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On local deformations of a Banach space of analytic functions on a Riemann surface

Dedicated to Professor Kôtaro Oikawa on his 60th birthday

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§1. Introduction.

Let $\mathcal{S}$ be the set consisting of all compact bordered Riemann surfaces. For $\bar{S}$ in $\mathcal{S}$, we denote its interior and its border by S and ∂S, respectively. Let us denote by p ($\geqq 0$) the genus of $\bar{S}$ and by q ($\geqq 1$) the number of boundary components of $\bar{S}$. We set

$$N=2p+q-1.$$

Furthermore we denote by $A(S)$ the set of all functions which are analytic in S and continuous on $\bar{S}$. It forms a Banach space with the supremum norm

$$\|f\|=\sup_{z\in S}|f(z)|.$$

For $\bar{S}$ and $\bar{S}'$ in $\mathcal{S}$, let $L(A(S), A(S'))$ denote the set of all continuous invertible linear mappings of $A(S)$ onto $A(S')$. It is shown by Rochberg [6] that $L(A(S), A(S'))$ is nonvoid if S and S' are homeomorphic. We set

$$c(T)=\|T\|\|T^{-1}\|$$

for T in $L(A(S), A(S'))$. We have always

$$c(T)\geqq 1,$$

and if $T1=1$, we see that

$$1\leqq\|T\|\leqq c(T), \qquad 1\leqq\|T^{-1}\|\leqq c(T)$$

and that

$$c(T)^{-1}\|f\|\leqq\|Tf\|\leqq c(T)\|f\|$$

for all f in $A(S)$. The above inequality implies that in the case $T1=1$, T is an isometry if and only if the value $c(T)$ attains its minimum 1. Therefore the value $\log c(T)$ is considered to be the quantity representing the deviation of T from isometries. This quantity was first studied by Banach and Mazur for more general cases (cf. [2]). It is well known that if there exists an isometry T in

$L(A(S), A(S'))$ with $T1=1$, then there exists a conformal mapping ϕ of S' onto S such that T is induced by ϕ, namely

$$Tf = f\circ\phi$$

for all f in $A(S)$ (cf. [**4**]).

In order to investigate deformations of the Banach space $A(S)$, we introduce a space consisting of isomorphisms of $A(S)$. We now fix an element $\bar{S}_0\in\mathcal{S}$ and denote by $L(S_0)$ the set of all $T\in L(A(S_0), A(S))$ with $T1=1$ for all $\bar{S}\in\mathcal{S}$ that are homeomorphic to $\bar{S}_0$. For T_1 and T_2 in $L(S_0)$, we say that T_1 is equivalent to T_2, if $T_2\circ T_1^{-1}$ is an isometry. This defines an equivalence relation in $L(S_0)$. We denote by $\mathcal{L}(S_0)$ the set of all equivalence classes $[T]$ for all $T\in L(S_0)$. We define a function $d(\cdot, \cdot)$ in $\mathcal{L}(S_0)\times\mathcal{L}(S_0)$ as follows;

$$d([T_1], [T_2]) = \log c(T_2\circ T_1^{-1})$$

for $[T_1]$ and $[T_2]\in\mathcal{L}(S_0)$. We can easily see that it is well defined independently of choices of representatives T_1 and T_2 and that it defines a metric on $\mathcal{L}(S_0)$. Thus $\mathcal{L}(S_0)$ is a metric space. When we take another $\bar{S}_1\in\mathcal{S}$ which is homeomorphic to $\bar{S}_0$, we can similarly define the metric space $\mathcal{L}(S_1)$. Obviously $\mathcal{L}(S_1)$ is isometric to $\mathcal{L}(S_0)$. Hence the metric space $\mathcal{L}(S_0)$ is determined independently of choices of $\bar{S}_0$. The purpose of the present paper is to investigate the local and topological structure of the space $\mathcal{L}(S_0)$, which describes slight changes of deformations of $A(S)$ by linear isomorphisms. In § 4 we construct a continuous mapping of a neighborhood $\mathcal{U}$ of every point of $\mathcal{L}(S_0)$ into the reduced Teichmüller space $\boldsymbol{T}^{\#}(S_0)$. By using this mapping we can resolve an element of $\mathcal{U}$, namely a slight deformation of $A(S)$ into a slight deformation of S in $\boldsymbol{T}^{\#}(S_0)$ and a linear automorphism of $A(S)$ which is very close to the identity. This is our main result, which is proved in § 6.

§ 2. Some results on almost isometries.

We state here certain continuity properties on almost isometries, that is, linear isomorphisms T with $c(T)$ very close to 1. They were proved by Rochberg.

THEOREM 1. *For every $\varepsilon>0$ there exists a constant $d>1$ having the following property:*

For $\bar{S}$, $\bar{S}'\in\mathcal{S}$ and for every $T\in L(A(S), A(S'))$ satisfying $c(T)<d$ and $T1=1$, there exists a homeomorphism h of ∂S onto $\partial S'$ such that

$$|f(z)-(Tf)(h(z))| \leqq \varepsilon\|f\|$$

for all $z\in\partial S$ and for all $f\in A(S)$ (cf. [**7**]).

The following theorems express in a natural form how close almost isometries are to an isomorphism induced by a conformal mapping. Theorem 2 is a corollary to Theorem A′ in [5], and Theorem 3 is a result which was implicitly contained in the proof of Proposition 5 and Theorem 3 in [7].

THEOREM 2. *For every* $\varepsilon>0$ *and for* $\bar{S}\in\mathcal{S}$, *there exists a constant* $d>1$ *having the following property:*

If $T\in L(A(S), A(S))$ *satisfies* $c(T)<d$ *and* $T1=1$, *then there exists a unique conformal automorphism* ϕ *of* S *such that*

$$\|Tf-f\circ\phi\| \leqq \varepsilon\|f\|$$

for all $f\in A(S)$.

THEOREM 3. *Let* $\bar{S}$ *be fixed in* $\mathcal{S}$. *Then, for every* $\varepsilon>0$ *and for every relatively compact subdomain* D *of* S, *there exists a constant* $d>1$ *having the following property:*

If $T\in L(A(S), A(S'))$ *satisfies* $c(T)<d$ *and* $T1=1$ *for* $\bar{S}'\in\mathcal{S}$, *then there exists a quasiconformal mapping* w *of* S *onto* S' *such that its maximal dilatation* $K(w)$ *is less than* $1+\varepsilon$ *and*

$$|f(z)-(Tf)(w(z))| \leqq \varepsilon\|f\|$$

for all $z\in D$ *and for all* $f\in A(S)$.

PROOF. Let $\varepsilon>0$ and relatively compact subdomain D of S be arbitrarily given. Then, by the same argument as the proof of Proposition 5 and Theorem 3 in [7], we can choose a constant $d>1$ such that if $T\in L(A(S), A(S'))$ satisfies $c(T)<d$ and $T1=1$ for $\bar{S}'\in\mathcal{S}$ there exist a relatively compact subdomain R of S and a homeomorphism w of S onto S' satisfying the following conditions:

(i) D is a subdomain of R. The boundary of R consists of a finite number of analytic contours and the set $\bar{S}\setminus R$ is a union of a finite number of annuli $A_1, \cdots, A_m$.

(ii) For a finite number of parametric disks $D_1, \cdots, D_n$ in S, w is conformal in $\bar{R}\setminus\bigcup_{i=1}^{n}D_i$.

(iii) w is quasiconformal in D_i $(i=1, \cdots, n)$ and in A_i $(i=1, \cdots, m)$.

