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Construction of the solutions of microhyperbolic pseudodifferential equations

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§ 0. Introduction.

In this paper, we give an explicit representation of the microfunction solutions of microhyperbolic pseudodifferential equations. Microhyperbolicity is an important notion introduced by M. Kashiwara and T. Kawai in [**4**], and such equations were investigated by [**4**] and [**5**]. These authors proved the existence of the solutions in an abstract manner, using a continuation theorem of holomorphic functions in complex domains. Our aim is to construct the solutions explicitly to the contrary. The basic idea of our theory is due to M. D. Bronstein [**3**]. We extend the arguments of [**3**] microlocally in the category of hyperfunctions, and define a very general class of operators, which will be called Bronstein operators (see § 2 for the precise definition). For that purpose, we employ a different formulation from that of [**3**]. We shall show how the arguments of [**3**] can be applied when one considers the defining functions of microfunctions. It will turn out that such an approach is successful, and we can directly construct the solutions of microhyperbolic pseudodifferential equations.

Recently K. Kataoka [**6**] gave an amelioration of our Proposition 4.7 below. He proved a more precise symbol formula for our operator theory extended in § 4. He also suggests that our theory can be understood from a wider point of view, and that it can then be applied to boundary value problems. S. Wakabayashi [**10**] investigated hyperbolic Cauchy problems in detail, also using Bronstein theory. Recently K. Kajitani and S. Wakabayashi [**11**] extended such a theory microlocally. The author thinks that our construction is more direct, although the basic idea is closely connected. They also gave a detailed result in the Gevrey category, and it seems that they also look for applications to boundary value problems.

To state the main theorem, we give some preliminaries. Let $x=(x_1, x')\in \boldsymbol{R}\times\boldsymbol{R}^{n-1}$ or $\boldsymbol{C}\times\boldsymbol{C}^{n-1}$, and let $D=\partial/\partial x$. If $q\in \boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$, $s\in\boldsymbol{R}$ and $s>1$, we denote by $C^q[\boldsymbol{R};\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$ the space of $\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})$ valued functions of C^q class with respect to the x_1 variable. Here $\mathscr{D}^{(s)\prime}(\boldsymbol{R}^{n-1})$ denotes the space of ultradistributions (see [8]). Let $f(x)$ be a section of $C^q[\boldsymbol{R};\mathscr{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$ defined on a neighborhood of the origin. We consider x_1 as a parameter, and take the spectrum of $f(x)$ with respect to x'. Then we obtain a microfunction on $\boldsymbol{R}\times\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$ with a parameter x_1, which we denote by $[f(x)]$. We denote by $SS'f(x)$ the support of $[f(x)]$. If $u_0(x')$ is a section of $\mathscr{D}^{(s)\prime}(\boldsymbol{R}^{n-1})$ defined on a neighborhood of the origin, we also denote by $[u_0(x')]$ (resp. $SS'u_0(x')$) the spectrum of $u_0(x')$ (resp. the support of $[u_0(x')]$).

To avoid confusion, it may be appropriate to give a brief review about pseudodifferential operators (i.e., microdifferential operators). Let $A_\alpha(x)$, $\alpha\in\boldsymbol{Z}_+^{n-1}\times\boldsymbol{Z}$, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n\leqq k$, be holomorphic functions defined on a complex neighborhood X of the origin. If there exists some $C>0$ such that

$$|A_\alpha(x)|\leqq C^{k-\alpha_n+1}(k-|\alpha|)!$$

on X, we can define a pseudodifferential operator $A(x, D)=\sum_{|\alpha|\leqq k}A_\alpha(x)D^\alpha$ defined at $(0;0,\cdots,0,\sqrt{-1})\in\sqrt{-1}\,S^*\boldsymbol{R}^n$.

The principal symbol $\sigma_k(A)(x, \xi)$ of $A(x, D)$ is defined by $\sigma_k(A)(x, \xi)=\sum_{|\alpha|=k}A_\alpha(x)\xi^\alpha$. Let $A(x, \xi)$ be some holomorphic function defined on a complex conic neighborhood of $(0;0,\cdots,0,\sqrt{-1})\in T^*\boldsymbol{C}^n$. If $A(x, \xi)\sim\sum_{|\alpha|\leqq k}A_\alpha(x)\xi^\alpha$, i.e., there exists some $C_1>0$ such that for any $k'\leqq k$ we have

$$|A(x, \xi)-\sum_{|\alpha|>k'}A_\alpha(x)\xi^\alpha|\leqq C_1^{k-k'+1}(k-k')!\,|\xi_n|^{k'}$$

on some complex conic neighborhood of $(0;0,\cdots,0,\sqrt{-1})\in T^*\boldsymbol{C}^n$ (which does not depend on k'), then we say that $A(x, \xi)$ is the total symbol of $A(x, D)$. If we have $A_\alpha(x)=0$ provided $\alpha_1\neq 0$, then we denote as $A(x, D)=A(x, D')$, $\sigma_k(A)(x, \xi)=\sigma_k(A)(x, \xi')$, and $A(x, \xi)=A(x, \xi')$. In this case, we consider that $A(x, D')$ is defined at $\mathring{x}^*=(0;0;0,\cdots,0,\sqrt{-1})\in\boldsymbol{R}\times\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$. From now on we always write as $A(x, \xi')\sim\sum_\alpha A_\alpha(x)\xi'^\alpha$ in the above sense (this time $\alpha\in\boldsymbol{Z}_+^{n-2}\times\boldsymbol{Z}$).

Now we state the main result. Let $P(x, D)$ be written as

$$P(x, D)=D_1^m+\sum_{j=0}^{m-1}P^{(j)}(x, D')D_1^j, \tag{0.1}$$

where $P^{(j)}(x, D')$, $0\leqq j\leqq m-1$, are pseudodifferential operators of order $m-j$ defined on a neighborhood of $\mathring{x}^*$. We assume that $m\geqq 2$, excluding the trivial case $m=1$. Let $\sigma_m(P)(x, \xi)=\xi_1^m+\sum_{j=0}^{m-1}\sigma_{m-j}(P^{(j)})(x, \xi')\xi_1^j$ be the principal symbol of $P(x, D)$, and let $\xi_1=\lambda_j(x, \xi')$, $1\leqq j\leqq m$, be the roots of $\sigma_m(P)(x, \xi)=0$ at each point (x, ξ'). Abbreviating the terminology of [4], we say that $P(x, D)$ is microhyperbolic in the direction x_1 at $\mathring{x}^*$ if

(0.2) $$\operatorname{Re}\lambda_j(x, \xi')=0, \quad 1\leqq j\leqq m$$

for any $(x, \xi')\in \boldsymbol{R}\times\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$ sufficiently close to $\mathring{x}^*$. Our main result is the following

THEOREM 0.1. *Assume that* $m\geqq 2$, $1<s<2$, $s\leqq m/(m-1)$. *Let* $P(x, D)$ *of the form* (0.1) *be microhyperbolic in the direction* x_1 *at* $\mathring{x}^*$. *Let* $f(x)$ *be a section of* $C^q[\boldsymbol{R}; \mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$, *and* $u_i(x')$, $0\leqq i\leqq m-1$, *be sections of* $\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})$ *defined on a neighborhood of the origin. Let* $\omega\subset \boldsymbol{R}\times\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$ *be a small neighborhood of* $\mathring{x}^*$, *and assume that* $SS'f(x)\subset\omega$ *and* $SS'u_i(x')\subset\omega\cap\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$, $0\leqq i\leqq m-1$. *Then there exist an open neighborhood* $\omega_0\subset\omega$ *of* $\mathring{x}^*$ *and a section* $u(x)$ *of* $C^{q+m}[\boldsymbol{R}; \mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$ *defined on a neighborhood of the origin such that*

(0.3) $$\begin{cases} P(x, D)[u(x)]=[f(x)] & \text{on } \omega_0, \\ D_1^i[u(0, x')]=[u_i(x')] & \text{on } \omega_0\cap\sqrt{-1}\,S^*\boldsymbol{R}^{n-1},\ 0\leqq i\leqq m-1. \end{cases}$$

REMARK. Here we emphasize the fact again that the above microfunction solution $[u(x)]$ can be explicitly constructed. It will be explained in detail in § 5.

Now we give the plan of this paper.

In § 1, we give a formal calculation following [**3**]. Then we obtain some symbol function the meaning of which will be explained in § 2.

In § 2, we first define a certain space of holomorphic functions, denoted by $\mathcal{O}(\Omega)$ there. Then we show that the symbol function obtained in § 1 defines an operator acting on $\mathcal{O}(\Omega)$. We call such an operator a Bronstein operator. For that purpose we need to prepare several calculations concerning the Fourier transformation in complex domains.

In § 3 and § 4, we give a theory concerning the composition of one pseudodifferential operator and one Bronstein operator. For that purpose, we investigate in § 3 how a pseudodifferential operator acts on the defining functions of microfunctions, following the argument of J. M. Bony and P. Schapira [**2**]. In § 4 we prove the symbol formula of such a composite operator.

In § 5 we show how our theory applies to solve the Cauchy problem (0.3).

§ 1. A formal calculation.

In this section, we prepare some formal calculation analogous to [**3**]. We do not explain the precise meaning for the moment. Several technical arguments are due to [**9**].

Let $P(x, D)$ be of the form (0.1) and microhyperbolic in the direction x_1 at $\mathring{x}^*=(0; 0; 0, \cdots, 0, \sqrt{-1})\in\boldsymbol{R}\times\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$. Using the local version of Bochner's tube theorem, it follows that if $C>0$ is large enough and $(x, \xi')\in \boldsymbol{C}^n\times\boldsymbol{C}^{n-1}$ satisfies

$$(1.1)\qquad \begin{cases} C|\mathrm{Re}\,x_j|<1, & 1\leqq j\leqq n, \qquad C|\mathrm{Im}\,x_j|<1, \quad 1\leqq j\leqq n, \\ C|\mathrm{Im}\,\xi_j|<\mathrm{Im}\,\xi_n, & 2\leqq j\leqq n-1, \qquad C(|\mathrm{Re}\,\xi_j|+1)<\mathrm{Im}\,\xi_n, \quad 2\leqq j\leqq n, \end{cases}$$

then we have

$$(1.2)\qquad |\mathrm{Re}\,\lambda_i(x,\xi')|\leqq \frac{C}{2}|\mathrm{Im}\,x|\cdot|\mathrm{Im}\,\xi_n|+\frac{C}{2}|\mathrm{Re}\,\xi'|$$

for $1\leqq i\leqq m$ (see [1], [7]).

Assume that $(x,\xi')\in \boldsymbol{C}^n\times\boldsymbol{C}^{n-1}$ satisfies (1.1) and

$$(1.3)\qquad \mathrm{Re}\,\xi_1>C|\mathrm{Re}\,\xi'|+C|\mathrm{Im}\,x|\cdot|\mathrm{Im}\,\xi_n|+C|\mathrm{Im}\,\xi_n|^{1/s} +C-C^{-1}(|\mathrm{Im}\,\xi_1|-C|\mathrm{Im}\,\xi_n|)_+$$

where $t_+=\max(t,0)$ for $t\in\boldsymbol{R}$. Throughout this paper we always assume that $m\geqq 2$, $1<s<2$, and $s\leqq m/(m-1)$. Since D_n is an invertible operator at $\mathring{x}^*$, we may consider $\tilde{P}(x,D)=P(x,D)D_n^{n+2}$ instead of $P(x,D)$. Let $\tilde{P}_0(x,D)=P_0(x,D)D_n^{n+2}$ be the principal part of $\tilde{P}(x,D)$, and let $\tilde{P}'(x,D)=\tilde{P}(x,D)-\tilde{P}_0(x,D)$. We have the following

LEMMA 1.1. *If* $(x,\xi)\in\boldsymbol{C}^n\times\boldsymbol{C}^{n-1}$ *satisfies* (1.1) *and* (1.3) *with large* $C>0$, *we have*

i) $|\tilde{P}_0(x,\xi)|^{-1}\leqq C|\xi_1|^{-m}|\xi_n|^{-n-1}$,

ii) $|\tilde{P}(x,\xi)|^{-1}\leqq C|\xi_1|^{-m}|\xi_n|^{-n-1}$,

iii) $|\tilde{P}'(x,\xi)|\leqq \frac{1}{2}|\tilde{P}_0(x,\xi)|$.

Here $\tilde{P}_0(x,\xi)$, $\tilde{P}'(x,\xi)$, *and* $\tilde{P}(x,\xi)$ *denote the total symbols of the corresponding operators.*

PROOF. If $C>0$ is large enough and $|\xi_1|\geqq C|\xi_n|$, we have

$$|\tilde{P}_0(x,\xi)|=|\xi_n|^{n+2}\prod_{j=1}^m|\xi_1-\lambda_j(x,\xi')|\geqq\frac{1}{2}|\xi_1|^m|\xi_n|^{n+2}$$

and

$$|\tilde{P}'(x,\xi)|\leqq C'|\xi_1|^{m-1}|\xi_n|^{n+2}\leqq\frac{1}{2}|\tilde{P}_0(x,\xi)|\,.$$

Thus we obtain

$$|\tilde{P}(x,\xi)|\geqq|\tilde{P}_0(x,\xi)|-|\tilde{P}'(x,\xi)|\geqq\frac{1}{2}|\tilde{P}_0(x,\xi)|\geqq\frac{1}{4}|\xi_1|^m|\xi_n|^{n+2}.$$

We next consider the case $|\xi_1|\leqq C|\xi_n|$. From (1.3) it follows that $|\mathrm{Im}\,\xi_1|\leqq C\,\mathrm{Im}\,\xi_n$ and $\mathrm{Re}\,\xi_1\geqq C(\mathrm{Im}\,\xi_n)^{1/s}$. We obtain

$$|\tilde{P}_0(x,\xi)|\geqq|\xi_n|^{n+2}\prod_{j=1}^m(\mathrm{Re}\,\xi_1-\mathrm{Re}\,\lambda_j(x,\xi'))\geqq 2^{-m}(\mathrm{Re}\,\xi_1)^m|\xi_n|^{n+2}$$

and

$$\begin{aligned}|\tilde{P}'(x, \xi)| &\leqq C'(|\mathrm{Re}\xi|^{m-1}+|\mathrm{Im}\xi|^{m-1})|\xi_n|^{n+2}\\ &\leqq C'(2(\mathrm{Re}\xi_1)^{m-1}+(2C+2)^{m-1}(\mathrm{Im}\xi_n)^{m-1})|\xi_n|^{n+2}\\ &\leqq C'(2(\mathrm{Re}\xi_1)^{m-1}+\left(\frac{2C+2}{C^s}\right)^{m-1}(\mathrm{Re}\xi_1)^{s(m-1)})|\xi_n|^{n+2}\end{aligned}$$

where $C'>0$ is some constant. Noting $s>1$ and $s(m-1)\leqq m$, we obtain $|\tilde{P}'(x, \xi)|\leqq 2^{-m-1}(\mathrm{Re}\xi_1)^m|\xi_n|^{n+2}\leqq(1/2)|\tilde{P}_0(x, \xi)|$ if $C>0$ is large enough. Now we have $|\tilde{P}(x, \xi)|\geqq(1/2)|\tilde{P}_0(x, \xi)|$ and

$$\begin{aligned}|\tilde{P}_0(x, \xi)| &\geqq 2^{-m}(\mathrm{Re}\xi_1)^m|\xi_n|^{n+2}\geqq 2^{-m}C^m(\mathrm{Im}\xi_n)^{m/s}|\xi_n|^{n+2}\\ &\geqq 2^{-2m}C^m|\xi_n|^{m+n+1}\geqq 2^{-2m}|\xi_1|^m|\xi_n|^{n+1}.\end{aligned}$$

