),$$
$$\phi_1(x) = 2^{-1}\left\{(z_1, x)^2 + \sum_{n=1}^{\infty} n^{-1}\varepsilon^n (e_n, x)^2\right\},$$
$$\phi_2(x) = \begin{cases} 2^{-1}M\{(z_2, x)^2-4\} & \text{if } (z_2, x)^2>4, \\ 0 & \text{if } (z_2, x)^2 \leqq 4, \end{cases}$$
$$f_2(t) = \rho(t) e_0$$

with

$$\rho(t) = \begin{cases} 1, & t \in [2mT, 2mT+r-\varepsilon_1), \\ -4\varepsilon_1^{-1}, & t \in [2mT+r-\varepsilon_1, 2mT+r), \\ -3(M+1), & t \in [2mT+r, (2m+2)T-\varepsilon_1), \\ 4\varepsilon_1^{-1} & t \in [(2m+2)T-\varepsilon_1, (2m+2)T). \end{cases}$$

Here r is the constant such that $\int_0^{2T} \rho(t)dt=0$ holds. Then both (1.1) and (1.6) hold.

We claim that (E; φ_2, f_2) has no periodic solution if ε, $\varepsilon_1>0$ are sufficiently small. Let $u(t)$ be the solution of

$$\text{(5.1)} \qquad \begin{cases} \dfrac{d}{dt} u(t) + \partial\varphi_2(u(t)) \ni f_2(t), & t>0, \\ u(0) = e_0. \end{cases}$$

We show that the set $\{u(t);\, t \geqq 0\}$ is unbounded.

By definition, one has

$$\text{(5.2)} \qquad \partial\varphi_2(x) = \begin{cases} y(0), & \text{if } (z_2, x)^2<4, \\ y(M), & \text{if } (z_2, x)^2>4, \\ \{y(\theta);\, \theta \in [0, M]\}, & \text{if } (z_2, x)^2=4, \end{cases}$$

where $x_k=(e_k, x)$, $k=0, 1, 2, \cdots$, $\alpha(x)=\sum_{k=1}^{\infty}\varepsilon^k x_k$ and

$$y(\theta) = \{(\theta+1)x_0+(\theta-1)\alpha(x)\}e_0+\sum_{n=1}^{\infty}\varepsilon^n\{(\theta-1)x_0+(\theta+1)\alpha(x)+n^{-1}x_n\}e_n.$$

Put $u_n(t)=(e_0, u(t))$, $n=0, 1, 2, \cdots$, and $a(t)=\alpha(u(t))$. Then by (5.2) one has the following;

(i) If $t\in[2mT, 2mT+r-\varepsilon_1)$ and $(z_2, u(t))<4$, then

$$\frac{d}{dt}u_0(t) = -u_0(t)+a(t)+1,$$

$$\frac{d}{dt}u_n(t) = \varepsilon^n\{u_0(t)-a(t)-n^{-1}u_n(t)\}, \qquad n\geqq 1.$$

(ii) If $t\in[2mT+r, 2(m+1)T-\varepsilon_1)$ and $(z_2, u(t))>4$, then

$$\frac{d}{dt}u_0(t) = -(M+1)u_0(t)+(M-1)a(t)-3(M+1),$$

$$\frac{d}{dt}u_n(t) = \varepsilon^n\{-(M-1)u_0(t)-(M+1)a(t)-n^{-1}u_n(t)\}, \qquad n\geqq 1.$$

Therefore, putting $I=\{t\geqq 0; |a(t)|\leqq 3^{-1}\delta\}$ for a fixed $\delta>0$, we obtain in the same way as in Section 4 that $I=[0, \infty)$. Moreover, as is seen in Section 4, it follows from (i) and (ii) that for each $n\geqq 1$ there is a positive number t_n satisfying

$$u_n(t_n) > n(1-\delta).$$

This estimate means that the set $\{u(t); t\geqq 0\}$ is unbounded in l^2, or equivalently that $(E; \varphi_2, f_2)$ has no periodic solution.

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