(iv) $K(w)<1+\varepsilon$.

(v) $|f(z)-(Tf)(w(z))| \leqq \varepsilon\|f\|$,

for all $z\in\bar{R}\setminus\bigcup_{i=1}^{n}D_i$ and for all $f\in A(S)$.

For $z_0\in D_i$, $z_1\in\partial D_i$ and $f\in A(S)$, we have

$$\begin{aligned}|f(z_0)-(Tf)(w(z_0))| \leqq |f(z_0)-f(z_1)|+|f(z_1)-(Tf)(w(z_1))|\\ +|(Tf)(w(z_1))-(Tf)(w(z_0))|.\end{aligned}$$

Since all functions $f/\|f\|$ and $(Tf)/\|f\|$ for $f\in A(S)$ are uniformly bounded, they are equicontinuous on every compact subset. Hence, if we choose beforehand sufficiently small D_i $(i=1, \cdots, n)$, then the first and the last terms of the right side of the above inequality are less than $\varepsilon\|f\|$. Furthermore, by (v), the second term is also less than $\varepsilon\|f\|$. Hence we obtain

$$|f(z_0)-(Tf)(w(z_0))| \leqq 3\varepsilon\|f\|$$

for all $f\in A(S)$. Therefore we may say that (v) holds for all $z\in\bar{R}$, consequently for all $z\in D$, and for all $f\in A(S)$.

§ 3. The reduced Teichmüller space.

Let $\bar{S}_0$ be fixed. If $\bar{S}\in\mathcal{S}$ is homeomorphic to $\bar{S}_0$, there exists a quasiconformal mapping w of S_0 onto S. We consider a pair (S, w). We say that two pairs (S_1, w_1) and (S_2, w_2) are equivalent, if there exists a conformal mapping ϕ of S_1 onto S_2 which is homotopic to $w_2\circ w_1^{-1}$. We denote by $[S, w]$ the equivalence class of (S, w). The set of all equivalence classes is denoted by $\boldsymbol{T}^{\#}(S_0)$. For two pairs (S_1, w_1) and (S_2, w_2), there exists a unique quasiconformal mapping w_0 whose maximal dilatation is the smallest in the family consisting of all homeomorphisms of S_1 onto S_2 homotopic to $w_2\circ w_1^{-1}$. Then we set

$$\rho([S_1, w_1], [S_2, w_2]) = \log K(w_0),$$

where $K(w_0)$ is the maximal dilatation of w_0. This quantity does not depend on choices of representatives (S_1, w_1) and (S_2, w_2). It defines a metric on $\boldsymbol{T}^{\#}(S_0)$, which is called the Teichmüller metric. The metric space $\boldsymbol{T}^{\#}(S_0)$ is called the reduced Teichmüller space of S_0.

We consider $m=\mu(z)d\bar{z}/dz$, a differential of type $(-1, 1)$ on a Riemann surface S, where $\mu(z)$ is a measurable function. Such a differential form is called a Beltrami differential on S. If

$$\|m\| = \sup_{z\in S}|\mu(z)| < 1,$$

a Riemannian metric

$$ds = |dz+\mu(z)d\bar{z}|$$

on S defines a new conformal structure on S as is well known. This new Riemann surface is denoted by S^m. The identity mapping of S onto itself is a quasiconformal mapping of S onto S^m, which is denoted by w^m, and it satisfies locally the Beltrami differential equation

$$w_{\bar{z}} = \mu w_z.$$

Let $[S_1, w_1]\in\boldsymbol{T}^{\#}(S_0)$ be an arbitrary point. The following fact is well known. There exists a $(3N-3)$-tuple $\boldsymbol{m}=(m_1, \cdots, m_{3N-3})$ of Beltrami differentials on S_1 such that every point $[S, w]$ in a neighborhood of $[S_1, w_1]$ is equal to $[S_1^{\boldsymbol{a}\cdot\boldsymbol{m}}, w^{\boldsymbol{a}\cdot\boldsymbol{m}}]$, where $\boldsymbol{a}=(a_1, \cdots, a_{3N-3})$ is a point of Euclidean space $\boldsymbol{R}^{3N-3}$ and

$$\boldsymbol{a}\cdot\boldsymbol{m} = a_1 m_1 + \cdots + a_{3N-3} m_{3N-3}.$$

We call $\boldsymbol{m}=(m_1, \cdots, m_{3N-3})$ a Beltrami basis on S_1. The mapping

$$\boldsymbol{a} = (a_1, \cdots, a_{3N-3}) \longmapsto [S_1^{\boldsymbol{a}\cdot\boldsymbol{m}}, w^{\boldsymbol{a}\cdot\boldsymbol{m}}]$$

is a homeomorphism of a neighborhood of the origin of $\boldsymbol{R}^{3N-3}$ onto a neighborhood of $[S_1, w_1] \in \boldsymbol{T}^{\#}(S_0)$. If we consider $(a_1, \cdots, a_{3N-3})$ local coordinates in the neighborhood of $[S_1, w_1]$, a real analytic structure is defined in $\boldsymbol{T}^{\#}(S_0)$. Moreover $\boldsymbol{T}^{\#}(S_0)$ is homeomorphic to $\boldsymbol{R}^{3N-3}$ (cf. [**3**]). We can construct a Beltrami basis $\boldsymbol{m}=(m_1, \cdots, m_{3N-3})$ so that m_i is infinitely differentiable in the real sense and the support of m_i is a compact subset of S_1 for $i=1, \cdots, 3N-3$ (cf. Prop. 6 in [**6**]).

§4. The construction of a mapping Φ of $\mathcal{L}(S_0)$ into $\boldsymbol{T}^{\#}(S_0)$.

Suppose that $\bar{S}_0$ and $\bar{S}_0^*$ in $\mathcal{S}$ are homeomorphic. Let T_0 be an element of $L(A(S_0), A(S_0^*))$ satisfying $T_0 1=1$, and let w_0 be a quasiconformal mapping of S_0 onto S_0^*. We consider the point $[S_0^*, w_0]$ in the reduced Teichmüller space $\boldsymbol{T}^{\#}(S_0)$. By using Theorem 3, for every $\varepsilon>0$ and relatively compact subdomain D_0 of S_0^*, there exists a constant $d>1$ having the following property: If $T\in L(A(S_0), A(S))$ satisfies $c(T\circ T_0^{-1})<d$ and $T1=1$, where $\bar{S}\in\mathcal{S}$ is homeomorphic to $\bar{S}_0$, then there exists a quasiconformal mapping w of S_0^* onto S such that $K(w) <1+\varepsilon$ and

$$|f(z)-(T\circ T_0^{-1}f)(w(z))| \leqq \varepsilon\|f\|$$

for all $z\in D_0$ and for all $f\in A(S_0^*)$. We now define a mapping Φ as follows;

$$\Phi(T) = (S, w\circ w_0).$$

In order to prove that Φ defines a mapping of a neighborhood of $[T_0]\in\mathcal{L}(S_0)$ into $\boldsymbol{T}^{\#}(S_0)$, we must show the following fact. There exist a positive number ε_0 and a relatively compact subdomain D_0 of S_0^* such that for every ε with $0<\varepsilon<\varepsilon_0$, for some $d>1$ and for any two equivalent isomorphisms $T, T'\in L(S_0)$ satisfying $c(T\circ T_0^{-1})<d$, $c(T'\circ T_0^{-1})<d$ and $T1=1$, $T'1=1$, $\Phi(T)$ and $\Phi(T')$ defined as above are equivalent, namely $\Phi(T)=(S, w\circ w_0)$ defines a point $[S, w\circ w_0]$ of $\boldsymbol{T}^{\#}(S_0)$ for $[T]\in\mathcal{L}(S_0)$.