Q. E. D.

LEMMA 1.2. *If C is large and (x, ξ) satisfies* (1.1) *and* (1.3), *we have*

i) $|\partial_{x_j}(1/\tilde{P}_0(x, \xi))|\leqq C^2|\xi_1|^{-m-1}|\xi_n|^{-n+(s-1)/s}, \quad 1\leqq j\leqq n,$

ii) $|\partial_{\xi_j}\tilde{P}_0(x, \xi)/\tilde{P}_0(x, \xi)|\leqq \frac{1}{3}|\xi_n|^{-1/s}, \quad 1\leqq j\leqq n.$

PROOF. From Rellich's theorem, it follows that each $\lambda_k(x, \xi')$ is directionary derivable, and

$$|\partial_{x_j}\lambda_k(x, \xi')|\leqq \text{const.}|\xi_n|, \qquad |\partial_{\xi_j}\lambda_k(x, \xi')|\leqq \text{const.}$$

for $1\leqq j\leqq n$, $1\leqq k\leqq m$, if (x, ξ') satisfies (1.1). (See [9] for the proof. There is also given a brief proof of Rellich's theorem. See Lemma 1.6 there.) If $|\xi_1|\geqq C|\xi_n|$, we have

$$\begin{aligned}\left|\partial_{x_j}\left(\frac{1}{\tilde{P}_0}\right)\right| &\leqq \frac{1}{|\tilde{P}_0|}\sum_{k=1}^m\frac{|\partial_{x_j}\lambda_k|}{|\xi_1-\lambda_k|}\leqq C|\xi_1|^{-m}|\xi_n|^{-n-1}\cdot 2mC'|\xi_n|\cdot|\xi_1|^{-1}\\ &\leqq C|\xi_1|^{-m-1}|\xi_n|^{-n}.\end{aligned}$$

On the other hand, if $|\xi_1|\leqq C|\xi_n|$ (and thus $|\mathrm{Im}\xi_1|\leqq C\,\mathrm{Im}\xi_n$), we have

$$\begin{aligned}\left|\partial_{x_j}\left(\frac{1}{\tilde{P}_0}\right)\right| &\leqq \frac{1}{|\tilde{P}_0|}\sum_{k=1}^m\frac{|\partial_{x_j}\lambda_k|}{\mathrm{Re}(\xi_1-\lambda_k)}\leqq C|\xi_1|^{-m}|\xi_n|^{-n-1}\cdot 2mC'|\xi_n|(\mathrm{Re}\xi_1)^{-1}\\ &\leqq 4mC'|\xi_1|^{-m}|\xi_n|^{-1-n+(s-1)/s}\\ &\leqq C^2|\xi_1|^{-m-1}|\xi_n|^{-n+(s-1)/s}\end{aligned}$$

if $C>0$ is large enough. Thus we obtain i), and the proof of ii) is similar.

Q. E. D.

LEMMA 1.3. *If $C>0$ is large enough and (x, ξ) satisfies* (1.1) *and* (1.3), *we have*

i) $|\partial_{x_j}(1/\tilde{P}(x, \xi))|\leqq C^3|\xi_1|^{-m-1}|\xi_n|^{-n+(s-1)/s}, \quad 1\leqq j\leqq n,$

ii) $|\partial_{\xi_j}\tilde{P}(x, \xi)/\tilde{P}(x, \xi)| \leqq |\xi_n|^{-1/s}, \qquad 1\leqq j\leqq n.$

PROOF. From Lemma 1.1, i), we have

$$\left|\partial_{x_j}\left(\frac{1}{\tilde{P}}\right)\right| \leqq \frac{1}{|\tilde{P}|}\cdot\frac{|\partial_{x_j}\tilde{P}|}{|\tilde{P}|} \leqq \frac{4}{|\tilde{P}_0|}\cdot\frac{|\partial_{x_j}\tilde{P}_0|}{|\tilde{P}_0|}+\frac{4}{|\tilde{P}_0|}\cdot\frac{|\partial_{x_j}\tilde{P}'|}{|\tilde{P}_0|}.$$

We need to estimate the second term in the right hand side. Considering the cases $|\xi_1|\geqq C|\xi_n|$ and $|\xi_1|\leqq C|\xi_n|$ separately, we can estimate this term similarly to Lemma 1.1. The proof of ii) is similar. Q.E.D.

If $k=1, 2, 3, \cdots$, we denote by $(\mathrm{H})_k$ the following condition:

CONDITION $(\mathrm{H})_k$. *$(x, \xi)\in \boldsymbol{C}^n\times \boldsymbol{C}^n$ satisfies*

$$kC|\mathrm{Re}\, x_j| < 1, \quad kC|\mathrm{Im}\, x_j| < 1, \qquad 1\leqq j\leqq n,$$
$$kC|\mathrm{Im}\xi_j| < \mathrm{Im}\xi_n, \qquad 2\leqq j\leqq n-1,$$
$$kC(|\mathrm{Re}\xi_j|+1) < \mathrm{Im}\xi_n, \qquad 2\leqq j\leqq n,$$
$$\mathrm{Re}\xi_1 > kC|\mathrm{Re}\xi'|+kC|\mathrm{Im}\, x|\cdot|\mathrm{Im}\xi_n|+kC|\mathrm{Im}\xi_n|^{1/s}+kC$$
$$-(1/kC)(|\mathrm{Im}\xi_1|-kC|\mathrm{Im}\xi_n|)_+ .$$

We have (1.1), (1.3)$\Leftrightarrow(\mathrm{H})_1$, and $(\mathrm{H})_i\Rightarrow(\mathrm{H})_j$ if $i\geqq j$. It is easy to see that we have

LEMMA 1.4. *Assume that $1\leqq k\leqq m-1$, (x, ξ) satisfies $(\mathrm{H})_{k+1}$, and $\tilde{\xi}\in \boldsymbol{C}^n$ satisfies $|\tilde{\xi}_j|\leqq C^{-1}|\mathrm{Im}\xi_n|^{1/s}$, $1\leqq j\leqq n$. Then $(x, \xi+\tilde{\xi})\in \boldsymbol{C}^n\times \boldsymbol{C}^n$ satisfies $(\mathrm{H})_k$.*

We leave the proof of this lemma to the reader. Now we have

COROLLARY 1.5. *If $1\leqq k\leqq m$ and (x, ξ) satisfies $(\mathrm{H})_k$, we have*

$$|\partial_\xi^\beta\tilde{P}(x, \xi)/\tilde{P}(x, \xi)| \leqq C^{2|\beta|}|\xi_n|^{-|\beta|/s}$$

if $\beta\in \boldsymbol{Z}_+^n$, $|\beta|=k$.

PROOF. We only need to prove for the case $2\leqq k\leqq m$. Assume that $2\leqq l\leqq m$, and that the statement is valid if $k=l-1$. If $|\beta|=l$ and $\beta_j\neq 0$, we define γ by $\gamma=(\beta_1, \cdots, \beta_j-1, \cdots, \beta_n)$. Using Cauchy integration theorem and Lemma 1.4, we obtain

$$\left|\frac{\partial_\xi^\beta\tilde{P}}{\tilde{P}}\right| \leqq \left|\partial_{\xi_j}\left(\frac{\partial_\xi^\gamma\tilde{P}}{\tilde{P}}\right)\right|+\left|\frac{\partial_\xi^\gamma\tilde{P}}{\tilde{P}}\right|\cdot\left|\frac{\partial_{\xi_j}\tilde{P}}{\tilde{P}}\right| \leqq C^{2|\beta|}|\xi_n|^{-|\beta|/s}.$$

Q.E.D.

LEMMA 1.6. *If (x, ξ) satisfies $(\mathrm{H})_{m+1}$, we have*

$$|\partial_\xi^\beta \tilde{P}(x, \xi)/\tilde{P}(x, \xi)| \leqq C^{2|\beta|}|\xi_n|^{-|\beta|/s}\beta!$$

for any $\beta \in \boldsymbol{Z}_+^n$.

PROOF. We only need to prove for the case $|\beta| \geqq m+1$. If $|\xi_1| \geqq C|\xi_n|$, from Cauchy integration theorem we have

$$\left|\frac{\partial_\xi^\beta \tilde{P}}{\tilde{P}}\right| \leqq \frac{2|\partial_\xi^\beta \tilde{P}|}{|\tilde{P}_0|} \leqq 2^{m+1}|\xi_1|^{-m}|\xi_n|^{-n-2}\cdot C^{|\beta|+1}\beta!|\xi|^m|\xi_n|^{n+2-|\beta|}$$

$$\leqq C^{2|\beta|}\beta!|\xi_n|^{-|\beta|/s}.$$

If $|\xi_1| \leqq C|\xi_n|$, we have

$$\left|\frac{\partial_\xi^\beta \tilde{P}}{\tilde{P}}\right| \leqq \frac{2|\partial_\xi^\beta \tilde{P}|}{|\tilde{P}_0|} \leqq 2^{m+1}(\mathrm{Re}\,\xi_1)^{-m}|\xi_n|^{-n-2}\cdot C^{|\beta|+1}\beta!|\xi|^m|\xi_n|^{n+2-|\beta|}$$

$$\leqq 2^{4m+1}C^{|\beta|+1}\beta!|\xi_n|^{-|\beta|+m(s-1)/s}$$

$$\leqq C^{2|\beta|}\beta!|\xi_n|^{-|\beta|/s}. \qquad \text{Q.E.D.}$$

The following lemma is also easy to prove and we leave the proof to the reader:

LEMMA 1.4′. *If (x, ξ) satisfies $(\mathrm{H})_{m+1}$ and $\tilde{x} \in \boldsymbol{C}^n$ satisfies $|\tilde{x}_j| \leqq C^{-1}|\mathrm{Im}\,\xi_n|^{(1-s)/s}$, $1 \leqq j \leqq n$, then $(x+\tilde{x}, \xi)$ satisfies $(\mathrm{H})_m$.*

COROLLARY 1.7. *If (x, ξ) satisfies $(\mathrm{H})_{m+1}$, we have*

i) $|\partial_x^\alpha(1/\tilde{P}(x, \xi))| \leqq C^{|\alpha|+3}\alpha!|\xi_1|^{-m-1}|\xi_n|^{-n+|\alpha|(s-1)/s}$

for any $\alpha \in \boldsymbol{Z}_+^{n-1}$, $\alpha \neq 0$,

ii) $|\partial_x^\alpha(\partial_\xi^\beta \tilde{P}(x, \xi)/\tilde{P}(x, \xi))| \leqq C^{|\alpha|+2|\beta|}\alpha!\,\beta!|\xi_n|^{((s-1)|\alpha|-|\beta|)/s}$

for any $\alpha, \beta \in \boldsymbol{Z}_+^n$.

Now let us define the formal parametrix of $\tilde{P}(x, D)$.

DEFINITION 1.8. We define $E_j(x, \xi)$, $j \in \boldsymbol{Z}_+$, inductively by

$$(1.4) \qquad E_j(x, \xi) = \begin{cases} 1/\tilde{P}(x, \xi), & j=0, \\ -\dfrac{1}{\tilde{P}(x, \xi)} \displaystyle\sum_{\substack{k+|\alpha|=j \\ k\neq j}} \frac{1}{\alpha!}\partial_\xi^\alpha \tilde{P}(x, \xi)\partial_x^\alpha E_k(x, \xi), & j \geqq 1. \end{cases}$$

We have the following

PROPOSITION 1.9. *If* (x, ξ) *satisfies* $(\mathrm{H})_{m+1}$, *we have*

i) $|E_0(x, \xi)| \leqq C|\xi_1|^{-m}|\xi_n|^{-n-1}$,

ii) $|\partial_x^\alpha E_j(x, \xi)| \leqq C^{5j+2|\alpha|+3}(j+|\alpha|)!|\xi_1|^{-1-m}|\xi_n|^{-n+((s-2)j+(s-1)|\alpha|)/s}$

for any $j \in \boldsymbol{Z}_+$, $\alpha \in \boldsymbol{Z}_+^n$, $j+|\alpha| \neq 0$.

PROOF. We can prove this proposition by induction on j, using Lemma 1.1 and Corollary 1.7. Q. E. D.

REMARK. The formal summation $\sum_{j=0}^{\infty} E_j(x, \xi)$ does not define a pseudo-differential operator or anything like that. We explain the meaning of the above symbol in the following sections.

§ 2. Definition of Bronstein operators.

We first define several function spaces. It will turn out that Bronstein operators defined below act on these spaces. The defining function of the solution of the Cauchy problem (0.3) will be constructed using such function spaces (see § 5).

