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On the number of exceptional values of the Gauss maps of minimal surfaces

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§ 1. Introduction.

In 1961, R. Osserman showed that the Gauss map of a complete non-flat minimal (immersed) surface in $\boldsymbol{R}^3$ cannot omit a set of positive logarithmic capacity ([8]). Moreover, he proved the following:

THEOREM 1.1 ([9]). *Let M be a minimal surface in $\boldsymbol{R}^m$ $(m \geqq 3)$, and p be a point of M. If all normals at points of M make angles of at least α with some fixed direction, then*

$$|K(p)| \leqq \frac{1}{d(p)^2} \cdot \frac{16(m-1)}{\sin^4 \alpha},$$

where $K(p)$ and $d(p)$ denote the Gauss curvature of M at p and the distance from p to the boundary of M respectively.

Afterwards, F. Xavier gave the following improvement of the former result of R. Osserman.

THEOREM 1.2 ([11]). *The Gauss map of a complete non-flat minimal surface in $\boldsymbol{R}^3$ can omit at most six points of the sphere.*

Recently, the author gave a generalization of this to the case of complete minimal surfaces in $\boldsymbol{R}^m$ $(m \geqq 4)$ ([4], [5]). He studied also the value distribution of the Gauss map of a complete submanifold M of $\boldsymbol{C}^m$ in the case where the universal covering of M is biholomorphic to the unit ball in $\boldsymbol{C}^n$ ([6]).

In this paper, relating to these results we shall give the following theorem.

THEOREM I. *Let M be a minimal surface in $\boldsymbol{R}^3$. Suppose that the Gauss map $G: M \to S^2$ omits at least five points $\alpha_1, \cdots, \alpha_5$. Then, there exists a positive constant C depending only on $\alpha_1, \cdots, \alpha_5$ such that*

$$|K(p)| \leqq \frac{C}{d(p)^2}$$

for an arbitrary point p of M.

Since $d(p)=\infty$ for any $p \in M$ in the case where M is complete, we have the following improvement of Theorem 1.2 as an immediate consequence of Theorem I.

COROLLARY 1.3. *The Gauss map of a complete non-flat minimal surface in* $\boldsymbol{R}^3$ *can omit at most four points of the sphere.*

We know some examples of complete non-flat minimal surfaces in $\boldsymbol{R}^3$ whose Gauss maps omit four points ([**8**], [**10**]). So, the number four of exceptional values of the Gauss map of Corollary 1.3 is best-possible.

We now consider a complete minimal surface M in $\boldsymbol{R}^4$. The Gauss map may be identified with a pair of meromorphic functions $g=(g_1, g_2)$ (cf. § 5). Relating to the results in [**2**] and [**5**], we shall prove the following:

THEOREM II. *Let* M *be a complete non-flat minimal surface in* $\boldsymbol{R}^4$ *and let* $g=(g_1, g_2)$ *be the Gauss map of* M.

(i) *In the case* $g_1 \not\equiv const.$ *and* $g_2 \not\equiv const.$, *if* g_1 *and* g_2 *omit* q_1 *points and* q_2 *points respectively, then* $q_1 \leqq 2$, *or* $q_2 \leqq 2$, *or*

$$\frac{1}{q_1-2}+\frac{1}{q_2-2} \geqq 1,$$

(ii) *In the case where one of* g_1 *and* g_2 *is constant, say* $g_2 \equiv const.$, *then* g_1 *can omit at most three points.*

After some preparations, we shall furnish a function-theoretic lemma in § 3 and give the proof of Theorem I in § 4. Theorem II will be proved in § 5.

It is a pleasure to thank the referee for his questions and comments, which led to improvements in the exposition.

§ 2. Preliminaries on Poincaré metrics.

In this section, we shall give some elementary properties of the Poincaré metric of a domain in the complex plane $\boldsymbol{C}$.

For a domain D of hyperbolic type in $\boldsymbol{C}$ we denote the Poincaré metric of D by $ds^2=\lambda_D(z)^2|dz|^2$. By definition, $\lambda_D(z)$ is a positive C^2-function satisfying the condition $\Delta \log \lambda_D=\lambda_D^2$. In particular, for a disc $\Delta(R):=\{z; |z|<R\}$ we have

$$\lambda_{\Delta(R)}(z)=\frac{2R}{R^2-|z|^2}.$$

We need later the following generalized Schwarz's lemma.

THEOREM 2.1. *Let* D *be a domain in* $\boldsymbol{C}$ *and* λ *be a positive* C^2*-function on* D *satisfying the condition* $\Delta \log \lambda \geqq \lambda^2$. *Then, for every holomorphic map* $f: \Delta(R) \to D$,

$$|f'(z)|\lambda(f(z)) \leqq \frac{2R}{R^2-|z|^2}.$$

For the proof, see, e.g., [1], p. 13.

Take q distinct points $\alpha_1, \cdots, \alpha_q$ in $\boldsymbol{C}$, where $q \geqq 2$. For brevity, we set

$$\lambda_{\alpha_1, \cdots, \alpha_q}(z) := \lambda_{C \setminus \{\alpha_1, \cdots, \alpha_q\}}(z).$$

PROPOSITION 2.2. *Take an arbitrary constant K_0 with $K_0 > \max(1, |\alpha_1|, \cdots, |\alpha_q|)$. Then, there exist positive constants A_i $(0 \leqq i \leqq q)$ depending only on $K_0, \alpha_1, \cdots, \alpha_q$ such that*

(i) $$\lambda_{\alpha_1, \cdots, \alpha_q}(z) \geqq \frac{A_0}{|z| \log |z|} \qquad \text{for } |z| \geqq K_0,$$

(ii) $$\lambda_{\alpha_1, \cdots, \alpha_q}(z) \geqq \frac{A_i}{|z-\alpha_i|\left(1+\log^+ \frac{1}{|z-\alpha_i|}\right)} \qquad (1 \leqq i \leqq q)$$

for $|z| \leqq K_0$ and $z \neq \alpha_1, \cdots, \alpha_q$, where $\log^+ x = \max(\log x, 0)$.