If it were not true, then there exist the following sequences:

(i) $\varepsilon_n>0$ such that $\varepsilon_n\to 0$,

(ii) relatively compact subdomains D_n of S_0^* which exhaust S_0^*,

(iii) $d_n>1$ such that $d_n\to 1$,

(iv) $\bar{S}_n, \bar{S}'_n\in\mathcal{S}$ which are homeomorphic to $\bar{S}_0^*$,

(v) $T_n\in L(A(S_0), A(S_n))$, $T'_n\in L(A(S_0), A(S'_n))$ with $T_n 1=1$, $T'_n 1=1$,

and

(vi) quasiconformal mappings w_n of S_0^* onto S_n, w'_n of S_0^* onto S'_n, satisfying the conditions

$$(1)\qquad c(T_n\circ T_0^{-1})<d_n\,,\qquad c(T'_n\circ T_0^{-1})<d_n\,,$$

(2) T_n is equivalent to T'_n,

$$(3)\qquad K(w_n)<1+\varepsilon_n\,,\qquad K(w'_n)<1+\varepsilon_n\,,$$

$$(4)\qquad \begin{cases} |f(z)-(T_n\circ T_0^{-1}f)(w_n(z))|\leqq\varepsilon_n\|f\| \\ |f(z)-(T'_n\circ T_0^{-1}f)(w'_n(z))|\leqq\varepsilon_n\|f\| \end{cases}$$

for all $z\in D_n$ and for all $f\in A(S_0^*)$, and

$$(5)\qquad \Phi(T_n)\neq\Phi(T'_n)\,,$$

namely $(S_n, w_n\circ w_0)$ is not equivalent to $(S'_n, w_n\circ w_0)$.

By the condition (2), there exists a conformal mapping ϕ_n of S'_n onto S_n such that

$$(6)\qquad T_n\circ(T'_n)^{-1}f=f\circ\phi_n^{-1}$$

for all $f\in A(S'_n)$. We set

$$\omega_n=\phi_n\circ w'_n\,.$$

Then, the condition (5) implies that w_n is not homotopic to ω_n. By the condition (3), we note that

$$(7)\qquad K(\omega_n)=K(w'_n)<1+\varepsilon_n\,.$$

Furthermore it follows from (6) and (4) that

$$(8)\qquad \begin{aligned} |f(z)-(T_n\circ T_0^{-1}f)(\omega_n(z))| &= |f(z)-[(T_n\circ T_n'^{-1})\circ(T'_n\circ T_0^{-1})f](\omega_n(z))| \\ &= |f(z)-[(T'_n\circ T_0^{-1}f)\circ\phi_n^{-1}](\phi_n\circ w'_n(z))| \\ &= |f(z)-(T'_n\circ T_0^{-1}f)(w'_n(z))| \\ &\leqq \varepsilon_n\|f\| \end{aligned}$$

for all $z\in D_n$ and for all $f\in A(S_0^*)$.

From now on, we use the method of uniformization. Let $\bar{S}$ be in $\mathcal{S}$. The universal covering surface of S are conformally equivalent to the unit disk $U=\{|\tilde{z}|<1\}$ in the complex plane. Hence we may consider U the universal covering surface of S and denote by Γ the group of cover transformations of U over S. Γ is a finitely generated Fuchsian group of the second kind and S can be identified with the orbit space U/Γ so that the natural projection $\pi:U\to U/\Gamma$ is analytic. A function $f\in A(S)$ determines a unique function $\tilde{f}$ on U such that $\tilde{f}=f\circ\pi$ on U. It satisfies $\tilde{f}\circ\gamma=\tilde{f}$ on U for every $\gamma\in\Gamma$. We denote by $A(U,\Gamma)$ the set of such functions $\tilde{f}$ for all $f\in A(S)$. It forms a Banach space with the supremum norm. Evidently, $\|\tilde{f}\|=\|f\|$ for $f\in A(S)$. Let $\bar{S}'$ be another element of $\mathcal{S}$, Γ' be a Fuchsian group such that $S'=U/\Gamma'$ and π' be the projection of U onto U/Γ'. For $T\in L(A(S), A(S'))$ we define a continuous linear isomorphism $\tilde{T}$ of $A(U, \Gamma)$ onto $A(U, \Gamma')$ by

$$\tilde{T}\tilde{f}=(Tf)^{\sim}$$

for all $\tilde{f}\in A(U, \Gamma)$. Evidently, $\|\tilde{T}\|=\|T\|$.

Let Γ_0 and Γ_n be the Fuchsian groups which represent S_0^* and S_n, respectively;

$$S_0^*=U/\Gamma_0, \qquad S_n=U/\Gamma_n.$$

We denote by $\tilde{w}_n$ a mapping of U onto itself which satisfies

$$\pi_n\circ\tilde{w}_n=w_n\circ\pi_0$$

where $\pi_0: U\to U/\Gamma_0$ and $\pi_n: U\to U/\Gamma_n$ are the projections. Then,

$$\Gamma_n=\tilde{w}_n\circ\Gamma_0\circ\tilde{w}_n^{-1}.$$

Without loss of generality, we may assume that

$$\tilde{w}_n(0)=0, \qquad \tilde{w}_n(1)=1. \tag{9}$$

We consider the normal polygon of Γ_n with the center at the origin, which is denoted by P_n. We denote by $\tilde{\omega}_n$ a mapping of U onto itself which satisfies

$$\pi_n\circ\tilde{\omega}_n=\omega_n\circ\pi_0$$

and

$$\tilde{\omega}_n(0)\in\bar{P}_n. \tag{10}$$

We note that

$$\Gamma_n=\tilde{w}_n\circ\Gamma_0\circ\tilde{w}_n^{-1}=\tilde{\omega}_n\circ\Gamma_0\circ\tilde{\omega}_n^{-1}. \tag{11}$$

Since $\tilde{w}_n$ is a quasiconformal mapping of U onto itself satisfying $\tilde{w}_n(0)=0$, we can symmetrically extend it to the whole complex plane. Hence we may consider $\tilde{w}_n$ a quasiconformal mapping of the complex plane onto itself. The mappings $\tilde{w}_n$ form a normal family in the complex plane, because $K(\tilde{w}_n)=K(w_n)$ and (3) implies that $\{K(\tilde{w}_n)\}$ is bounded. Hence we may assume that $\{\tilde{w}_n\}$ converges uniformly on $\bar{U}$. Similarly, we may assume that $\{\tilde{w}_n^{-1}\}$ converges uniformly on $\bar{U}$. Since (3) implies $K(\tilde{w}_n)\to 1$, by noting (9), we see that the limit function of $\{\tilde{w}_n\}$ is a conformal automorphism of U which fixes 0 and 1, namely the identity mapping. Similarly, the limit function of $\{\tilde{w}_n^{-1}\}$ is the identity mapping. Therefore

$$\lim_{n\to\infty}\tilde{w}_n=\mathrm{id}, \qquad \lim_{n\to\infty}\tilde{w}_n^{-1}=\mathrm{id} \tag{12}$$

uniformly on $\bar{U}$. For brevity, we set

$$T_n^* = T_n \circ T_0^{-1}.$$

It follows from (4) and (8) that

$$|\tilde{f}(\tilde{z}) - (\tilde{T}_n^* \tilde{f})(\tilde{w}_n(\tilde{z}))| \leqq \varepsilon_n \|f\| \tag{13}$$