In the rest of this paper, we fix some constants a, a_0, b, and R such that $1 \ll a \ll a_0$, $aa_0^{s/(s-1)} \ll b$ and $0 < R \ll a_0^{-s/(s-1)}$. We define linear functions $y = y(x) = (y_1(x_1), y_2(x'), \cdots, y_n(x'))$ and $\eta = \eta(\xi) = (\eta_1(\xi_1), \eta_2(\xi'), \cdots, \eta_n(\xi'))$ by

$$y_j(x) = \begin{cases} x_1, & j=1, \\ \dfrac{1}{(n-1)a}\sum_{k=2}^{n-1} x_k - \dfrac{1}{a}x_j + \dfrac{1}{n-1}x_n, & 2 \leqq j \leqq n-1, \\ \dfrac{1}{(n-1)a}\sum_{k=2}^{n-1} x_k + \dfrac{1}{n-1}x_n, & j=n, \end{cases} \tag{2.1}$$

and

$$\eta_j(\xi) = \begin{cases} \xi_1, & j=1, \\ \xi_n - a\xi_j, & 2 \leqq j \leqq n-1, \\ \xi_n + a\sum_{k=2}^{n-1} \xi_k, & j=n. \end{cases} \tag{2.2}$$

It is easy to see that $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n = y(x) \cdot \eta(\xi)$ and $x' \cdot \xi' = x_2\xi_2 + \cdots + x_n\xi_n = y'(x') \cdot \eta'(\xi')$, and the inverse functions are given by

$$x_j = \begin{cases} y_1, & j=1, \\ ay_n - ay_j, & 2 \leqq j \leqq n-1, \\ \sum_{k=2}^{n} y_k, & j=n, \end{cases} \tag{2.3}$$

and

$$\xi_j=\begin{cases}\eta_1, & j=1,\\ \dfrac{1}{a(n-1)}\sum_{k=2}^{n}\eta_k-\dfrac{1}{a}\eta_j, & 2\leqq j\leqq n-1,\\ \dfrac{1}{n-1}\sum_{k=2}^{n}\eta_k, & j=n,\end{cases}\tag{2.4}$$

respectively.

REMARK. From the definition, we have

$$\begin{aligned}&\{\xi'\in\sqrt{-1}\,\boldsymbol{R}^{n-1};\ \ \mathrm{Im}\,\xi_n\geqq(n-1)a\,|\mathrm{Im}\,\xi_j|,\ 2\leqq j\leqq n-1\}\\ \subset\ &\{\xi'\in\sqrt{-1}\,\boldsymbol{R}^{n-1};\ \ \mathrm{Im}\,\eta_j(\xi')\geqq 0,\ 2\leqq j\leqq n\}\\ \subset\ &\left\{\xi'\in\sqrt{-1}\,\boldsymbol{R}^{n-1};\ \ \mathrm{Im}\,\xi_n\geqq\frac{a}{n-1}|\mathrm{Im}\,\xi_j|,\ 2\leqq j\leqq n-1\right\}.\end{aligned}$$

This means that if $u_0(x')\in\mathcal{B}(\boldsymbol{R}^{n-1})$ has its singular support contained in a small neighborhood of $(0;0,\cdots,0,\sqrt{-1})\in\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$, $u_0(x')$ can be represented as the boundary value of some holomorphic function defined on the "first octant" $\{x'\in\boldsymbol{C}^{n-1};\ |x'|\ll 1,\ \mathrm{Im}\,y_j(x')>0,\ 2\leqq j\leqq n\}$. Conversely, if $u_0(x')$ is represented as the boundary value from this domain, then $SS'u_0(x')\cap\sqrt{-1}\,S^*_{x'}\boldsymbol{R}^{n-1}$ is contained in a small neighborhood of $(0,\cdots,0,\sqrt{-1})\in\sqrt{-1}\,S^*_{x'}\boldsymbol{R}^{n-1}$ at each $x'\in\boldsymbol{R}^{n-1}$.

Let J_i, $i=1,2,3$, be subsets of $\{2,3,\cdots,n\}$ such that $J_1\cup J_2\cup J_3=\{2,3,\cdots,n\}$ is a disjoint union, and let $J=(J_1,J_2,J_3)$ be a 3-tuple of such subsets. For each J, we define $\tilde{\Omega}\in\boldsymbol{C}^n$ by $\tilde{\Omega}=\bigcup_J\tilde{\Omega}_J$, where

$$\begin{aligned}\tilde{\Omega}_J=\Big\{x\in\boldsymbol{R}\times\boldsymbol{C}^{n-1};\ \ &a a_0^{s/(s-1)}|x_1|<1,\\ &0<a_0\mathrm{Im}\,y_j(x')<2,\ a_0|\mathrm{Re}\,y_j(x')|<2,\quad 2\leqq j\leqq n,\\ &0<a_0\mathrm{Im}\,x_n<1,\ a_0\mathrm{Im}\,x_n+a_0|\mathrm{Re}\,x_n|<1,\\ &\mathrm{Im}\,y_j(x')>a_0(x_1+R)\max\nolimits_{2\leq i\leq n}|\mathrm{Im}\,y_i(x')|,\ j\in J_1,\\ &\mathrm{Im}\,y_j(x')+(1/a_0)\mathrm{Re}\,y_j(x')-x_1-R>0,\ j\in J_2,\\ &\mathrm{Im}\,y_j(x')-(1/a_0)\mathrm{Re}\,y_j(x')-x_1-R>0,\ j\in J_3\Big\}.\end{aligned}$$

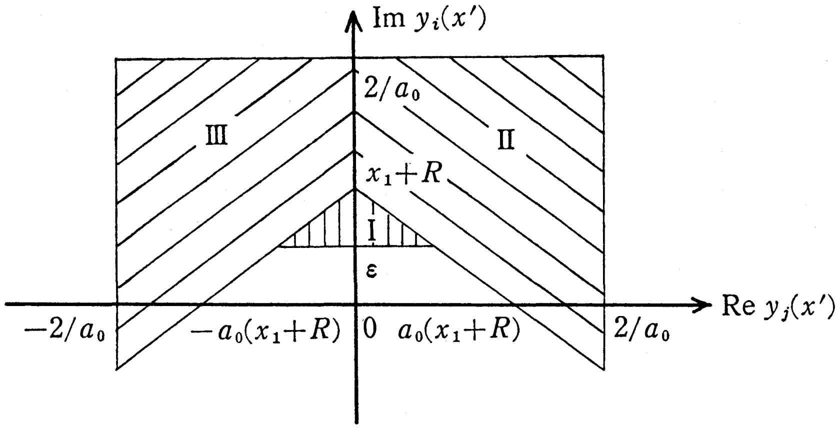
We define $\Omega=\bigcup_J\Omega_J$ where

$$\Omega_J=\{x\in\tilde{\Omega}_J;\ 0<a_0\mathrm{Im}\,y_j(x')<1,\ a_0|\mathrm{Re}\,y_j(x')|<1,\ 2\leqq j\leqq n\}.$$

REMARK. We need to explain the meaning of these domains. For the sake of simplicity, let us forget the condition

$$0<a_0\mathrm{Im}\,x_n<1,\quad a_0\mathrm{Im}\,x_n+a_0|\mathrm{Re}\,x_n|<1$$

in the above definition of $\tilde{\Omega}_J$ (and thus of $\tilde{\Omega}$). Then the cross section of $\tilde{\Omega}$ in the $y_j(x')$-plane is the shaded domain in Figure 1. In fact, assume that for each j, $2\leqq j\leqq n$, $y_j(x')$ belongs to the shaded domain. Then $y_j(x')$ belongs to one of the three domains I, II, and III, described there. For such a point x, we define $J=(J_1, J_2, J_3)$ as follows: Let $2\leqq j\leqq n$. If $y_j(x')$ belongs to the domain I, we let j belong to J_1. If $y_j(x')$ belongs to the domain II (resp. III), we let j belong to J_2 (resp. J_3). Then we have $x\in\tilde{\Omega}_J$, omitting the other conditions. Roughly speaking, $\tilde{\Omega}$ and Ω look like the "first octant" $\{x\in\boldsymbol{R}\times\boldsymbol{C}^{n-1};\ |x|\ll 1,\ \mathrm{Im}\,y_j(x')>0,\ 2\leqq j\leqq n\}$. Although $\tilde{\Omega}$ and Ω do not cover the first octant, they are very close to it. Let $u(x)$ be a continuous function defined on $\tilde{\Omega}$ or Ω, holomorphic in x', and consider the corresponding microfunction (with a real parameter x_1). Let $x\in\boldsymbol{R}^n$ be near the origin, and consider the stalk at x of the support of this microfunction. This stalk is contained in a cone which is a little larger than $\{\xi'\in\sqrt{-1}\,\boldsymbol{R}^{n-1};\ \mathrm{Im}\,\eta_j(\xi')\geqq 0,\ 2\leqq j\leqq n\}$, and this cone again is a small neighborhood of $(0, \cdots, 0, \sqrt{-1})\in\sqrt{-1}\,\boldsymbol{R}^{n-1}$, in the original coordinate system.



$$\varepsilon = \Big(a_0(x_1+R)\max_{\substack{2\leqq i\leqq n\\ i\neq j}}|\mathrm{Im}\,y_i(x')|\Big)_+$$

Figure 1.

If $s>1$, we denote by $\mathcal{O}^{0,s}(\Omega)$ the set of all continuous functions $f(x)$ defined on Ω which satisfy the following conditions:

i) For each x_1 fixed, $f(x)$ is holomorphic in x' if $x\in\Omega$.

ii) $f(x)=0$ if $x_1\leqq 0$.

iii) There exists some constant $C>0$ such that

$$|f(x)| \leqq C\exp\{b(x_1+R)^{s/(s-1)}\max_{j\in J_1}(|\mathrm{Im}\,y_j(x')|^{-1/(s-1)})\}$$

if $x\in\Omega_J$, for each J.

We define $\max_{j\in J_1}(|\mathrm{Im}\, y_j(x')|^{-1/(s-1)})=0$ if $J_1=\emptyset$. Thus the estimate required in the above condition iii) means $|f(x)|\leqq C$ if $J_1=\emptyset$. $\mathcal{O}^{0,s}(\Omega)$ is a Banach space whose norm is defined by

$$\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}=\max_J\Big(\sup_{x\in\Omega_J}\Big(|f(x)|\exp\Big\{-b(x_1+R)^{s/(s-1)}\max_{j\in J_1}(|\mathrm{Im}\, y_j(x')|^{-1/(s-1)})\Big\}\Big)\Big).$$

If $q\in \boldsymbol{Z}_+$, we define $\mathcal{O}^{-q,s}(\Omega)$ by

$$\mathcal{O}^{-q,s}(\Omega)=\{D_1^q f(x)\,;\, f(x)\in\mathcal{O}^{0,s}(\Omega)\}\,.$$

Here the derivation should be taken in the sense of distribution. We define $\mathcal{O}^{-q,s}(\tilde{\Omega})$, $q\in\boldsymbol{Z}_+$, just in the same way (Ω replaced by $\tilde{\Omega}$).

REMARK. Assume that $u(x)\in C^0[\boldsymbol{R}\,;\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$ satisfies $u(x)=0$ if $x_1\leqq 0$, and that $SS'u(x)$ is contained in a small neighborhood of $\mathring{x}^*$. If b is large enough, $u(x)$ can be represented as the boundary value of some function $v(x)\in\mathcal{O}^{0,s}(\Omega)$. Conversely, the boundary value $[v(x)]$ of $v(x)\in\mathcal{O}^{0,s}(\Omega)$ is a section of $C^0[\boldsymbol{R}\,;\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$. Furthermore, $SS'[v(x)]\cap(\{x_1\}\times\sqrt{-1}\,S^*_{x'}\boldsymbol{R}^{n-1})$ is contained in a small neighborhood of $(x\,;0,\cdots,0,\sqrt{-1})\in\{x_1\}\times\sqrt{-1}\,S^*_{x'}\boldsymbol{R}^{n-1}\cong\sqrt{-1}\,S^*_{x'}\boldsymbol{R}^{n-1}$ for each $x\in\boldsymbol{R}^n$.

Let us denote the spectrum of $[v(x)]$ also by $[v(x)]$. Assume that a pseudo-differential operator $P(x, D)$ of the form (0.1) is microhyperbolic in the direction x_1 at $\mathring{x}^*$. In §5, we will prove that for any $f(x)\in\mathcal{O}^{0,s}(\Omega)$, there exists some $v(x)\in\mathcal{O}^{0,s}(\Omega)$ which satisfies $P(x, D)D_n^{n+2}[v(x)]=[f(x)]$ at $\mathring{x}^*$. Using this fact we can construct the solution of (0.3).

The space $\mathcal{O}^{0,s}(\tilde{\Omega})$ has essentially the same property as $\mathcal{O}^{0,s}(\Omega)$.

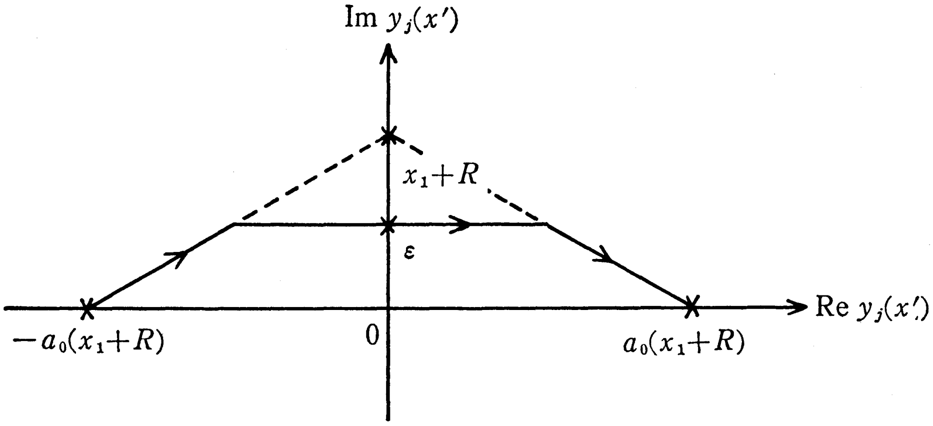
We define the partial Fourier transformation $\hat{f}(x_1,\xi')=(\mathcal{F}_{x'\to\xi'}f)(x_1,\xi')$ of $f(x)\in\mathcal{O}^{0,s}(\Omega)$ by

$$\hat{f}(x_1,\xi')=\int_{\Gamma_K}e^{-x'\cdot\xi'}f(x)dx'$$

where $K=(K_1, K_2, K_3)$ is a 3-tuple as above, and

$$\Gamma_K=\Big\{x'\in\boldsymbol{C}^{n-1}\,;\ |\mathrm{Re}\, y_j(x')|<a_0x_1+a_0R,$$
$$\mathrm{Im}\, y_j(x')=\min\Big(x_1+R-\frac{1}{a_0}|\mathrm{Re}\, y_j(x')|,$$
$$(x_1+R)\Big(\frac{b}{(s-1)\sum_{k\in K_1}|\mathrm{Im}\,\eta_k(\xi')|}\Big)^{(s-1)/s}\Big),\qquad 2\leqq j\leqq n\Big\},$$

(see Figure 2).



$$\varepsilon=(x_1+R)\left(\frac{b}{(s-1)\sum_{k\in K_1}|\operatorname{Im}\eta_k(\xi')|}\right)^{(s-1)/s}$$

Figure 2.