and

$$|\tilde{f}(\tilde{z}) - (\tilde{T}_n^* \tilde{f})(\tilde{\omega}_n(\tilde{z}))| \leqq \varepsilon_n \|\tilde{f}\| \tag{14}$$

for all $\tilde{z} \in \tilde{D}_n$ and for all $\tilde{f} \in A(U, \Gamma_0)$, where $\tilde{D}_n$ is the inverse image of D_n under π_n. We fix $\tilde{f} \in A(U, \Gamma_0)$ arbitrarily. Since, by using (1), we have

$$\|\tilde{T}_n^* \tilde{f}\| = \|T_n^* f\| \leqq \|T_n^*\| \|f\| \leqq c(T_n^*) \|f\| \leqq d_n \|f\|$$

the functions $\tilde{T}_n^* \tilde{f}$ are uniformly bounded. Hence we may assume that $\{\tilde{T}_n^* \tilde{f}\}$ converges uniformly on every compact subset of U. We see from (12) and (13) that the limit function of $\tilde{T}_n^* \tilde{f}$ is equal to $\tilde{f}$. We have thus obtained that

$$\lim_{n \to \infty} \tilde{T}_n^* \tilde{f} = \tilde{f} \tag{15}$$

uniformly on every compact subset of U for every $\tilde{f} \in A(U, \Gamma_0)$.

We can find a function of $A(S_0^*)$ which has a simple zero at $\zeta = \pi_0(0)$ and has no zeros in $\bar{S}_0^*$ except ζ (cf. [7]). Let us denote this function by f_ζ. By setting $\tilde{f} = \tilde{f}_\zeta$ in (14), we obtain

$$|\tilde{f}_\zeta(\tilde{z}) - (\tilde{T}_n^* \tilde{f}_\zeta)(\tilde{\omega}_n(\tilde{z}))| \leqq \varepsilon_n \|\tilde{f}_\zeta\|$$

for all $\tilde{z} \in \tilde{D}_n$. Since $\tilde{f}_\zeta(0) = 0$,

$$|(\tilde{T}_n^* \tilde{f}_\zeta)(\tilde{\omega}_n(0))| \leqq \varepsilon_n \|\tilde{f}_\zeta\|.$$

Hence

$$\lim_{n \to \infty} (\tilde{T}_n^* \tilde{f}_\zeta)(\tilde{\omega}_n(0)) = 0. \tag{16}$$

Now we fix a number r_0 with $0 < r_0 < 1$ such that $\tilde{f}_\zeta$ has no zeros on $|\tilde{z}| = r_0$, and set

$$C_n = \pi_n(\bar{P}_n \cap \{|\tilde{z}| = r_0\}), \qquad R_n = \pi_n(\bar{P}_n \cap \{r_0 < |\tilde{z}| < 1\}).$$

If r_0 is sufficiently close to 1, C_n is a union of several loops on S_n and R_n is a union of several ring domains on S_n whose boundary consist of C_n and a component of ∂S_n. Since f_ζ has no zeros on $\bar{S}_0^*$ except ζ, we can choose an r_0 sufficiently close to 1 and a sufficiently small $\delta > 0$ such that

$$|f_\zeta(z)| \geqq \delta \tag{17}$$

on $\bar{R}_n$, consequently

$$|\tilde{f}_\zeta(\tilde{z})| \geqq \delta$$

for all $\tilde{z} \in \bar{P}_n \cap \{r_0 \leqq |\tilde{z}| \leqq 1\}$. Therefore we obtain

$$|(\tilde{T}_n^* \tilde{f}_\zeta)(\tilde{z})| \geqq \delta/2$$

on $|\tilde{z}|=r_0$, that is,

(18) $$|(T_n^* f_\zeta)(z)| \geqq \delta/2$$

on C_n for all sufficiently large n. Because, by (15), $\tilde{T}_n^* \tilde{f}_\zeta$ converges to $\tilde{f}_\zeta$ uniformly on $|\tilde{z}|=r_0$. On the other hand, by Theorem 1, there exists a homeomorphism h_n of ∂S_0^* onto ∂S_n such that

$$|f_\zeta(z)-(T_n^* f_\zeta)(h_n(z))| \leqq \varepsilon_n \|f_\zeta\|$$

for all $z \in \partial S_0^*$. Hence

(19) $$|(T_n^* f_\zeta)(z)-f_\zeta(h_n^{-1}(z))| < \delta/2$$

on ∂S_n for all sufficiently large n. Consequently, by using (17),

(20) $$|(T_n^* f_\zeta)(z)| \geqq |f_\zeta(h_n^{-1}(z))|-\delta/2 \geqq \delta/2$$

on ∂S_n for all sufficiently large n. It follows from (19) and (20) that the change of argument of $T_n^* f_\zeta$ around ∂S_n is the same as that of f_ζ around ∂S_0^* for every sufficiently large n. Therefore, by the argument principle, $T_n^* f_\zeta$ has the same number of zeros as f_ζ, namely exactly one zero in $\bar{S}_n$ for every sufficiently large n. Since $\tilde{T}_n^* \tilde{f}_\zeta$ converges to $\tilde{f}_\zeta$ uniformly on every compact subset of U, Hurwitz's theorem shows us that $\tilde{T}_n^* \tilde{f}_\zeta$ has a zero in a neighborhood Δ of $\tilde{z}=0$ for every sufficiently large n. Since P_n converges to the normal polygon of Γ_0 with the center at the origin, we can choose Δ so that

$$\Delta \subset \bigcap_{n=1}^{\infty} P_n .$$

Hence $\tilde{T}_n^* \tilde{f}_\zeta$ has no zeros in $\bar{P}_n \cap \{r_0 \leqq |\tilde{z}| \leqq 1\}$, that is, $T_n f_\zeta$ has no zeros in R_n for every sufficiently large n. Then, by using (18) and (20), the minimum principle guarantees that

$$|(T_n^* f_\zeta)(z)| \geqq \delta/2$$

in R_n, namely

(21) $$|(\tilde{T}_n^* \tilde{f}_\zeta)(\tilde{z})| \geqq \delta/2$$

in $\bar{P}_n \cap \{r_0 \leqq |\tilde{z}| \leqq 1\}$.

By (10), (16) and (21) we see that the sequence $\{\tilde{\omega}_n(0)\}$ has no accumulating points on $|\tilde{z}|=1$. Consequently, there exists a number r_1 with $0<r_1<1$ such that

(22) $$|\tilde{\omega}_n(0)| < r_1$$

for all n. On the other hand, the mappings $\tilde{\omega}_n$ form a normal family, for (7) implies that $\{K(\tilde{\omega}_n)\}$ is bounded. Accordingly, we may assume that $\{\tilde{\omega}_n\}$ converges uniformly on every compact subset of U. The limit function

$$\alpha = \lim_{n\to\infty} \tilde{\omega}_n$$

is not a constant, for (22) holds for every n. Since (7) implies that $K(\tilde{\omega}_n)\to 1$, α is a conformal automorphism of U, consequently it is a Möbius transformation. By considering the limit of (14) as $n\to\infty$, we obtain

$$\tilde{f}(\alpha(\tilde{z})) = \tilde{f}(\tilde{z})$$

in U for all $\tilde{f}\in A(U, \Gamma_0)$. If $\alpha\notin\Gamma_0$, then there is a $\tilde{z}_0\in U$ such that

$$\pi_0(\alpha(\tilde{z}_0)) \neq \pi_0(\tilde{z}_0)\,.$$

Since $A(S_0^*)$ separates points on S_0^*, there exists a function $f\in A(S_0^*)$ such that

$$f(\pi_0(\alpha(\tilde{z}_0))) \neq f(\pi_0(\tilde{z}_0))\,,$$

namely

$$\tilde{f}(\alpha(\tilde{z}_0)) \neq \tilde{f}(\tilde{z}_0)\,.$$

This is a contradiction. Therefore we know

$$\alpha \in \Gamma_0\,.$$

Now we set

$$\chi_n(\gamma) = \tilde{w}_n^{-1}\circ\tilde{\omega}_n\circ\gamma\circ\tilde{\omega}_n^{-1}\circ\tilde{w}_n$$

for every $\gamma\in\Gamma_0$. By noting (11) we see that χ_n is an automorphism of Γ_0. Since (12) implies that

$$\lim_{n\to\infty} \tilde{w}_n^{-1}\circ\tilde{\omega}_n = \alpha$$

we obtain

$$\lim_{n\to\infty} \chi_n(\gamma) = \alpha\circ\gamma\circ\alpha^{-1}$$

for all $\gamma\in\Gamma_0$. Hence, by the discontinuity of Γ_0,

$$\chi_n(\gamma) = \alpha\circ\gamma\circ\alpha^{-1} \tag{23}$$

for every $\gamma\in\Gamma_0$ and for every sufficiently large n, because $\chi_n(\gamma)\in\Gamma_0$ and $\alpha\circ\gamma\circ\alpha^{-1}\in\Gamma_0$. Since Γ_0 is finitely generated, we can choose a number n_0 independent of $\gamma\in\Gamma_0$ such that (23) holds for every n with $n\geqq n_0$ and for every $\gamma\in\Gamma_0$. Therefore

$$\tilde{w}_n^{-1}\circ\tilde{\omega}_n\circ\gamma\circ\tilde{\omega}_n^{-1}\circ\tilde{w}_n = \alpha\circ\gamma\circ\alpha^{-1}$$

for every $\gamma\in\Gamma_0$ and for every n with $n\geqq n_0$. This implies that $w_n^{-1}\circ\omega_n$ is homotopic to the identity mapping of S_0^*, namely w_n is homotopic to ω_n for every n with $n\geqq n_0$. This contradicts the before mentioned fact that w_n is not homotopic to ω_n for every n. Thus we have proved that $\Phi(T)=(S, w\circ w_0)$ defines a point $[S, w\circ w_0]$ of $\boldsymbol{T}^{\#}(S_0)$ for $[T]\in\mathcal{L}(S_0)$. Therefore we obtain a mapping Φ of a neighborhood of $[T_0]\in\mathcal{L}(S_0)$ into $\boldsymbol{T}^{\#}(S_0)$.

§ 5. Some properties of the mapping Φ.

We use the same notations as in the previous section. Let $\mathcal{U}$ be the neighborhood of $[T_0]$ in which the mapping Φ is defined. $\mathcal{U}$ consists of all $[T]\in \mathcal{L}(S_0)$ satisfying

$$d([T], [T_0]) < \log d .$$

Then, for every $[T]\in\mathcal{U}$, we see that

$$\begin{aligned}\rho(\Phi([T]), \Phi([T_0])) &= \rho([S, w\circ w_0], [S_0^*, w_0]) \\ &\leqq \log K(w) \\ &< \log(1+\varepsilon).\end{aligned}$$

This implies that Φ is continuous at $[T_0]$. In order to prove the continuity of Φ at another point $[T_1]\in\mathcal{U}$ ($[T_1]\neq[T_0]$), we consider the mapping Φ_1 of a neighborhood $\mathcal{U}_1(\subset\mathcal{U})$ of $[T_1]$ which is defined in the same way as used in defining Φ. We can show that $\Phi=\Phi_1$ in $\mathcal{U}_1$ if $\mathcal{U}$ and $\mathcal{U}_1$ are sufficiently small. Indeed, if $\Phi\neq\Phi_1$ in any small neighborhood $\mathcal{U}_1$ of $[T_1]$ for any small neighborhood $\mathcal{U}$ of $[T_0]$, there exists a sequence $[T_n^*]$ converging to $[T_0]$ such that

$$\Phi([T_n^*]) \neq \Phi_1([T_n^*]).$$

Then, we can derive a contradiction by the same argument as in the previous section. Since Φ_1 is continuous at $[T_1]$, Φ is also continuous at $[T_1]$. Hence Φ is continuous in $\mathcal{U}$.

Next we prove that Φ is a mapping of $\mathcal{U}$ onto a neighborhood $\mathcal{N}$ of $[S_0^*, w_0]$. We take an ε with $0<\varepsilon<\varepsilon_0$ and a d with $d>1$ as before. Let $\mathcal{U}$ be the neighborhood of $[T_0]$ consisting of all $[T]$ such that $d([T], [T_0])<\log d$, and let $\mathcal{N}$ be the neighborhood of $[S_0^*, w_0]$ consisting of all $[S, w]\in \boldsymbol{T}^{\#}(S_0)$ such that

$$\rho([S, w], [S_0^*, w_0]) < \log(1+\varepsilon). \tag{24}$$

We fix an arbitrary point $[S, w]\in\mathcal{N}$. We may assume that

$$K(w\circ w_0^{-1}) < 1+\varepsilon ,$$

because (24) means that there exists a quasiconformal mapping ϕ of S_0^* onto S homotopic to $w\circ w_0^{-1}$ satisfying

$$K(\phi) < 1+\varepsilon .$$

As we mentioned in § 3, we choose a Beltrami basis

$$\boldsymbol{m} = (m_1, \cdots , m_{3N-3})$$

on S_0^* such that each m_i is infinitely differentiable in the real sense and the support of m_i is a compact subset of S_0^* for every i with $1\leqq i\leqq 3N-3$. If ε is sufficiently small, there is an $\eta>0$ such that

$$[S, w\circ w_0^{-1}]=[(S_0^*)^{a\cdot m}, w^{a\cdot m}]$$

holds as a point of $T^{\#}(S_0^*)$ for some $a\in R^{3N-3}$ with $|a|<\eta$. Then,

$$[S, w]=[(S_0^*)^{a\cdot m}, w^{a\cdot m}\circ w_0] \tag{25}$$

holds as a point of $T^{\#}(S_0)$. Since every Beltrami differential m_i of m has a compact support in S_0^*, $w^{a\cdot m}$ is conformal in some neighborhood of ∂S. Let $\gamma_1, \cdots, \gamma_N$ be smooth contours in $(S_0^*)^{a\cdot m}$ which form a homology basis on $(S_0^*)^{a\cdot m}$. We denote by $P_i(\omega)$ the period of a closed differential ω along γ_i, that is,

$$P_i(\omega)=\int_{\gamma_i}\omega$$

for every i with $1\leqq i\leqq N$. Let $\omega_1, \cdots, \omega_N$ be a basis of the space of analytic Schottky differentials on $(S_0^*)^{a\cdot m}$ satisfying

$$P_i(\omega_j)=\delta_{ij} \quad (i, j=1, \cdots, N)$$

(cf. [1]). Let z_0 be a point of $(S_0^*)^{a\cdot m}$. We now construct an isomorphism $T^{a\cdot m}\in L(A(S_0^*), A((S_0^*)^{a\cdot m}))$ as follows. For $f\in A(S_0^*)$, let us denote by f_1 the unique harmonic function on $(S_0^*)^{a\cdot m}$ with boundary values $f\circ(w^{a\cdot m})^{-1}$. We set

$$\theta=(df_1+i^*df_1)/2.$$

It is an analytic differential on $(S_0^*)^{a\cdot m}$. For $z\in(S_0^*)^{a\cdot m}$ we set

$$(T^{a\cdot m}f)(z)=f((w^{a\cdot m})^{-1}(z_0))+\int_{z_0}^{z}\left\{\theta-\sum_{j=1}^{N}P_i(\theta)\omega_j\right\}. \tag{26}$$