If $x_1\leqq 0$, we have $\hat{f}(x_1, \xi')=0$. Assume that $x_1>0$, $x'\in\Gamma_K$, and $\operatorname{Im}\eta_j(\xi')\geqq a_0|\operatorname{Re}\eta_j(\xi')|$, $2\leqq j\leqq n$. Then we have

$$|e^{-x'\cdot\xi'}f(x)|\leqq\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}\exp\Big\{b(x_1+R)\Big(\frac{b}{s-1}\Big)^{-1/s}\Big(\sum_{j\in K_1}\operatorname{Im}\eta_j(\xi')\Big)^{1/s}$$
$$+\sum_{j\in K_1\cup K_2\cup K_3}(\operatorname{Im}y_j(x')\operatorname{Im}\eta_j(\xi')-\operatorname{Re}y_j(x')\operatorname{Re}\eta_j(\xi'))\Big\}.$$

Using

$$\sum_{K_1}(\operatorname{Im}y_j\operatorname{Im}\eta_j-\operatorname{Re}y_j\operatorname{Re}\eta_j)$$
$$\leqq\Big(\frac{b}{s-1}\Big)^{(s-1)/s}(x_1+R)\Big(\sum_{K_1}\operatorname{Im}\eta_j\Big)^{1/s}+a_0(x_1+R)\sum_{K_1}|\operatorname{Re}\eta_j|$$

and

$$\sum_{K_2\cup K_3}(\operatorname{Im}y_j\operatorname{Im}\eta_j-\operatorname{Re}y_j\operatorname{Re}\eta_j)$$
$$\leqq\sum_{K_2\cup K_3}\Big(x_1+R-\frac{1}{a_0}|\operatorname{Re}y_j|\Big)\operatorname{Im}\eta_j-\sum_{K_2\cup K_3}\operatorname{Re}y_j\operatorname{Re}\eta_j$$
$$\leqq(x_1+R)\sum_{K_2\cup K_3}\operatorname{Im}\eta_j,$$

we have the following

PROPOSITION 2.1. *If* $f(x)\in\mathcal{O}^{0,s}(\Omega)$, $aa_0^{s/(s-1)}|x_1|<1$, *and* $\operatorname{Im}\eta_j(\xi')\geqq a_0|\operatorname{Re}\eta_j(\xi')|$, $2\leqq j\leqq n$, *we have*

$$(2.5)\qquad|\hat{f}(x_1,\xi')|\leqq\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}\exp\Big\{\Big(\frac{b}{s-1}\Big)^{(s-1)/s}s(x_1+R)\Big(\sum_{K_1}\operatorname{Im}\eta_j(\xi')\Big)^{1/s}$$

$$+a_0(x_1+R)\sum_{K_1}|\mathrm{Re}\,\eta_j(\xi')|+(x_1+R)\sum_{K_2\cup K_3}|\mathrm{Im}\,\eta_j(\xi')|\Big\}$$

for any K. $\hat{f}(x_1, \xi')=0$ *if* $x_1\leqq 0$.

Let us define $U\subset C^n\times C^n$ by

$$U=\Big\{(x, \xi)\in C^n\times C^n;\ a_0|\mathrm{Re}\,y_j(x)|<4,\ a_0|\mathrm{Im}\,y_j(x)|<4,\ 1\leqq j\leqq n,$$
$$\mathrm{Im}\,\eta_j(\xi')>\frac{1}{4}a_0(|\mathrm{Re}\,\eta_j(\xi')|+1),\ 2\leqq j\leqq n,$$
$$\mathrm{Re}\,\xi_1>\phi(y(x), \eta'(\xi'))-a_0^{-3}\Big(|\mathrm{Im}\,\xi_1|-a_0\sum_{k=2}^{n}|\mathrm{Im}\,\eta_k(\xi')|\Big)_+\Big\},$$

where

$$\phi(y, \eta')=\frac{a_0}{4}\sum_{j=2}^{n}|\mathrm{Re}\,\eta_j|+\frac{a_0}{4}\Big(\max_{1\leq i\leq n}|\mathrm{Im}\,y_i|\Big)\sum_{j=2}^{n}|\mathrm{Im}\,\eta_j|$$
$$+\frac{a_0}{4}\Big(\sum_{j=2}^{n}|\mathrm{Im}\,\eta_j|\Big)^{1/s}+\frac{a_0}{4}.$$

Roughly speaking, $(x, \xi)\in U$ when (x, ξ') belongs to a small conic neighborhood of $\mathring{x}^*$ and $\mathrm{Re}\,\xi_1$ is very large.

We say that $Q(x, \xi)$ is *a symbol* (*of a Bronstein operator*) if $Q(x, \xi)$ is holomorphic on U and satisfies

$$|Q(x, \xi)|\leqq C_0|\xi_1|^{-1+l}|\xi_n|^{-n} \tag{2.6}$$

with some $C_0>0$ and $l\in Z_+$ on U.

REMARK. If a and a_0 are large enough, the functions $E_j(x, \xi)$, $j\in Z_+$, defined by Definition 1.8, are symbols in the above sense. If $Q(x, \xi)$ is holomorphic on U and satisfies

$$|Q(x, \xi)|\leqq C_0|\xi_1|^{m'}|\xi_n|^{m''} \tag{2.6$'$}$$

with some $C_0>0$ and $m', m''\in Z_+$, $Q(x, \xi)$ is a symbol because we have $|\xi_1|>|\xi_n|^{1/s}$ on U, and thus (2.6)′ means

$$|Q(x, \xi)|\leqq C_0|\xi_1|^{m'+s(m''+n)}|\xi_n|^{-n}. \tag{2.6$''$}$$

Let $Q(x, \xi)$ satisfy (2.6) with $l=0$. We define the partial inverse Fourier transformation $\check{Q}(x, \tilde{x}_1, \xi')=(\mathcal{F}_{\xi_1\to\tilde{x}_1}Q)(x, \tilde{x}_1, \xi')$ by

$$\check{Q}(x, \tilde{x}_1, \xi')=\frac{1}{2\pi\sqrt{-1}}\int_\delta e^{\tilde{x}_1\xi_1}Q(x, \xi)d\xi_1 \tag{2.7}$$

where $\delta=\{\xi\in C;\ \mathrm{Re}\,\xi_1=\phi(y(x), \eta'(\xi'))+1\}$. It is easy to see $\check{Q}(x, \tilde{x}_1, \xi')=0$ if $\tilde{x}_1<0$, and that we may replace this path with

$$\delta_\varepsilon=\left\{\xi_1\in \boldsymbol{C};\ \mathrm{Re}\,\xi_1=\psi(y(x),\ \eta'(\xi'))-a_0^{-3}\Big(|\mathrm{Im}\,\xi_1|-a_0\sum_{j=2}^{n}\mathrm{Im}\,\eta_j(\xi')\Big)_+ +\varepsilon,\ \mathrm{Im}\,\xi_1\in \boldsymbol{R}\right\}$$

for any $\varepsilon>0$, if $\tilde{x}_1\geqq 0$. Here $\tilde{x}_1$ is a copy of x_1. We define $\check{U}\subset \boldsymbol{C}^n\times \boldsymbol{R}\times \boldsymbol{C}^{n-1}$ by

$$\check{U}=\Big\{(x,\ \tilde{x}_1,\ \xi')\in \boldsymbol{C}^n\times \boldsymbol{R}\times \boldsymbol{C}^{n-1};\ a_0|\mathrm{Re}\,y_j(x)|<4,\ a_0|\mathrm{Im}\,y_j(x)|<4,\ 1\leqq j\leqq n,$$
$$a_0|\tilde{x}_1|<4,\ \mathrm{Im}\,\eta_j(\xi')>\frac{a_0}{4}(|\mathrm{Re}\,\eta_j(\xi')|+1),\ 2\leqq j\leqq n\Big\}.$$

Then we have the following

PROPOSITION 2.2. *If $Q(x,\xi)$ satisfies* (2.6) *on U with $l=0$, we have*

$$|\check{Q}(x,\ \tilde{x}_1,\ \xi')|\leqq 4na_0^{3/n}C_0|\tilde{x}_1|^{-1/n}|\xi_n|^{-n+1/n}\exp\{\tilde{x}_1\psi(y(x),\ \eta'(\xi'))\} \tag{2.8}$$

on $\check{U}$. We have $\check{Q}(x,\ \tilde{x}_1,\ \xi')=0$ if $\tilde{x}_1<0$.

PROOF. The second statement is trivial, and we assume $\tilde{x}_1\geqq 0$. Let $\varepsilon>0$ be arbitrary. If $1+|\mathrm{Im}\,\xi_1|\leqq 2a_0\sum_{j=2}^{n}|\mathrm{Im}\,\eta_j(\xi')|$, we have $|\xi_n|\geqq(1/2(n-1)a_0)\times(1+|\mathrm{Im}\,\xi_1|)$, and thus

$$|e^{\tilde{x}_1\xi_1}Q(x,\ \xi)|\leqq 2(2(n-1)a_0)^{1/n}C_0(1+|\mathrm{Im}\,\xi_1|)^{-1-1/n}\times|\xi_n|^{-n+1/n}\exp\{\tilde{x}_1\psi(y(x),\ \eta'(\xi'))+\varepsilon\tilde{x}_1\}. \tag{2.9}$$

On the other hand, if $1+|\mathrm{Im}\,\xi_1|\geqq 2a_0\sum_{j=2}^{n}|\mathrm{Im}\,\eta_j(\xi')|$, we have $|\mathrm{Im}\,\xi_1|\geqq a_0\sum_{j=2}^{n}|\mathrm{Im}\,\eta_j(\xi')|$ and thus

$$\begin{aligned}(2.10)\qquad &|e^{\tilde{x}_1\xi_1}Q(x,\ \xi)|\\ &\leqq C_0|\xi_1|^{-1}|\xi_n|^{-n}\exp\Big\{\tilde{x}_1\psi(y(x),\ \eta'(\xi'))\\ &\qquad -\tilde{x}_1a_0^{-3}(|\mathrm{Im}\,\xi_1|-a_0\sum_{j=2}^{n}|\mathrm{Im}\,\eta_j(\xi')|)+\varepsilon\tilde{x}_1\Big\}\\ &\leqq C_0|\xi_1|^{-1}|\xi_n|^{-n}\exp\Big\{\tilde{x}_1\psi(y(x),\ \eta'(\xi'))-\frac{\tilde{x}_1}{3}a_0^{-3}(|\mathrm{Im}\,\xi_1|+1)+\varepsilon\tilde{x}_1\Big\}\\ &\leqq C_0a_0^{3/n}|\tilde{x}_1|^{-1/n}(1+|\mathrm{Im}\,\xi_1|)^{-1/n}|\xi_1|^{-1}|\xi_n|^{-n}\exp\{\tilde{x}_1\psi(y(x),\ \eta'(\xi'))+\varepsilon\tilde{x}_1\}.\end{aligned}$$

Combining (2.9) and (2.10), we obtain

$$|e^{\tilde{x}_1\xi_1}Q(x,\ \xi)|\leqq 2C_0a_0^{3/n}|\tilde{x}_1|^{-1/n}(1+|\mathrm{Im}\,\xi_1|)^{-1-1/n}|\xi_n|^{-n+1/n}\times\exp\{\tilde{x}_1\psi(y(x),\ \eta'(\xi'))+\varepsilon\tilde{x}_1\}.$$

Letting $\varepsilon\to 0$, we can prove (2.8) directly from this estimate. Q.E.D.

Now we can define Bronstein operators. If $f(x)\in\mathcal{O}^{0,s}(\Omega)$ and $Q(x,\xi)$ satisfies (2.6) on U, we define $Q_B(x,D)f(x)$ by

$$(2.11)\qquad Q_B(x, D)f(x)=\frac{1}{(2\pi\sqrt{-1})^{n-1}}\int_{\Delta_J}\int_R e^{x'\cdot\xi'}\check{Q}(x, \tilde{x}_1, \xi')\hat{f}(x_1-\tilde{x}_1, \xi')d\tilde{x}_1 d\xi'$$

where

$$\Delta_J=\Big\{\xi'\in C^{n-1};\ \mathrm{Im}\,\eta_j(\xi')>a_0,\ \mathrm{Re}\,\eta_j(\xi')=0,\ j\in J_1,$$
$$\mathrm{Im}\,\eta_j(\xi')>a_0,\ \mathrm{Re}\,\eta_j(\xi')=-(1/a_0)(\mathrm{Im}\,\eta_j(\xi')-a_0),\ j\in J_2,$$
$$\mathrm{Im}\,\eta_j(\xi')>a_0,\ \mathrm{Re}\,\eta_j(\xi')=(1/a_0)(\mathrm{Im}\,\eta_j(\xi')-a_0),\ j\in J_3\Big\}.$$

Since $\check{Q}(x, \tilde{x}_1, \xi')=0$ if $\tilde{x}_1\leqq 0$ and $\hat{f}(x_1-\tilde{x}_1, \xi')=0$ if $x_1-\tilde{x}_1\leqq 0$, the integration is taken over $0<\tilde{x}_1<x_1$, and thus $Q_B(x, D)f(x)=0$ if $x_1\leqq 0$. If $0<\tilde{x}_1<x_1$, $x\in\tilde{\Omega}_J$ and $\xi'\in\Delta_J$, from (2.5) and (2.8) we have

$$|e^{x'\cdot\xi'}\check{Q}(x, \tilde{x}_1, \xi')\hat{f}(x_1-\tilde{x}_1, \xi')|$$
$$\leqq 4na_0^{3/n}C_0\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}|\tilde{x}_1|^{-1/n}|\xi_n|^{-n+1/n}$$
$$\times\exp\Big\{\Big(\frac{b}{s-1}\Big)^{(s-1)/s}s(x_1-\tilde{x}_1+R)\Big(\sum_{J_1}|\mathrm{Im}\,\eta_j(\xi')|\Big)^{1/s}$$
$$+(x_1-\tilde{x}_1+R)\sum_{J_2\cup J_3}|\mathrm{Im}\,\eta_j(\xi')|+\frac{a_0}{4}\tilde{x}_1\sum_{J_2\cup J_3}|\mathrm{Re}\,\eta_j(\xi')|$$
$$+\frac{a_0}{4}\tilde{x}_1\Big(\max_{2\leq i\leq n}|\mathrm{Im}\,y_i(x')|\Big)\sum_{J_1\cup J_2\cup J_3}\mathrm{Im}\,\eta_j(\xi')$$
$$+\frac{a_0}{4}\tilde{x}_1\Big(\sum_{J_1\cup J_2\cup J_3}|\mathrm{Im}\,\eta_j(\xi')|\Big)^{1/s}+\frac{a_0}{4}\tilde{x}_1$$
$$-\sum_{J_1\cup J_2\cup J_3}\mathrm{Im}\,y_j(x')\mathrm{Im}\,\eta_j(\xi')+\sum_{J_2\cup J_3}\mathrm{Re}\,y_j(x')\mathrm{Re}\,\eta_j(\xi')\Big\}$$
$$\leqq 4ne^n a_0^{3/n}C_0\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}|\tilde{x}_1|^{-1/n}|\xi_n|^{-n+1/n}$$
$$\times\exp\Big\{\Big(\frac{b}{s-1}\Big)^{(s-1)/s}s(x_1-\tilde{x}_1+R)\Big(\sum_{J_1}|\mathrm{Im}\,\eta_j(\xi')|\Big)^{1/s}$$
$$+\frac{a_0}{4}\tilde{x}_1\Big(\sum_{J_1}|\mathrm{Im}\,\eta_j(\xi')|\Big)^{1/s}-\sum_{J_1}\mathrm{Im}\,y_j(x')\mathrm{Im}\,\eta_j(\xi')$$
$$+\frac{a_0}{4}\tilde{x}_1\Big(\max_{2\leq i\leq n}|\mathrm{Im}\,y_i(x')|\Big)\sum_{J_1}\mathrm{Im}\,\eta_j(\xi')\Big\}$$
$$\times\exp\Big\{-\frac{1}{4}\tilde{x}_1\sum_{J_2\cup J_3}\mathrm{Im}\,\eta_j(\xi')+\frac{a_0}{4}\tilde{x}_1\Big(\sum_{J_2\cup J_3}|\mathrm{Im}\,\eta_j(\xi')|\Big)^{1/s}\Big\}.$$