Since the integrand is exact, the integral is independent of the path. It was proved that $T^{a\cdot m}$ is a continuous and invertible linear mapping of $A(S_0^*)$ onto $A((S_0^*)^{a\cdot m})$, namely, $T^{a\cdot m}\in L(A(S_0^*), A((S_0^*)^{a\cdot m}))$ (cf. [6]). We note that $T^{a\cdot m}1=1$. If we set

$$T=T^{a\cdot m}\circ T_0,$$

T is in $L(A(S_0), A((S_0^*)^{a\cdot m}))$. We can make $K(w^{a\cdot m})$ close to 1, consequently we can make $\rho([(S_0^*)^{a\cdot m}, w^{a\cdot m}\circ w_0], [S_0^*, w_0])$ close to 0 by choosing a sufficiently small $\eta>0$. Hence, by using the same argument as the proof of Proposition 8 in [6] we can deduce that if we choose a sufficiently small $\eta>0$ for every $\varepsilon_1>0$

$$|f(z)-(T^{a\cdot m}f)(w^{a\cdot m}(z))|\leqq\varepsilon_1\|f\| \tag{27}$$

for all $z\in S_0^*$, for all $f\in A(S_0^*)$ and for all a with $|a|<\eta$. By setting $\varepsilon_1=\sqrt{d}-1$, (27) yields

$$\|T^{a\cdot m}f\|\leqq\sqrt{d}\,\|f\|, \qquad \|(T^{a\cdot m})^{-1}f\|\leqq\sqrt{d}\,\|f\|,$$

consequently

$$c(T\circ T_0^{-1})=c(T^{a\cdot m})=\|T^{a\cdot m}\|\ \|(T^{a\cdot m})^{-1}\|<d$$

if we choose a sufficiently small $\eta>0$. Then we see that $[T]\in\mathcal{V}$, and by the definition of Φ and (25) we obtain

$$\Phi([T])=[(S_0^*)^{a\cdot m},\ w^{a\cdot m}\circ w_0]=[S,\ w]\,.$$

Therefore Φ is a mapping of $\mathcal{V}$ onto $\mathfrak{N}$.

REMARK. *Φ is not injective.* To show this, we take the identity mapping $I\in L(A(S_0), A(S_0))$. Then we see that

$$\Phi([I])=[S_0,\ \mathrm{id}]\,.$$

For $\varepsilon>0$, Theorem 2 guarantees the existence of a $d>1$ having the following property: If $T\in L(A(S_0), A(S_0))$ satisfies $c(T)<d$ and $T1=1$, then there exists a unique conformal automorphism ϕ of S_0 such that

$$|f(z)-(Tf)(\phi(z))|\leqq\varepsilon\|f\|$$

for all $z\in S_0$ and for all $f\in A(S_0)$. However, there exists a $T\in L(A(S_0), A(S_0))$ which is not an isometry and satisfies $c(T)<d$ and $T1=1$. Then, $[T]\neq[I]$ and by the definition of Φ we have

$$\Phi([T])=[S_0,\ \phi]=[S_0,\ \mathrm{id}]=\Phi([I])\,.$$

Hence Φ is not injective.

We now look for the inverse image of an arbitrary point $[S_1, w_1]\in\mathfrak{N}$ under Φ. There is a $[T_1]\in\mathcal{V}$ such that $\Phi([T_1])=[S_1, w_1]$. We may assume $T_1\in L(A(S_0), A(S_1))$. Since $c(T_1\circ T_0^{-1})<d$, by Theorem 3, for every $\varepsilon>0$ there exists a quasiconformal mapping w_1' of S_0^* onto S_1 such that $K(w_1')<1+\varepsilon$ and

$$|f(z)-(T_1\circ T_0^{-1}f)(w_1'(z))|\leqq\varepsilon\|f\|$$

for all z in a relatively compact subset D_0 of S_0^* and for all $f\in A(S_0^*)$. Then, by the definition of Φ,

$$\Phi([T_1])=[S_1,\ w_1'\circ w_0]=[S_1,\ w_1]\,.$$

We take an element $[T]\in\mathfrak{N}$ such that

$$\Phi([T])=[S_1,\ w_1]\,.$$

Let T be in $L(A(S_0), A(S))$. Similarly, for every $\varepsilon>0$, there exists a quasiconformal mapping w of S_0^* onto S such that $K(w)<1+\varepsilon$ and

$$|f(z)-(T\circ T_0^{-1}f)(w(z))|\leqq\varepsilon\|f\|$$

for all $z\in D_0$ and for all $f\in A(S_0^*)$. Then,

$$\Phi([T])=[S,\ w\circ w_0]=[S_1,\ w_1'\circ w_0]\,.$$

Hence there is a conformal mapping ϕ of S_1 onto S such that ϕ is homotopic to $w\circ(w_1')^{-1}$. Let us denote by T_ϕ the isometry in $L(A(S_1), A(S))$ induced by ϕ, that is,

$$T_\phi f = f\circ\phi^{-1}$$

for all $f\in A(S_1)$. We set

$$T' = T_\phi^{-1}\circ T\circ T_1^{-1}.$$

Then, $T'\in L(A(S_1), A(S_1))$ and $[T'\circ T_1]=[T]$. Moreover,

$$c(T') = c(T\circ T_1^{-1}) \leqq c(T\circ T_0^{-1})c(T_1\circ T_0^{-1}) < d^2,$$

that is, $c(T')$ is close to 1.

Conversely, we take a $T'\in L(A(S_1), A(S_1))$ with $c(T')<d$. By Theorem 2, for every $\varepsilon>0$, there exists a conformal automorphism ϕ of S_1 such that

$$|g(z)-(T'g)(\phi(z))| \leqq \varepsilon\|g\|$$

for all $z\in S_1$ and for all $g\in A(S_1)$. Hence

$$\begin{aligned} &|f(z)-(T'\circ T_1\circ T_0^{-1}f)(\phi\circ w_1'(z))| \\ &\leqq |f(z)-(T_1\circ T_0^{-1}f)(w_1'(z))|+|(T_1\circ T_0^{-1}f)(w_1'(z))-(T'(T_1\circ T_0^{-1}f))(\phi(w'(z)))| \\ &\leqq \varepsilon\|f\|+\varepsilon\|T_1\circ T_0^{-1}f\| \\ &\leqq \varepsilon(1+d)\|f\| \end{aligned}$$

for all $z\in D_0$ and for all $f\in A(S_0^*)$. Therefore, by the definition of Φ, we see that

$$\Phi([T'\circ T_1]) = [S_1, \phi\circ w_1'\circ w_0] = [S_1, w_1'\circ w_0] = [S_1, w_1].$$

Thus we can conclude that if we choose a $d>1$ sufficiently close to 1, the inverse image $\Phi^{-1}([S_1, w_1])$ is the set of $[T'\circ T_1]$ such that $T'\in L(A(S_1), A(S_1))$ and $c(T')<d$. Particularly, $\Phi^{-1}([S_0^*, w_0])$ is the set of $[T'\circ T_0]$ such that $T'\in L(A(S_0^*), A(S_0^*))$ and $c(T')<d$.