Note that if $c_1, c_2>0$, we have

$$(2.12)\qquad \max_{t\geq 0}(-c_1t+c_2t^{1/s})=(s-1)s^{-s/(s-1)}c_1^{-1/(s-1)}c_2^{s/(s-1)}.$$

Using this inequality, we obtain

$$\left(\frac{b}{s-1}\right)^{(s-1)/s} s(x_1-\tilde{x}_1+R)\Big(\sum_{J_1}|\operatorname{Im}\eta_j(\xi')|\Big)^{1/s}+\frac{a_0}{4}\tilde{x}_1\Big(\sum_{J_1}|\operatorname{Im}\eta_j(\xi')|\Big)^{1/s}$$
$$-\sum_{J_1}\operatorname{Im}y_j(x')\operatorname{Im}\eta_j(\xi')+\frac{a_0}{4}\tilde{x}_1\Big(\max_{2\leq i\leq n}|\operatorname{Im}y_i(x')|\Big)\sum_{J_1}\operatorname{Im}\eta_j(\xi')$$
$$\leqq \frac{x_1-\tilde{x}_1+R}{x_1+R}\Big\{\Big(\frac{b}{s-1}\Big)^{(s-1)/s} s(x_1+R)\Big(\sum_{J_1}|\operatorname{Im}\eta_j(\xi')|\Big)^{1/s}-\sum_{J_1}\operatorname{Im}y_j(x')\operatorname{Im}\eta_j(\xi')\Big\}$$
$$+\frac{\tilde{x}_1}{x_1+R}\Big\{\frac{a_0}{4}(x_1+R)\Big(\sum_{J_1}|\operatorname{Im}\eta_j(\xi')|\Big)^{1/s}-\frac{3}{4}\sum_{J_1}\operatorname{Im}y_i(x')\operatorname{Im}\eta_j(\xi')\Big\}$$
$$\leqq b(x_1+R)^{s/(s-1)}\max_{J_1}(|\operatorname{Im}y_j(x')|^{-1/(s-1)}).$$

Here we have used the assumption $b\gg a_0^{s/(s-1)}$. Similarly, we have

$$-\frac{1}{4}\tilde{x}_1\sum_{J_2\cup J_3}\operatorname{Im}\eta_j(\xi')+\frac{a_0}{4}\tilde{x}_1\Big(\sum_{J_2\cup J_3}|\operatorname{Im}\eta_j(\xi')|\Big)^{1/s}$$
$$\leqq \frac{1}{4}(s-1)s^{-s/(s-1)}a_0^{s/(s-1)}\tilde{x}_1\leqq 1.$$

We note that

$$|\xi_n|^{-n+1/n}\leqq(\operatorname{Im}\xi_n)^{-n+1/n}\leqq\prod_{j=2}^{n}\Big(\Big(\frac{\operatorname{Im}\eta_j(\xi')}{n-1}\Big)^{-1-1/n}\Big).$$

Summing up, we obtain

$$|e^{x'\cdot\xi'}\check{Q}(x,\tilde{x}_1,\xi')\hat{f}(x_1-\tilde{x}_1,\xi')|$$
$$\leqq 4ne^{n+1}a_0^{3/n}C_0\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}|\tilde{x}_1|^{-1/n}\prod_{j=2}^{n}\Big(\Big(\frac{\operatorname{Im}\eta_j(\xi')}{n-1}\Big)^{-1-1/n}\Big)$$
$$\times\exp\Big\{b(x_1+R)^{s/(s-1)}\max_{J_1}(|\operatorname{Im}y_j(x')|^{-1/(s-1)})\Big\}.$$

If $x\in\tilde{\Omega}$, we can thus integrate (2.11), and it follows that $Q_B(x,D)f(x)\in\mathcal{O}^{0,s}(\tilde{\Omega})$ $\subset\mathcal{O}^{0,s}(\Omega)$ if $f(x)\in\mathcal{O}^{0,s}(\Omega)$. Furthermore, the above estimate shows that

$$\|Q_B(x,D)f(x)\|_{\mathcal{O}^{0,s}(\tilde{\Omega})}\leqq a_0^{3/n}C_0\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}. \tag{2.13}$$

We call $Q_B(x,D):\mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\tilde{\Omega})$ $(\subset\mathcal{O}^{0,s}(\Omega))$ *a Bronstein operator* (*in the strict sense*). More generally, let $\check{Q}(x,\tilde{x}_1,\xi')$ be a continuous function defined on $\check{U}_0=\{(x,\tilde{x}_1,\xi')\in\check{U};\,x\in\Omega\}$ which is holomorphic in (x,ξ'). Furthermore, assume that i) $\check{Q}(x,\tilde{x}_1,\xi')$ satisfies (2.8) on $\check{U}_0$, and ii) $\check{Q}(x,\tilde{x}_1,\xi')=0$ if $\tilde{x}_1\leqq 0$. Then the integration (2.11) is well defined if $x\in\Omega_J\subset\tilde{\Omega}_J$, and we obtain a function $Q_B(x,D)f(x)\in\mathcal{O}^{0,s}(\Omega)$. In this case we have

$$\|Q_B(x,D)f(x)\|_{\mathcal{O}^{0,s}(\Omega)}\leqq a_0^{3/n}C_0\|f(x)\|_{\mathcal{O}^{0,s}(\Omega)}. \tag{2.13)'$$

This time we call the operator $Q_B(x, D): \mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)$ thus obtained *a Bronstein operator in the wider sense.*

Now let $Q(x, \xi)$ be holomorphic on U and satisfy (2.6) with a general $l\in \boldsymbol{Z}_+$. Then the integrations (2.7) and (2.11) can be defined in distribution sense with respect to $\tilde{x}_1$. A similar argument as above shows that $Q_B(x, D)f(x)$ thus defined belongs to $\mathcal{O}^{-l,s}(\tilde{\Omega})$. We also call this operator from $\mathcal{O}^{0,s}(\Omega)$ to $\mathcal{O}^{-l,s}(\tilde{\Omega})$ *a Bronstein operator* (*in the strict sense*). We have proved the following

PROPOSITION 2.3. i) *Let $Q(x, \xi)$ be holomorphic on U and satisfy* (2.6) *with $l=0$. Then we have $Q_B(x, D)f(x)\in\mathcal{O}^{0,s}(\tilde{\Omega})$ and* (2.13) *holds.*

ii) *Let $\check{Q}(x, \tilde{x}_1, \xi')$ be a continuous function defined on $\check{U}_0$ which is holomorphic in (x, ξ'). If $\check{Q}(x, \tilde{x}_1, \xi')$ satisfies* (2.8) *on $\check{U}_0$ and $\check{Q}(x, \tilde{x}_1, \xi')=0$ for $\tilde{x}_1\leqq 0$, we have $Q_B(x, D)f(x)\in\mathcal{O}^{0,s}(\Omega)$ and* (2.13)$'$ *holds.*

iii) *Let $Q(x, \xi)$ be holomorphic on U and satisfy* (2.6). *Then we have $Q_B(x, D)f(x)\in\mathcal{O}^{-l,s}(\tilde{\Omega})$.*

REMARK. If $f(x)\in\mathcal{O}^{0,s}(\Omega)$, we can also define the Fourier transformation $(\mathcal{F}_{x\to\xi}f)(\xi)$ of $f(x)$ with respect to all the variables. We might have defined $Q_B(x, D)$ more simply by

$$Q_B(x, D)f(x) = \frac{1}{(2\pi\sqrt{-1})^n}\int e^{x\cdot\xi}Q(x, \xi)(\mathcal{F}_{x\to\xi}f)(\xi)d\xi. \tag{2.14}$$

(This definition coincides with the above one.) But to get a slightly better estimate, we prefer the above definition.

§3. Action of pseudodifferential operators.

Let $A(x, \xi')$ be a total symbol of some pseudodifferential operator $A(x, D')$ defined at $\mathring{x}^*\in \boldsymbol{R}\times\sqrt{-1}\,S^*\boldsymbol{R}^{n-1}$. Since $A(x, \xi')$ is holomorphic on a domain containing U, $A(x, \xi')$ defines a Bronstein operator $A_B(x, D'): \mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{-l,s}(\tilde{\Omega})$ with some $l\in\boldsymbol{Z}_+$, as was explained in §2. In this section we consider another action of pseudodifferential operators, also on the level of the defining functions of microfunctions. This action was defined by [2] and [4], and we discuss following the arguments of [2].

Let $l'\in\boldsymbol{Z}_+$. Let $A_{\alpha'}(x)$, $\alpha'\in\boldsymbol{Z}_+^{n-2}\times\boldsymbol{Z}$, $|\alpha'|=\alpha_2+\cdots+\alpha_n\leqq l'$, be a sequence of holomorphic functions defined on $\{C|x|<1\}$ such that

$$|A_{\alpha'}(x)| \leqq C^{l'-\alpha_n+1}(l'-|\alpha'|)! \tag{3.1}$$

there. Here $C>0$ is some large constant. Let $A(x, \xi')$ be a holomorphic function defined on $\{(x, \xi')\in\boldsymbol{C}\times\sqrt{-1}\,S^*\boldsymbol{R}^{n-1};\ C|x|<1,\ 4C|\xi_j|<|\xi_n|,\ 2\leqq j\leqq n-1\}$ such that

(3.2) $$A(x, \xi') \sim \sum_{|\alpha'|\leq l'} A_{\alpha'}(x)\xi^{\alpha'} .$$

(See § 0.) Let $f(x)\in\mathcal{O}^{0,s}(\tilde{\Omega})$ and $\mathring{x}_n=\sqrt{-1}\,a_0^{-1}$. We define $A_\Sigma(x, D')f(x)$ formally by

(3.3) $$\begin{aligned} &A_\Sigma(x, D')f(x) \\ &= \sum_{\substack{|\alpha'|\leq l' \\ \alpha_n\geq 0}} A_{\alpha'}(x)D^{\alpha'}f(x)+\sum_{\substack{|\alpha'|\leq l' \\ \alpha_n<0}} A_{\alpha'}(x)\int_{\mathring{x}_n}^{x_n}\cdots\int_{\mathring{x}_n}^{x_n} D^{\alpha''}f(x)dx_n\cdots dx_n . \end{aligned}$$

In the first summation, $D^{\alpha'}$ denotes the usual differential operator. In the second summation $D^{\alpha''}$ denotes the usual differential operator with respect to $x''=(x_2, \cdots, x_{n-1})$, and the integration is repeated $-\alpha_n>0$ times. Here we inherit the notation $A_\Sigma(x, D')$ from [2] (Σ denotes the hypersurface $x_n=\sqrt{-1}\,a_0^{-1}$). Let us prove that (3.3) converges on $\Omega\subset\tilde{\Omega}$. The first summation is a finite summation of the derivatives of $f(x)$, and thus we only need to prove the convergence of the second summation. We first prepare the following

Lemma 3.1. *If $x\in\Omega$ and $\tilde{x}'\in \boldsymbol{C}^{n-1}$ satisfy*

(3.4) $$\tilde{x}_n=r(\mathring{x}_n-x_n), \qquad 0<r<1,$$

(3.5) $$|\tilde{x}_j|\leq\frac{\sqrt{a}}{n}\operatorname{Im}\tilde{x}_n, \qquad 2\leq j\leq n-1,$$

then we have

i) $\operatorname{Im}y_j(\tilde{x}')>0, \quad 2\leq j\leq n,$

ii) $(x_1, x'+\tilde{x}')\in\tilde{\Omega}.$

Proof. i) From (2.1) and (3.5), it follows that

$$\operatorname{Im}y_j(\tilde{x}')\geq-\frac{1}{a}\sum_{k=2}^{n-1}|\operatorname{Im}\tilde{x}_k|+\frac{1}{n-1}\operatorname{Im}\tilde{x}_n\geq\Big(\frac{1}{n-1}-\frac{n-2}{n\sqrt{a}}\Big)\operatorname{Im}\tilde{x}_n>0$$

for $2\leq j\leq n$.

ii) From (2.1), (3.4), and (3.5), we have

(3.6) $$|\operatorname{Re}y_j(\tilde{x}')|\leq\operatorname{Im}\tilde{x}_n, \qquad |\operatorname{Im}y_j(\tilde{x}')|\leq\operatorname{Im}\tilde{x}_n,$$

for $2\leq j\leq n$. For each J, let us prove that if $x\in\Omega_J$ and $\tilde{x}'$ satisfy (3.4) and (3.5), then $(x_1, x'+\tilde{x}')\in\tilde{\Omega}_J$. Using i), (3.4), and (3.6), we obtain

(3.7) $$0<a_0\operatorname{Im}y_j(x'+\tilde{x}')<2, \quad a_0|\operatorname{Re}y_j(x'+\tilde{x}')|<2,$$

for $2\leq j\leq n$, and

(3.8) $$0<a_0\operatorname{Im}(x_n+\tilde{x}_n)<1, \quad a_0\operatorname{Im}(x_n+\tilde{x}_n)+a_0|\operatorname{Re}(x_n+\tilde{x}_n)|<1.$$

From (2.1), (3.5), (3.6), and the definition of $\tilde{\Omega}_J$, we obtain

(3.9)
$$\begin{aligned}&\operatorname{Im} y_j(x'+\tilde{x}')-a_0(x_1+R)|\operatorname{Im} y_i(x'+\tilde{x}')|\\ &\geqq \{\operatorname{Im} y_j(x')-a_0(x_1+R)|\operatorname{Im} y_i(x')|\}+\{\operatorname{Im} y_j(\tilde{x}')-a_0(x_1+R)|\operatorname{Im} y_i(\tilde{x}')|\}\\ &>\operatorname{Im} y_j(\tilde{x}')-a_0(x_1+R)|\operatorname{Im} y_i(\tilde{x}')|\\ &\geqq\Big(\frac{1}{n-1}-\frac{n-2}{n\sqrt{a}}-\frac{1}{a_0}\Big)\operatorname{Im}\tilde{x}_n>0,\end{aligned}$$

for $j\in J_1$, $2\leqq i\leqq n$. Similarly we can prove

$$\operatorname{Im} y_j(x'+\tilde{x}')+\frac{1}{a_0}\operatorname{Re} y_j(x'+\tilde{x}')-x_1-R>0 \tag{3.10}$$

for $j\in J_2$, and

$$\operatorname{Im} y_j(x'+\tilde{x}')-\frac{1}{a_0}\operatorname{Re} y_j(x'+\tilde{x}')-x_1-R>0 \tag{3.11}$$

for $j\in J_3$. (3.7)-(3.11) means that $(x_1, x'+\tilde{x}')\in\tilde{\Omega}_J$. Q. E. D.

Using Cauchy integration theory, we obtain the following

COROLLARY 3.2. *If* $x\in\Omega$ *and* $t=r(\mathring{x}_n-x_n)+x_n$, $0<r<1$, *we have*

$$\begin{aligned}|D^{\alpha''}f(x_1, x_2, \cdots, x_{n-1}, t)|\leqq&\Big(\frac{n}{\sqrt{a}\cdot\operatorname{Im}(t-x_n)}\Big)^{|\alpha''|}\alpha''!\|f(x)\|_{\mathcal{O}^{0,s}(\tilde{\Omega})}\\ &\times\exp\Big\{b(x_1+R)^{s/(s-1)}\max_{J_1}(|\operatorname{Im} y_j(x')|^{-1/(s-1)})\Big\}.\end{aligned} \tag{3.12}$$

We define $(A_j)_\Sigma(x, D')f(x)$, $j\in\boldsymbol{Z}_+$, by

$$\begin{aligned}(A_j)_\Sigma(x, D')f(x)&=\sum_{\substack{|\alpha'|=l'-j\\ \alpha_n<0}}A_{\alpha'}(x)\int_{\mathring{x}_n}^{x_n}\cdots\int_{\mathring{x}_n}^{x_n}D^{\alpha''}f(x)dx_n\cdots dx_n\\ &=\sum_{\substack{|\alpha'|=l'-j\\ \alpha_n<0}}A_{\alpha'}(x)\int_{\mathring{x}_n}^{x_n}\frac{(x_n-t)^{-\alpha_n-1}}{(-\alpha_n-1)!}D^{\alpha''}f(x_1, \cdots, x_{n-1}, t)dt.\end{aligned}$$

Here we have omitted the finite number of terms with $\alpha_n\geqq 0$. If $j\geqq l'+1$ and $x\in\Omega_J$, from (3.12) we have

$$\begin{aligned}&|(A_j)_\Sigma(x, D')f(x)|\\ &\leqq\sum_{\substack{|\alpha'|=l'-j\\ \alpha_n<0}}C^{j+|\alpha''|+1}j!\Big(\frac{n}{\sqrt{a}}\Big)^{|\alpha''|}\alpha''!\|f(x)\|_{\mathcal{O}^{0,s}(\tilde{\Omega})}\Big|\int_{\mathring{x}_n}^{x_n}\frac{|x_n-t|^{-\alpha_n-1}}{(-\alpha_n-1)!}(\operatorname{Im}(t-x_n))^{-|\alpha''|}dt\Big|\\ &\quad\times\exp\Big\{b(x_1+R)^{s/(s-1)}\max_{J_1}(|\operatorname{Im} y_j(x')|^{-1/(s-1)})\Big\}.\end{aligned}$$

Since we can take the path of integration in such a way that $|x_n-t|\leqq 2\operatorname{Im}(t-x_n)$ on that path, and since we have $|\mathring{x}_n-x_n|\leqq 2a_0^{-1}$, we obtain

$$|(A_j)_\Sigma(x, D')f(x)| \leqq \sum_{\alpha''\in Z_+^{n-2}} a_0^{l'-j} 2^j C^{j+1}\left(\frac{2nC}{\sqrt{a}}\right)^{|\alpha''|} \|f(x)\|_{\mathcal{O}^{0,s}(\tilde{\Omega})} l'!$$
$$\times \exp\left\{b(x_1+R)^{s/(s-1)} \max_{J_1}(|\mathrm{Im}\, y_j(x')|^{-1/(s-1)})\right\}$$
$$\leqq (2C)^{l'+1}\left(\frac{2C}{a_0}\right)^{j-l'} \|f(x)\|_{\mathcal{O}^{0,s}(\tilde{\Omega})} l'!$$
$$\times \exp\left\{b(x_1+R)^{s/(s-1)} \max_{J_1}(|\mathrm{Im}\, y_j(x')|^{-1/(s-1)})\right\},$$

if $x\in\Omega_J$ and a_0 is large enough. Thus $(A_j)_\Sigma(x, D')f(x)\in\mathcal{O}^{0,s}(\Omega)$ and we have

$$\|(A_j)_\Sigma(x, D')f(x)\|_{\mathcal{O}^{0,s}(\Omega)} \leqq (2C)^{l'+1}\left(\frac{2C}{a_0}\right)^{j-l'} l'!\|f(x)\|_{\mathcal{O}^{0,s}(\tilde{\Omega})} \tag{3.13}$$

if $j\geqq l'+1$. Since $a_0\gg a\gg C$, $\sum_{j=l'+1}^{\infty}(A_j)_\Sigma(x, D')f(x)$ converges in $\mathcal{O}^{0,s}(\Omega)$. Thus we have proved that if $A(x, \xi')$ is a symbol of a pseudodifferential operator of order $\leqq -1$, $A_\Sigma(x, D')f(x)$ is well defined on the domain Ω. If $l'\geqq 0$ and $A(x, \xi')$ is a symbol satisfying (3.2) it is easy to see that

$$A_\Sigma(x, D')f(x) = \sum_{\substack{\beta'\in Z_+^{n-1}\\ |\beta'|\leq l'+1}} D^{\beta'}((C_{\beta'})_\Sigma(x, D')f(x))$$

where $C_{\beta'}(x, \xi')$, $|\beta'|\leqq l'+1$, are finite number of symbols of some pseudodifferential operators whose orders are -1. Thus $A_\Sigma(x, D')f(x)$ converges on Ω, and we have the following

PROPOSITION 3.3. *If $f(x)\in\mathcal{O}^{0,s}(\tilde{\Omega})$, $A_\Sigma(x, D')f(x)$ is well defined on Ω, and is a continuous function holomorphic in x' (we denote by $\mathcal{O}(\Omega)$ the space of such functions).*

From (3.13), we have the following

PROPOSITION 3.4. *If $N>0$ is large, we have*

$$\sum_{j=N+1}^{\infty}(A_j)_\Sigma(x, D')f(x) \in \mathcal{O}^{0,s}(\Omega).$$

Furthermore, for any $\varepsilon>0$ there exist some N and a_0 (which are large) such that we have

$$\left\|\sum_{j=N+1}^{\infty}(A_j)_\Sigma(x, D')f(x)\right\|_{\mathcal{O}^{0,s}(\Omega)} \leqq \varepsilon\|f(x)\|_{\mathcal{O}^{0,s}(\tilde{\Omega})}.$$

REMARK. In the above arguments, we need to have shrunk the domain of definition of the functions. This is because we do not have chosen the domain $\tilde{\Omega}$ to be "flat" in the terminology of [2]. But this loss does not trouble us at all.

Proposition 3.4 insists that the summation of very low order terms defines a very small operator from $\mathcal{O}^{0,s}(\tilde{\Omega})$ to $\mathcal{O}^{0,s}(\Omega)$.

Let $A(x, \xi')$ satisfy (3.2), and let $l \geqq sl'+sn$. We have proved in § 2 that $A(x, \xi')$ defines a Bronstein operator

$$A_B(x, D') : \mathcal{O}^{0,s}(\tilde{\Omega}) \longrightarrow \mathcal{O}^{-l,s}(\tilde{\Omega}) \subset \mathcal{O}^{-l,s}(\Omega).$$
$$\begin{array}{c} \cap \\ \mathcal{O}^{0,s}(\Omega) \nearrow \end{array}$$

Thus both $A_B(x, D')f(x)$ and $A_\Sigma(x, D')f(x)$ are well defined on Ω, if $f(x) \in \mathcal{O}^{0,s}(\tilde{\Omega})$. We can prove that taking the spectrum with respect to x', these two functions coincides as a microfunction at $\mathring{x}^*$. In [2] and [4], it is proved that $[A_\Sigma(x, D')f(x)] = A(x, D')[f(x)]$ at $\mathring{x}^*$, where the latter $A(x, D')$ denotes the usual pseudodifferential operator acting to the microfunction $[f(x)]$. Thus we have

$$[A_B(x, D')f(x)] = A(x, D')[f(x)] \tag{3.14}$$

at $\mathring{x}^*$. Since we do not use such a general fact in this paper, we omit the proof of the above statement. Only a very special case of (3.14) when $A(x, \xi')=1$ is necessary for us. Let us consider this case more precisely. We define $\partial_k\Omega$, $2 \leqq k \leqq n$, by

$$\begin{aligned} \partial_k\Omega = \{x \in \boldsymbol{R} \times \boldsymbol{C}^{n-1};\ & a a_0^{s/(s-1)}|x_1| < 1,\ |\mathrm{Re}\, y_j(x')| < a_0(x_1+R),\ 2 \leqq j \leqq n, \\ & a_0(x_1+R) \max_{2 \leqq i \leqq n} |\mathrm{Im}\, y_i(x')| < \mathrm{Im}\, y_j(x') < R,\ 2 \leqq j \leqq n,\ j \neq k, \\ & |\mathrm{Im}\, y_k(x')| < R\}. \end{aligned}$$

Let us denote by $\mathcal{O}(\partial_k\Omega)$, $2 \leqq k \leqq n$, the spaces of all continuous functions on $\partial_k\Omega$, holomorphic in x', respectively. If one takes the spectrum of such a function, with respect to x', one obtains a microfunction which is zero in a neighborhood of $\mathring{x}^*$. Let us denote by $1_B(x, D)$ the Bronstein operator defined by the symbol which is identically equal to 1. Then we have the following

PROPOSITION 3.5. *If* $f(x) \in \mathcal{O}^{0,s}(\Omega)$, *we have*

$$1_B(x, D)f(x) \equiv f(x) \qquad \textit{modulo} \bigoplus_{k=2}^{n} \mathcal{O}(\partial_k\Omega).$$

We can directly prove this proposition. Since the proof is easy, we omit it.

§ 4. Composition of a pseudodifferential operator and a Bronstein operator.

Let $A(x, D')$ be a pseudodifferential operator of order l' and let $Q_B(x, D)$ be a Bronstein operator (in the strict sense). Assume that $Q(x, \xi)$ satisfies (2.6) with $l=0$ on U. Then we have

$$\mathcal{O}^{0,s}(\Omega) \xrightarrow{Q_B(x, D)} \mathcal{O}^{0,s}(\tilde{\Omega}) \xrightarrow{A_\Sigma(x, D')} \mathcal{O}(\Omega).$$

The purpose of this section is to give a calculation of the composite operator $A_\Sigma(x, D')Q_B(x, D)$. We first give a preparation.

Assume that $Q(x, \xi)$ is holomorphic on the following domain $\tilde{U}\supset U$:

$$\tilde{U} = \Big\{(x, \xi)\in \boldsymbol{C}^n\times \boldsymbol{C}^n;\ a_0^{(s-1)/s}|\mathrm{Re}\,y_j(x)|<1,\ 1\leqq j\leqq n,$$

$$a_0^{(s-1)/s}|\mathrm{Im}\,y_j(x)|<1,\ 1\leqq j\leqq n,$$

$$\mathrm{Im}\,\eta_j(\xi')>(a_0/4)|\mathrm{Re}\,\eta_j(\xi')|+(a_0/4),\ 2\leqq j\leqq n,$$

$$\mathrm{Re}\,\xi_1>(1/2)\phi(y(x), \eta'(\xi'))-a_0^{-3}\Big(|\mathrm{Im}\,\xi_1|-a_0\textstyle\sum_{k=2}^n|\mathrm{Im}\,\eta_k(\xi')|\Big)_+\Big\}.$$

(Roughly speaking, $(x, \xi)\in\tilde{U}$ when (x, ξ') belongs to a small conic neighborhood of $\mathring{x}^*$ and $\mathrm{Re}\,\xi_1$ is very large.) We have the following

LEMMA 4.1. *If $(x, \xi)\in U$ and $\tilde{x}'\in \boldsymbol{C}^{n-1}$ satisfies*

$$\begin{cases} |\tilde{x}_j| \leqq \dfrac{a}{n-1}\Big(\max\limits_{2\leqq i\leqq n}|\mathrm{Im}\,y_i(x')|\Big)+\sqrt{a}\Big(\sum\limits_{k=2}^n|\xi_k|\Big)^{(1-s)/s}, & 2\leqq j\leqq n-1, \\ |\tilde{x}_n| \leqq \dfrac{1}{n-1}\Big(\max\limits_{2\leqq i\leqq n}|\mathrm{Im}\,y_i(x')|\Big)+\dfrac{1}{\sqrt{a}}\Big(\sum\limits_{k=2}^n|\xi_k|\Big)^{(1-s)/s}, & \end{cases}$$

then we have $(x_1, x'+\tilde{x}', \xi)\in\tilde{U}$.