§ 6. Main theorem.

From the fact stated in the previous section we can conjecture the following theorem. It means that a slight deformation of $A(S)$ is composed of a slight deformation of S in the reduced Teichmüller space and a linear automorphism of $A(S)$ which is very close to the identity. It is our aim to prove this theorem.

THEOREM. *Suppose that $\bar{S}_0$ and $\bar{S}_0^*\in\mathcal{S}$ are homeomorphic and that S_0^* has no conformal automorphisms except the identity mapping. Let T_0 be an arbitrary element of $L(A(S_0), A(S_0^*))$ and $[S_0^*, w_0]$ be a point of the reduced Teichmüller space $\boldsymbol{T}^\#(S_0)$, where w_0 is a quasiconformal mapping of S_0 onto S_0^*. Furthermore we denote by $\mathcal{K}$ the subset of $\mathcal{L}(S_0^*)$ consisting of all equivalence classes of elements of $L(A(S_0^*), A(S_0^*))$. Then there exist a neighborhood $\mathcal{U}$ of $[T_0]$ in $\mathcal{L}(S_0)$, a* *neighborhood $\mathcal{N}$ of $[S_0^*, w_0]$ in $\boldsymbol{T}^\#(S_0)$ and a neighborhood $\mathcal{V}$ of the equivalence class $[I]$ of the identity $I \in L(A(S_0^*), A(S_0^*))$ in $\mathcal{L}(S_0^*)$ such that $\mathcal{U}$ is homeomorphic to the direct product $\mathcal{N} \times (\mathcal{K} \cap \mathcal{V})$.*

PROOF. We use the same notations as before. In the previous section we proved that Φ is a mapping of a neighborhood $\mathcal{U}$ of $[T_0]$ onto a neighborhood $\mathcal{N}$ of $[S_0^*, w_0]$. For every $[T] \in \mathcal{U}$ with $T \in L(A(S_0), A(S))$, we set

$$\Phi([T]) = [S, w],$$

where w is a quasiconformal mapping of S_0 onto S. We choose a Beltrami basis $\boldsymbol{m} = (m_1, \cdots, m_{3N-3})$ on S_0^* such that m_i is infinitely differentiable in the real sense and the support of m_i is a compact subset of S_0^* for every i with $1 \leqq i \leqq 3N-3$. Then there exists a unique $\boldsymbol{a} = (a_1, \cdots, a_{3N-3})$ about the origin of $\boldsymbol{R}^{3N-3}$ such that (25) holds. Accordingly, there exists a conformal mapping ϕ of S onto $(S_0^*)^{\boldsymbol{a}\cdot\boldsymbol{m}}$ such that $\phi \circ w$ is homotopic to $w^{\boldsymbol{a}\cdot\boldsymbol{m}} \circ w_0$. We define

$$\Psi(T) = (T_\phi \circ T \circ T_0^{-1})^{-1} \circ T^{\boldsymbol{a}\cdot\boldsymbol{m}}, \tag{28}$$

where T_ϕ is the isometry induced by ϕ and $T^{\boldsymbol{a}\cdot\boldsymbol{m}}$ is the isomorphism defined by (26). We note that $\Psi(T) \in L(A(S_0^*), A(S_0^*))$ and

$$c(\Psi(T)) \leqq c(T_\phi) c(T \circ T_0^{-1}) c(T^{\boldsymbol{a}\cdot\boldsymbol{m}}) < d^2.$$

We now take another $T' \in L(A(S_0), A(S'))$ such that $[T'] \in \mathcal{U}$ and

$$\Phi([T']) = [S', w'] = [(S_0^*)^{\boldsymbol{a}'\cdot\boldsymbol{m}}, w^{\boldsymbol{a}'\cdot\boldsymbol{m}} \circ w_0]$$

for a unique $\boldsymbol{a}'$ about the origin of $\boldsymbol{R}^{3N-3}$. Let ϕ' be a conformal mapping of S' onto $(S_0^*)^{\boldsymbol{a}'\cdot\boldsymbol{m}}$ such that $\phi' \circ w'$ is homotopic to $w^{\boldsymbol{a}'\cdot\boldsymbol{m}} \circ w_0$. Then

$$\Psi(T') = (T_{\phi'} \circ T' \circ T_0^{-1})^{-1} \circ T^{\boldsymbol{a}'\cdot\boldsymbol{m}}.$$

Suppose that $[T] = [T']$. This implies that $T' \circ T^{-1}$ is induced by a conformal mapping. Hence

$$[T_\phi \circ T \circ T_0^{-1}] = [T_{\phi'} \circ T' \circ T_0^{-1}]. \tag{29}$$

Since $\Phi([T]) = \Phi([T'])$, namely $[S, w] = [S', w']$, we see that $\boldsymbol{a} = \boldsymbol{a}'$. By assumption, S_0^* has no conformal automorphisms except the identity. Since $\boldsymbol{a}$ is close to the origin, $(S_0^*)^{\boldsymbol{a}\cdot\boldsymbol{m}}$ is known to have the same property. It follows from (29) that

$$T_\phi \circ T \circ T_0^{-1} = T_{\phi'} \circ T' \circ T_0^{-1}. \tag{30}$$

Because $(T_{\phi'} \circ T' \circ T_0^{-1}) \circ (T_\phi \circ T \circ T_0^{-1})^{-1}$ is induced by a conformal automorphism of $(S_0^*)^{\boldsymbol{a}\cdot\boldsymbol{m}}$, namely the identity mapping. Hence

(31) $$(T_{\phi}\circ T\circ T_0^{-1})^{-1}\circ T^{a\cdot m}=(T_{\phi'}\circ T'\circ T_0^{-1})^{-1}\circ T^{a\cdot m}, \quad \text{i.e. } \Psi(T)=\Psi(T').$$

Thus Ψ is well defined in $\mathcal{U}$ by (28), that is,

$$\Psi([T])=[(T_{\phi}\circ T\circ T_0^{-1})^{-1}\circ T^{a\cdot m}].$$

We denote by $\mathcal{K}$ the subset of $\mathcal{L}(S_0^*)$ consisting of all equivalence classes of elements of $L(A(S_0^*), A(S_0^*))$ and denote by $\mathcal{V}$ the neighborhood of $[I]$ in $\mathcal{L}(S_0^*)$ consisting of all $[T]$ with $T\in L(A(S_0^*), A(S_0^*))$ such that $c(T)<d^2$. We have shown that Ψ is a mapping of $\mathcal{U}$ into $\mathcal{K}\cap\mathcal{V}$. We now define a new mapping σ of $\mathcal{U}$ into the direct product $\mathfrak{N}\times(\mathcal{K}\cap\mathcal{V})$ by

$$\sigma([T])=(\Phi([T]), \Psi([T])).$$

If $\sigma([T])=\sigma([T'])$, then $\Phi([T])=\Phi([T'])$ and $\Psi([T])=\Psi([T'])$. Hence, as shown before, we have $\boldsymbol{a}=\boldsymbol{a}'$ and we obtain (30) from (31). Therefore we see that $[T]=[T']$, namely σ is injective. In order to prove that σ is surjective, we represent every element $[S, w]$ of $\mathfrak{N}$ in the form of (25). Let

$$([S, w], [T])=([(S_0^*)^{a\cdot m}, w^{a\cdot m}\circ w_0], [T])$$

be an arbitrary element of $\mathfrak{N}\times(\mathcal{K}\cap\mathcal{V})$, where $T\in L(A(S_0^*), A(S_0^*))$ and $c(T)<d^2$. By Theorem 2, if $\varepsilon>0$ is given,