We leave the proof of this lemma to the reader. Cauchy integration theorem gives the following

COROLLARY 4.2. *If $Q(x, \xi)$ is holomorphic on $\tilde{U}$ and satisfies (2.6) with $l=0$ on $\tilde{U}$, we have*

$$(4.1)\qquad |\partial_{x'}^{\alpha'}Q(x, \xi)| \leqq C_0|\xi_1|^{-1}|\xi_n|^{-n}a^{-|\alpha''|}\alpha'!$$
$$\times\Big(\frac{1}{n-1}\Big(\max_{2\leqq i\leqq n}|\mathrm{Im}\,y_i(x')|\Big)+\frac{1}{\sqrt{a}}\Big(\sum_{k=2}^n|\xi_k|\Big)^{(1-s)/s}\Big)^{-|\alpha'|}$$

for any $\alpha'\in \boldsymbol{Z}_+^{n-1}$ on U.

A trivial modification of the proof of Proposition 2.2 gives the following

PROPOSITION 4.3. *If $Q(x, \xi)$ is holomorphic on $\tilde{U}$ and satisfies (2.6) with $l=0$ on $\tilde{U}$, we have*

$$(4.2)\qquad |\partial_{\tilde{x}'}^{\alpha'}\check{Q}(x, \tilde{x}_1, \xi')| \leqq 4na_0^{3/n}C_0|\tilde{x}_1|^{-1/n}|\xi_n|^{-n+1/n}a^{-|\alpha''|}\alpha'! \times\exp\{\tilde{x}_1\phi(y(x), \eta'(\xi'))\} \times\left(\frac{1}{n-1}\left(\max_{2\leq i\leq n}|\mathrm{Im}\, y_i(x')|\right)+\frac{1}{\sqrt{a}}\left(\sum_{k=2}^{n}|\xi_k|\right)^{(1-s)/s}\right)^{-|\alpha'|}$$

for any $\alpha'\in \boldsymbol{Z}_+^{n-1}$ on $\check{U}$.

Let $A(x, \xi')$ satisfy the asymptotic expansion (3.2). We assume that $A_{\alpha'}(x)=0$ if $|\alpha'|\leqq l'-N-1$, for the moment ($N\geqq l'+1$ is large enough). If $f(x)\in\mathcal{O}^{0,s}(\Omega)$, we have

$$(4.3)\qquad A_\Sigma(x, D')Q_B(x, D)f(x)$$
$$=\sum_{\substack{0\leq|\alpha'|\leq l'\\ \alpha_n\geq 0}}A_{\alpha'}(x)D^{\alpha'}\left(\int_{\Delta_J}\int_{\boldsymbol{R}}e^{x'\cdot\xi'}\check{Q}(x, \tilde{x}_1, \xi')\hat{f}(x_1-\tilde{x}_1, \xi')d\tilde{x}_1d\xi'\right)$$
$$+\sum_{\substack{l'-N\leq|\alpha'|\leq l'\\ \alpha_n<0}}A_{\alpha'}(x)\int_{\mathring{x}_n}^{x_n}\cdots\int_{\mathring{x}_n}^{x_n}D^{\alpha''}\left(\int_{\Delta_J}\int_{\boldsymbol{R}}e^{x'\cdot\xi'}\check{Q}(x, \tilde{x}_1, \xi')\hat{f}(x_1-\tilde{x}_1, \xi')d\tilde{x}_1d\xi'\right)\times dx_n\cdots dx_n\,.$$

It is easy to see that we can apply the chain rule to the first term, and thus this term equals to $Q'_B(x, D)f(x)$, where $Q'_B(x, D)$ is a Bronstein operator defined by the symbol

$$Q'(x, \xi)=\sum_{\gamma'\in\boldsymbol{Z}_+^{n-1}}\sum_{\substack{0\leq|\alpha'|\leq l'\\ \alpha_n\geq 0}}\frac{1}{\gamma'!}\partial_\xi^{\gamma'}(A_{\alpha'}(x)\xi^{\alpha'})\partial_x^{\gamma'}Q(x, \xi)\,.$$

We can thus easily calculate the symbol of the first term in (4.3), and we only need to consider the second term. For this reason, we assume that $\alpha_n<0$ in the asymptotic expansion (3.2), from the beginning. Then we have

$$(4.4)\qquad A_\Sigma Q_B f(x)$$
$$=\sum_{\substack{l'-N\leq|\alpha'|\leq l'\\ \alpha_n<0}}A_{\alpha'}(x)\int_{\mathring{x}_n}^{x_n}\int_{\Delta_J}\int_{\boldsymbol{R}}e^{x''\cdot\xi''+t\xi_n}\frac{(x_n-t)^{-\alpha_n-1}}{(-\alpha_n-1)!}\times(\xi+\partial_x)^{\alpha''}\check{Q}(x_1, \cdots, x_{n-1}, t, \tilde{x}_1, \xi')\hat{f}(x_1-\tilde{x}_1, \xi')d\tilde{x}_1d\xi'dt$$
$$=\sum_{\substack{l'-N\leq|\alpha'|\leq l'\\ \alpha_n<0\\ \beta''+\gamma''=\alpha''}}\int_{\Delta_J}\int_{\boldsymbol{R}}e^{x'\cdot\xi'}\Phi_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi')\hat{f}(x_1-\tilde{x}_1, \xi')d\tilde{x}_1d\xi'\,.$$

Here we have defined $\Phi_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi')$, $\alpha\in\boldsymbol{Z}_+^{n-2}\times\boldsymbol{Z}_-$, β'', $\gamma''\in\boldsymbol{Z}_+^{n-2}$, by

$$\Phi_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi')$$
$$= \frac{\alpha''!}{\beta''!\gamma''!} \cdot A_{\alpha'}(x)\xi^{\gamma''}\int_{\mathring{x}_n}^{x_n} \frac{(x_n-t)^{-\alpha_n-1}}{(-\alpha_n-1)!} e^{(t-x_n)\xi_n}\partial_x^{\beta''}\check{Q}(x_1, \cdots, x_{n-1}, t, \tilde{x}_1, \xi')dt.$$

Let us write $\check{Q}(t)=\check{Q}(x_1, \cdots, x_{n-1}, t, \tilde{x}_1, \xi')$, to avoid a heavy notation. Integrating by parts, we obtain

$$(4.5)\qquad \Phi_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi')$$
$$= \sum_{k=0}^{N+|\gamma''|-1-l'} \frac{\alpha''!}{\beta''!\gamma''!} \cdot A_{\alpha'}(x)\xi^{\gamma''}\Big[e^{(t-x_n)\xi_n}\xi_n^{-1-k}(-\partial_t)^k\Big(\frac{(x_n-t)^{-\alpha_n-1}}{(-\alpha_n-1)!}\partial_x^{\beta''}\check{Q}(t)\Big)\Big]_{t=\mathring{x}_n}^{t=x_n}$$
$$+\frac{\alpha''!}{\beta''!\gamma''!} \cdot A_{\alpha'}(x)\xi^{\gamma''}\xi_n^{-N-|\gamma''|+l'}$$
$$\times\int_{\mathring{x}_n}^{x_n} e^{(t-x_n)\xi_n}(-\partial_t)^{N+|\gamma''|-l'}\Big(\frac{(x_n-t)^{-\alpha_n-1}}{(-\alpha_n-1)!}\partial_x^{\beta''}\check{Q}(t)\Big)dt$$
$$= \Phi^1_{\alpha',\beta'',\gamma''}+\Phi^2_{\alpha',\beta'',\gamma''}+\Phi^3_{\alpha',\beta'',\gamma''},$$

where

$$(4.6)\qquad \Phi^1_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi') = \sum_{\substack{0\le k\le N+|\gamma''|-1-l' \\ k+\alpha_n+1\ge 0}} \frac{1}{\beta''!(k+\alpha_n+1)!}$$
$$\times\partial_\xi^{\beta''}\partial_{\xi_n}^{k+\alpha_n+1}(A_{\alpha'}(x)\xi^{\alpha'})\partial_x^{\beta''}\partial_{x_n}^{k+\alpha_n+1}\check{Q}(x_n),$$

$$(4.7)\qquad \Phi^2_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi') = -\sum_{k=0}^{N+|\gamma''|-1-l'} \frac{\alpha''!}{\beta''!\gamma''!} A_{\alpha'}(x)\xi^{\gamma''}e^{(\mathring{x}_n-x_n)\xi_n}\xi_n^{-1-k}$$
$$\times\Big[(-\partial_t)^k\Big(\frac{(x_n-t)^{-\alpha_n-1}}{(-\alpha_n-1)!}\partial_x^{\beta''}\check{Q}(t)\Big)\Big]_{t=\mathring{x}_n},$$

$$(4.8)\qquad \Phi^3_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi') = \frac{\alpha''!}{\beta''!\gamma''!} A_{\alpha'}(x)\xi^{\gamma''}\xi_n^{-N-|\gamma''|+l'}$$
$$\times\int_{\mathring{x}_n}^{x_n} e^{(t-x_n)\xi_n}(-\partial_t)^{N+|\gamma''|-l'}\Big(\frac{(x_n-t)^{-\alpha_n-1}}{(-\alpha_n-1)!}\partial_x^{\beta''}\check{Q}(t)\Big)dt.$$

Let us define $\Phi_N^j(x, \tilde{x}_1, \xi')$, $j=1, 2, 3$, by

$$(4.9)\qquad \Phi_N^j(x, \tilde{x}_1, \xi') = \sum_{\substack{-N+l'\le|\alpha'|\le l' \\ \alpha_n<0 \\ \beta''+\gamma''=\alpha''}} \Phi^j_{\alpha',\beta'',\gamma''}(x, \tilde{x}_1, \xi').$$

From (4.4)-(4.9) we have

$$A_\Sigma(x, D')Q_B(x, D)f(x) = f_1(x)+f_2(x)+f_3(x)$$

where

$$f_j(x) = \int_{\Delta_J}\int_{\boldsymbol{R}} e^{x'\cdot\xi'}\Phi_N^j(x, \tilde{x}_1, \xi')f(x_1-\tilde{x}_1, \xi')d\tilde{x}_1 d\xi',$$

$j=1, 2, 3$. From (4.6) and (4.9) we obtain

(4.10) $$\Phi^1_{\alpha',\beta'',\gamma'}(x, \tilde{x}_1, \xi') = \sum_{k=0}^{N-l'+|\alpha'|-|\beta''|} \frac{1}{\beta''!\,k!}\partial_\xi^{\beta''}\partial_{\xi_n}^k(A_{\alpha'}(x)\xi^{\alpha'})\partial_x^{\beta''}\partial_{x_n}^k\check{Q}(x_n).$$

(4.10) shows that $f_1(x)=Q_B^{(N)}(x, D)f(x)$ where $Q_B^{(N)}(x, D)$ is a Bronstein operator (in the strict sense) defined by the symbol

(4.11) $$Q^{(N)}(x, \xi) = \sum_{\substack{-N+l'\leq|\alpha'|\leq l' \\ -|\alpha'|+|\beta'|\leq N-l'}} \frac{1}{\beta'!}\partial_\xi^{\beta'}(A_{\alpha'}(x)\xi^{\alpha'})\partial_x^{\beta'}Q(x, \xi).$$

We next consider $f_2(x)$. Note that if $(x, \tilde{x}_1, \xi')\in\check{U}_0$, we have

(4.12) $$|e^{(\mathring{x}_n-x_n)\xi_n}| \leqq \exp\left\{-\frac{1}{4}|\mathring{x}_n-x_n|\cdot|\xi_n|\right\}.$$

A direct calculation using (3.1), (4.2), (4.7), (4.9), and (4.12) shows that

(4.13) $$|\Phi_N^2(x, \tilde{x}_1, \xi')| \leqq a_0^{3/n}C_0N!|\tilde{x}_1|^{-1/n}|\xi_n|^{l'-n+1/n}|\mathring{x}_n-x_n|^{-N}$$
$$\times\exp\left\{-\frac{1}{8}|\mathring{x}_n-x_n||\xi_n|+[\tilde{x}_1\phi(y(x), \eta'(\xi'))]_{x_n=\mathring{x}_n}\right\}.$$

If $x\in\partial_2\Omega$, $(x, \tilde{x}_1, \xi')\in\check{U}_0$, and $0<\tilde{x}_1<x_1$, we have

(4.14) $$|\mathring{x}_n-x_n| \geqq \frac{1}{2a_0},$$

and

(4.15) $$[\tilde{x}_1\phi(y(x), \eta'(\xi'))]_{x_n=\mathring{x}_n} \leqq \sum_{j=2}^n |\mathrm{Re}\,\eta_j(\xi')|+a_0^{-1/(s-1)}\sum_{j=2}^n |\mathrm{Im}\,\eta_j(\xi')|.$$

From (4.13)-(4.15) we have

(4.16) $$|\Phi_N^2(x, \tilde{x}_1, \xi')| \leqq a_0^{3/n}C_0(2a_0)^NN!|\tilde{x}_1|^{-1/n}|\xi_n|^{l'}$$
$$\times\exp\left\{\sum_{j=2}^n |\mathrm{Re}\,\eta_j(\xi')|-R\sum_{j=2}^n |\mathrm{Im}\,\eta_j(\xi')|\right\}.$$

If $\Delta_0=\{\xi'\in\sqrt{-1}\,\boldsymbol{R}^{n-1}; \mathrm{Im}\,\eta_j(\xi')>a_0,\ 2\leqq j\leqq n\}$, we have

$$f_2(x) = \int_{\Delta_0}\int_{\boldsymbol{R}} e^{x'\cdot\xi'}\Phi_N^2(x, \tilde{x}_1, \xi')\hat{f}(x_1-\tilde{x}_1, \xi')d\tilde{x}_1 d\xi'.$$

From (4.14) and Proposition 2.1 with $J_2=J_3=\emptyset$, we can easily prove that $f_2(x)\in\mathcal{O}(\partial_2\Omega)$, and thus this is a negligible function (in fact we can easily prove that $f_2(x)$ is real analytic).

Finally, let us consider $f_3(x)$. From (3.1), (4.2), (4.8) and (4.9), we can directly prove

$$|\Phi_N^3(x, \tilde{x}_1, \xi')| \leqq a_0^{3/n}C_0(8\sqrt{a})^{N+1}(N+1)!|\tilde{x}_1|^{-1/n}|\xi_n|^{-n+1/n-(N-l')/s+(l'+1)(s-1)/s}$$
$$\times\exp\{\tilde{x}_1\phi(y(x), \eta'(\xi'))\}$$

on $\check{U}_0$. Since $|\xi_n| \geqq a_0$ on $\check{U}_0$, we have

$$|\Phi_N^3(x, \tilde{x}_1, \xi')| \leqq a_0^{3/n} C_0 \Big(\frac{8\sqrt{a}}{a_0^{1/s}}\Big)^{N+1} a_0^{l'+1}(N+1)!$$
$$\times |\tilde{x}_1|^{-1/n} |\xi_n|^{-n+1/n} \exp\{\tilde{x}_1 \phi(y(x), \eta'(\xi'))\}.$$

We can choose N and a_0 large enough in such a manner that we have

$$|\Phi_N^3(x, \tilde{x}_1, \xi')| \leqq a_0^{-1} |\tilde{x}_1|^{-1/n} |\xi_n|^{-n+1/n} \exp\{\tilde{x}_1 \phi(y(x), \eta'(\xi'))\}. \tag{4.17}$$

From Proposition 2.3 it follows that $f_3(x) = R_B^{(N)}(x, D) f(x)$ where $R_B^{(N)}(x, D)$ is a Bronstein operator in the wider sense, and we have $\|R_B^{(N)}(x, D)\|_{\mathcal{O}^{0,s}(\Omega) \to \mathcal{O}^{0,s}(\Omega)} \leqq a_0^{-1}$. We have thus proved the following

Lemma 4.4. *If $A_{\alpha'}(x)=0$, $|\alpha'| \leqq -N+l'-1$, we can choose N and a_0 (which are very large) in such a manner that we have*

$$A_\Sigma(x, D') Q_B(x, D) f(x) \equiv Q_B^{(N)}(x, D) f(x) + R_B^{(N)}(x, D) f(x) \qquad \textit{modulo } \bigoplus_{k=2}^{n} \mathcal{O}(\partial_k \Omega).$$

Here $Q_B^{(N)}(x, D)$ is a Bronstein operator (in the strict sense) whose symbol is defined by (4.11) *and $R_B^{(N)}(x, D)$ is a Bronstein operator in the generalized sense which satisfies*

$$\|R_B^{(N)}(x, D)\|_{\mathcal{O}^{0,s}(\Omega) \to \mathcal{O}^{0,s}(\Omega)} \leqq a_0^{-1}. \tag{4.18}$$

Now we consider the general case without the condition $A_{\alpha'}(x)=0$, $|\alpha'| \leqq -N+l'-1$, which was assumed thus far. We define $A'_\Sigma(x, D')$ by

$$A'_\Sigma(x, D') = \sum_{j=0}^{N} (A_j)_\Sigma(x, D')$$

and $A''_\Sigma(x, D')$ by $A''_\Sigma(x, D') = A_\Sigma(x, D') - A'_\Sigma(x, D')$. $A'_\Sigma(x, D')$ satisfies the assumption of Lemma 4.4, and we can calculate $A'_\Sigma(x, D') Q_B(x, D)$ as above. On the other hand from Proposition 3.4 we have the following

Lemma 4.5. *If N and a_0 are large enough, we have*

$$\|A''_\Sigma(x, D') Q_B(x, D)\|_{\mathcal{O}^{0,s}(\Omega) \to \mathcal{O}^{0,s}(\Omega)} \leqq a_0^{-1}.$$

Combining Lemma 4.4 and Lemma 4.5, we obtain the following

Proposition 4.6. *Let $Q(x, \xi')$ satisfy* (2.6) *with $l=0$ on $\tilde{U}$. We can choose large N and a_0 in such a manner that if $f(x) \in \mathcal{O}^{0,s}(\Omega)$, we have*

$$A_\Sigma(x, D') Q_B(x, D) f(x) \equiv Q_B^{(N)}(x, D) f(x) + R^{(N)'}(x, D) f(x) \qquad \textit{modulo } \bigoplus_{k=2}^{n} \mathcal{O}(\partial_k \Omega). \tag{4.19}$$

Here the symbol of $Q_B^{(N)}(x, D)$ *is defined by* (4.10), *and we have*

$$\|R^{(N)\prime}(x, D)\|_{\mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)} \leqq 2a_0^{-1}.$$

We write $A_\Sigma(x, D')Q_B(x, D)\equiv Q_B^{(N)}(x, D)+R^{(N)\prime}(x, D)$ if (4.19) holds. Proposition 4.6 means that we can calculate the composite operator of $A_\Sigma(x, D')$ and $Q_B(x, D)$ "approximately" by the usual chain rule. We have another version of this proposition which is more convenient for us.

PROPOSITION 4.7. *Let* $Q(x, \xi')$ *satisfy* (2.6) *with* $l=0$ *on* $\tilde{U}$. *We can choose large* N *and* a_0 *in such a manner that if* $f(x)\in\mathcal{O}^{0,s}(\Omega)$, *we have*

$$A_\Sigma(x, D')Q_B(x, D)f(x)\equiv \bar{Q}_B^{(N)}(x, D)f(x)+\bar{R}^{(N)}(x, D)f(x) \qquad \textit{modulo} \bigoplus_{k=2}^{n}\mathcal{O}(\partial_k\Omega).$$

Here $\bar{Q}_B^{(N)}(x, D)$ *is a Bronstein operator* (*in the strict sense*) *defined by the symbol*

$$\bar{Q}^{(N)}(x, \xi)=\sum_{|\beta'|\leqq N}\frac{1}{\beta'!}\partial_\xi^{\beta'}A(x, \xi')\partial_x^{\beta'}Q(x, \xi), \tag{4.11$'$}$$

and $\bar{R}^{(N)}(x, D)\colon \mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)$ *satisfies*

$$\|\bar{R}^{(N)}(x, D)\|_{\mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)} \leqq 3a_0^{-1}.$$

PROOF. From (4.11) and (4.11)$'$ it follows that $Q_B^{(N)}(x, D)-\bar{Q}_B^{(N)}(x, D)$ is a Bronstein operator whose symbol is

$$\begin{aligned} &Q^{(N)}(x, \xi)-\bar{Q}^{(N)}(x, \xi)\\ &=\sum_{\substack{|\beta'|\leqq N\\ \beta\in Z_+^{n-1}}}\frac{1}{\beta'!}\partial_\xi^{\beta'}\Big(A(x, \xi')-\sum_{|\beta'|-N+l'\leqq|\alpha'|\leqq l'}A_{\alpha'}(x)\xi^{\alpha'}\Big)\partial_x^{\beta'}Q(x, \xi). \end{aligned}$$

Since $A(x, \xi')$ satisfies the asymptotic expansion (3.2), from (4.1) it follows that there exists some $C_1>0$ such that

$$|Q^{(N)}(x, \xi)-\bar{Q}^{(N)}(x, \xi)|\leqq C_1^{N+1}N!|\xi_1|^{-1}|\xi_n|^{l'-N/s}.$$

on U. Since $|\xi_n|\geqq a_0$ on U, we have

$$|Q^{(N)}(x, \xi)-\bar{Q}^{(N)}(x, \xi)|\leqq a_0^{-1-3/n}|\xi_1|^{-1}|\xi_n|^{-n}$$

on U, if a_0 is large enough. From Proposition 2.3, we have

$$\|Q_B^{(N)}(x, D)-\bar{Q}_B^{(N)}(x, D)\|_{\mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)}\leqq a_0^{-1}.$$

We only need to define $\bar{R}^{(N)}(x, D)$ by $\bar{R}^{(N)}(x, D)=R^{(N)}(x, D)+Q_B^{(N)}(x, D)-\bar{Q}_B^{(N)}(x, D)$. Q.E.D.

§ 5. Construction of the solutions.

We first state the following

THEOREM 5.1. *Let a_0 be large enough. For any $f(x)\in\mathcal{O}^{0,s}(\Omega)$ there exists some $u(x)\in\mathcal{O}^{0,s}(\tilde{\Omega})$ which satisfies*

$$\tilde{P}_\Sigma(x, D)u(x) \equiv f(x) \qquad \textit{modulo} \bigoplus_{k=2}^{n} \mathcal{O}(\partial_k\Omega).$$

Before proving this theorem, we explain the precise meaning of the formal calculation of § 1. Let $N>0$ be large enough. We consider a finite summation $E(x, \xi)=\sum_{j=0}^{N} E_j(x, \xi)$. This symbol $E(x, \xi)$ defines a Bronstein operator (in the strict sense) $E_B(x, D)$. It follows that if $f(x)\in\mathcal{O}^{0,s}(\Omega)$, $\tilde{P}_\Sigma(x, D)E_B(x, D)f(x)$ is well defined on Ω. Let us consider the composite operator $\tilde{P}_\Sigma(x, D)E_B(x, D)$. We remind the reader that

$$\tilde{P}_\Sigma(x, D) = D_n^{n+2}D_1^m + \sum_{j=0}^{m-1}(\tilde{P}^{(j)})_\Sigma(x, D')D_1^j$$

where $(\tilde{P}^{(j)})_\Sigma(x, D')=(P^{(j)})_\Sigma(x, D')D_n^{n+2}$, $0\leqq j\leqq m-1$. Since $D_n^{n+2}D_1^m$ is only a differential operator, the symbol of the composite operator $D_n^{n+2}D_1^m E_B(x, D)$ can be given by the usual chain rule. If $0\leqq j\leqq m-1$, the symbol of the operator $D_1^j E_B(x, D)$ is also given by the chain rule. This symbol satisfies (4.1) on U because of Proposition 1.9, and thus we can calculate the symbol of $(\tilde{P}^{(j)})_\Sigma(x, D')D_1^j E_B(x, D)$ using Proposition 4.7. We can choose large N and a_0 in such a manner that we have

$$\tilde{P}_\Sigma(x, D)E_B(x, D) \equiv T_B(x, D)+T'(x, D)$$

where $T_B(x, D)$ is a Bronstein operator defined by the symbol

$$T(x, \xi) = \sum_{\substack{0\leq\alpha_1\leq m\\ |\alpha'|\leq N}} \frac{1}{\alpha!}\partial_\xi^\alpha \tilde{P}(x, \xi)\partial_x^\alpha E(x, \xi) \tag{5.1}$$

and we have $\|T'(x, D)\|_{\mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)}\leqq 1/4$. From Definition 1.8 and (5.1) we have

$$T(x, \xi) = 1+ \sum_{\substack{0\leq\alpha_1\leq m\\ 0\leq|\alpha'|\leq N\\ N-|\alpha|+1\leq j\leq N}} \frac{1}{\alpha!}\partial_\xi^\alpha \tilde{P}(x, \xi)\partial_x^\alpha E_j(x, \xi).$$

From Proposition 1.9 we obtain

$$|T(x, \xi)-1| \leqq 2^{m+n+1}(2N+m)!C^{5N+4}|\xi_1|^{-1}|\xi_n|^{2+m+(s-2)(N+1)/s}.$$

Since $s<2$ and $|\xi_n|\geqq a_0^{-1}$, we can choose large N and a_0 such that

$$|T(x, \xi)-1| \leqq 2^{m+n+1}(2N+m)!C^{5N+4}a_0^{m+n+2+(s-2)(N+1)/s}|\xi_1|^{-1}|\xi_n|^{-n}$$

$$\leqq a_0^{-1-3/n}|\xi_1|^{-1}|\xi_n|^{-n}.$$

From Proposition 2.3 it follows that

$$\|1_B(x, D)-T_B(x, D)\|_{\mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)} \leqq 1/4.$$

Thus we obtain

$$\tilde{P}_\Sigma(x, D)E_B(x, D) \equiv 1_B(x, D)-F(x, D)$$

where $\|F(x, D)\|_{\mathcal{O}^{0,s}(\Omega)\to\mathcal{O}^{0,s}(\Omega)}\leqq 1/2$. We are ready to give the following

PROOF OF THEOREM 5.1. Let $f(x)\in\mathcal{O}^{0,s}(\Omega)$. We define $u(x)\in\mathcal{O}^{0,s}(\tilde{\Omega})$ by

$$u(x) = E_B(x, D)\sum_{j=0}^{\infty}(F(x, D))^j f(x).$$

Then we have

$$\tilde{P}_\Sigma(x, D)u(x) \equiv (1_B(x, D)-F(x, D))\sum_{j=0}^{\infty}(F(x, D))^j f(x)$$

$$\equiv f(x) \qquad \text{modulo } \bigoplus_{k=2}^{n}\mathcal{O}(\partial_k\Omega). \qquad \text{Q.E.D.}$$

Now the proof of Theorem 0.1 is immediate.

PROOF OF THEOREM 0.1. Let $f(x)$ be a section of $C^q[\boldsymbol{R};\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$, and let $u_i(x')$, $0\leqq i\leqq m-1$, be sections of $\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})$. We may consider $u(x)-\sum_{i=0}^{m} x_1^i u_i'(x')$ with some $u_i'(x')\in\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})$, $0\leqq i\leqq m$, instead of $u(x)$, and thus we may assume that $f(0, x')=0$, $u_i(x')=0$, $0\leqq i\leqq m-1$. We define $f_\pm(x)$ by

$$f_\pm(x) = \begin{cases} 0 & \pm x_1\leqq 0, \\ f(x) & \pm x_1\geqq 0. \end{cases}$$

Then $f_+(x)$ can be represented as the boundary value of some $g_+(x)\in\mathcal{O}^{0,s}(\Omega)$. By Theorem 5.1, there exists some $v_+(x)\in\mathcal{O}^{0,s}(\tilde{\Omega})$ which satisfies

$$\tilde{P}_\Sigma(x, D)v_+(x) \equiv g_+(x) \qquad \text{modulo } \bigoplus_{k=2}^{n}\mathcal{O}(\partial_k\Omega).$$

Since $\tilde{P}_\Sigma(x, D)=P_\Sigma(x, D)D_n^{n+2}$, we have

$$P_\Sigma(x, D)u_+(x) = g_+(x) \qquad \text{modulo } \bigoplus_{k=2}^{n}\mathcal{O}(\partial_k\Omega)$$

with $u_+(x)=D_n^{n+2}v_+(x)$. Taking the spectrum, we obtain

$$P(x, D)[u_+(x)] = [g_+(x)] = [f_+(x)]$$

on a neighborhood of $\mathring{x}^*$. Reversing the time variable x_1, we can also construct a microfunction $[u_-(x)]$ which satisfies

$$P(x, D)[u_-(x)] = [f_-(x)]$$

on a neighborhood of $\mathring{x}^*$. $u(x)=u_+(x)+u_-(x)$ defines a section of $C^0[\boldsymbol{R};\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$ which satisfies

$$P(x, D)[u(x)] = [f(x)]$$

on a neighborhood of $\mathring{x}^*$. Since $P(x, D)$ is noncharacteristic with respect to x_1, the boundary value of $u(x)$ is in fact a section of $C^{q+m}[\boldsymbol{R};\mathcal{D}^{(s)\prime}(\boldsymbol{R}^{n-1})]$, and we have

$$[D_1^i u(0, x')] = 0, \qquad 0 \leqq i \leqq m-1. \qquad \text{Q.E.D.}$$

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