(32) $$|f(z)-(T^{-1}f)(z)|\leqq\varepsilon\|f\|$$

for all $z\in S_0^*$ and for all $f\in A(S_0^*)$. If we set

(33) $$T^*=T_{\phi}^{-1}\circ T^{a\cdot m}\circ T^{-1}\circ T_0,$$

then $T^*\in L(A(S_0), A(S))$ and

$$c(T^*)\leqq c(T^{a\cdot m})c(T\circ T_0^{-1})<d^2.$$

We obtain from (28) and (33)

$$\Psi([T^*])=[T].$$

Moreover it follows from (27) and (32) that if $\varepsilon>0$ is given,

$$\begin{aligned}
&|f(z)-(T^*\circ T_0^{-1}f)(\phi^{-1}\circ w^{a\cdot m}(z))|\\
&=|f(z)-(T_{\phi}^{-1}\circ T^{a\cdot m}\circ T^{-1}f)(\phi^{-1}\circ w^{a\cdot n}(z))|\\
&=|f(z)-(T^{a\cdot m}\circ T^{-1}f)(w^{a\cdot m}(z))|\\
&=|f(z)-(T^{-1}f)(z)|+|(T^{-1}f)(z)-(T^{a\cdot m}(T^{-1}f))(w^{a\cdot m}(z))|\\
&\leqq\varepsilon\|f\|+\varepsilon\|T^{-1}f\|\\
&\leqq\varepsilon(1+\|T^{-1}\|)\|f\|
\end{aligned}$$

for all $z \in S_0^*$ and for all $f \in A(S_0^*)$. Hence, by the definition of Φ, we obtain

$$\Phi([T^*]) = [S, \phi^{-1} \circ w^{a \cdot m} \circ w_0] = [(S_0^*)^{a \cdot m}, w^{a \cdot m} \circ w_0].$$

Therefore

$$\sigma([T^*]) = ([(S_0^*)^{a \cdot m}, w^{a \cdot m} \circ w_0], [T]),$$

that is, σ is surjective.

In order to prove that σ is continuous, we must show that Ψ is continuous. Let $[T]$ be an arbitrary element of $\mathcal{U}$. We set

$$\Phi([T]) = [S, w] = [(S_0^*)^{a \cdot m}, w^{a \cdot m} \circ w_0].$$

It is sufficient to show that

$$\lim_{n \to \infty} c(\Psi(T_n) \circ \Psi(T)^{-1}) = 1 \tag{34}$$

for every sequence $[T_n] \in \mathcal{U}$ satisfying

$$\lim_{n \to \infty} c(T_n \circ T^{-1}) = 1. \tag{35}$$

Let us set

$$\Phi([T_n]) = [S_n, w_n].$$

Then there exist a unique $\boldsymbol{a}_n \in \boldsymbol{R}^{3N-3}$ such that

$$[S_n, w_n] = [(S_0^*)^{a_n \cdot m}, w^{a_n \cdot m} \circ w_0]$$

and a conformal mapping ϕ_n of S_n onto $(S_0^*)^{a_n \cdot m}$ such that $\phi_n \circ w_n$ is homotopic to $w^{a_n \cdot m} \circ w_0$. Since Φ is continuous, (35) implies that $\Phi([T_n]) = [S_n, w_n]$ converges to $\Phi([T]) = [S, w]$. Hence we know that $\boldsymbol{a}_n \to \boldsymbol{a}$. As we defined Ψ by (28),

$$\Psi(T_n) = (T_{\phi_n} \circ T_n \circ T_0^{-1})^{-1} \circ T^{a_n \cdot m}.$$

Hence we obtain

$$\begin{aligned} \Psi(T_n) \circ \Psi(T)^{-1} &= (T_{\phi_n} \circ T_n \circ T_0^{-1})^{-1} \circ T^{a_n \cdot m} \circ (T^{a \cdot m})^{-1} \circ T_\phi \circ T \circ T_0^{-1} \\ &= (T_0^*)^{-1} \circ T_n^* \circ T_0^* \end{aligned} \tag{36}$$

where

$$T_0^* = T \circ T_0^{-1} \in L(A(S_0^*), A(S))$$

and

$$T_n^* = T \circ T_n^{-1} \circ T_{\phi_n}^{-1} \circ T^{a_n \cdot m} \circ (T^{a \cdot m})^{-1} \circ T_\phi \in L(A(S), A(S)).$$

We note that

$$c(T_n^*) \leq c(T_n \circ T^{-1}) c(T^{a_n \cdot m} \circ (T^{a \cdot m})^{-1}). \tag{37}$$

Since $\boldsymbol{a}_n \to \boldsymbol{a}$, by the same argument as the proof of Proposition 8 in [**6**], we can show that

$$\lim_{n\to\infty} c(T^{a_n\cdot m}\circ(T^{a\cdot m})^{-1}) = 1.$$

Hence it follows from (35) and (37) that

$$\lim_{n\to\infty} c(T_n^*) = 1. \tag{38}$$

Since S has no conformal automorphisms except the identity mapping we see from (38) and Theorem 2 that

$$\lim_{n\to\infty}\frac{\|T_n^* f - f\|}{\|f\|} = 0$$

uniformly for $f\in A(S)$. Consequently

$$\lim_{n\to\infty}\frac{\|(T_n^*\circ T_0^*)f - T_0^* f\|}{\|f\|} = 0$$

uniformly for $f\in A(S_0^*)$. Hence, by the continuity of $(T_0^*)^{-1}$,

$$\lim_{n\to\infty}\frac{\|((T_0^*)^{-1}\circ T_n^*\circ T_0^*)f - f\|}{\|f\|} = 0$$

uniformly for $f\in A(S_0^*)$. This implies that

$$\lim_{n\to\infty}\|(T_0^*)^{-1}\circ T_n^*\circ T_0^*\| = 1.$$

Similarly,

$$\lim_{n\to\infty}\|(T_0^*)^{-1}\circ (T_n^*)^{-1}\circ T_0^*\| = 1.$$

Hence

$$\lim_{n\to\infty} c((T_0^*)^{-1}\circ T_n^*\circ T_0^*) = 1.$$

Therefore (34) follows from (36).

Conversely, by the same argument as above, we can prove that (35) follows from (34), namely the inverse mapping σ^{-1} is continuous. Thus σ is a homeomorphism of $\mathcal{U}$ onto $\mathcal{N}\times(\mathcal{K}\cap\mathcal{V})$. Our proof has been completed.

References

[1] L. V. Ahlfors, Open Riemann surfaces and extremal problems on compact subregions, Comment. Math. Helv., **24** (1950), 100-134.

[2] S. Banach, Théorie des Opèrations Linéaires, Warsaw, 1932.

[3] L. Bers, Universal Teichmüller space, Analytic methods in mathematical physics, Gordon and Breach, London, 1968, pp. 65-83.

[4] M. Nagasawa, Isomorphisms between commutative Banach algebras with an application to rings of analytic functions, Kôdai Math. Sem. Rep., **11** (1959), 182-188.

[5] R. Rochberg, Almost isometries of Banach spaces and moduli of planar domains,

Pacific J. Math., **49** (1973), 445-466.
[6] R. Rochberg, Almost isometries of Banach spaces and moduli of Riemann surfaces, Duke Math. J., **40** (1973), 41-52.
[7] R. Rochberg. Almost isometries of Banach spaces and moduli of Riemann surfaces II, Duke Math. J., **42** (1975), 167-182.

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