For the proof, we use the following fact shown by L. V. Ahlfors ([1], p. 17).

(2.3) *Set* $D := \{z; |z| \leqq 1, |z| \leqq |z-1|\}$ *and*

$$\zeta(z) := \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1} \qquad (z \in D),$$

where $\sqrt{1-z}$ means the branch with $\mathrm{Re}\sqrt{1-z} > 0$ *for $z \in D$. Then,*

$$\lambda_{0,1}(z) \geqq \left|\frac{\zeta'(z)}{\zeta(z)}\right| \frac{1}{4-\log|\zeta(z)|} \qquad (z \in D).$$

PROOF OF PROPOSITION 2.2. We shall show first

$$\liminf_{z \to 0} \lambda_{0,1}(z)|z| \log \frac{1}{|z|} \geqq 1. \tag{1}$$

Since $|\zeta'(z)/\zeta(z)| = |z|^{-1}|z-1|^{-1/2}$, we have by (2.3)

$$\lambda_{0,1}(z)|z| \log |1/z| \geqq \frac{\log|1/z|}{|z-1|^{1/2}(4+\log(|\sqrt{1-z}+1|^2/|z|))}$$

$$= \frac{\log|1/z|}{|z-1|^{1/2}(\log|1/z|+4+2\log|\sqrt{1-z}+1|)},$$

which tends to 1 as z tends to 0. So, we get (1).

Since Poincaré metrics are invariant under biholomorphic transformations and $u = 1/z$ maps $\boldsymbol{C} \setminus \{0, 1\}$ biholomorphically onto itself,

$$\lambda_{0,1}(z)|dz| = \frac{1}{|z|^2}\lambda_{0,1}\Big(\frac{1}{z}\Big)|dz|.$$

Therefore, we obtain from (1)

$$\liminf_{z\to\infty} \lambda_{0,1}(z)|z|\log|z| = \liminf_{u\to 0} \lambda_{0,1}(u)|u|\log\frac{1}{|u|} \geqq 1. \tag{2}$$

For each index i $(1\leqq i\leqq q)$ we take another index j. Applying the distance decreasing property of Poincaré metrics to the inclusion map of $\boldsymbol{C}\setminus\{\alpha_1, \cdots, \alpha_q\}$ into $\boldsymbol{C}\setminus\{\alpha_i, \alpha_j\}$, we see

$$\lambda_{\alpha_1,\cdots,\alpha_q}(z) \geqq \lambda_{\alpha_i\alpha_j}(z) \qquad (z\in\boldsymbol{C}\setminus\{\alpha_1, \cdots, \alpha_q\}). \tag{3}$$

Moreover, we have

$$\lambda_{\alpha_i\alpha_j}(z) = \frac{1}{|\alpha_j-\alpha_i|}\lambda_{0,1}\Big(\frac{z-\alpha_i}{\alpha_j-\alpha_i}\Big), \tag{4}$$

because $w=(z-\alpha_i)/(\alpha_j-\alpha_i)$ maps $\boldsymbol{C}\setminus\{\alpha_i, \alpha_j\}$ biholomorphically onto $\boldsymbol{C}\setminus\{0, 1\}$. Therefore, we conclude from (3), (4) and (1)

$$\liminf_{z\to\alpha_i} \lambda_{\alpha_1,\cdots,\alpha_q}(z)|z-\alpha_i|\Big(1+\log^+\frac{1}{|z-\alpha_i|}\Big)$$

$$\geqq \liminf_{u\to 0} \lambda_{0,1}(u)|u|\log\frac{1}{|u|}\Big(1-\frac{\log^+|\alpha_i-\alpha_j|}{\log|1/u|}\Big) \geqq 1.$$

We now consider the function

$$h_i(z) := \lambda_{\alpha_1,\cdots,\alpha_q}(z)|z-\alpha_i|\Big(1+\log^+\frac{1}{|z-\alpha_i|}\Big)$$

on the set $\Delta' := \{z\,;\,|z|\leqq K_0\}\setminus\{\alpha_1, \cdots, \alpha_q\}$ for each i $(1\leqq i\leqq q)$. We can easily conclude $A_i := \inf_{z\in\Delta'} h_i(z)>0$ because h_i is continuous and $\liminf_{z\to\alpha_j} h_i(z)>0$ for each $j=1, 2, \cdots, q$. The constants A_i satisfy the inequality (ii) of Proposition 2.2.

Next, we consider the function

$$h_0(z) := \lambda_{\alpha_1,\cdots,\alpha_q}(z)|z|\log|z|$$

on the set $\Delta'' := \{z\,;\,|z|\geqq K_0\}$. By (2), (3) and (4),

$$\begin{aligned}
&\liminf_{z\to\infty} \lambda_{\alpha_1,\cdots,\alpha_q}(z)|z|\log|z| \\
&\geqq \liminf_{z\to\infty} \lambda_{\alpha_1\alpha_2}(z)|z|\log|z| \\
&= \liminf_{z\to\infty} \frac{1}{|\alpha_2-\alpha_1|}\lambda_{0,1}\Big(\frac{z-\alpha_1}{\alpha_2-\alpha_1}\Big)|z|\log|z| \\
&= \liminf_{u\to\infty} \lambda_{0,1}(u)|u|\log|u| \geqq 1.
\end{aligned}$$

Therefore, $A_0 := \inf_{z \in \Delta''} h_0(z) > 0$ and A_0 satisfies the desired inequality (i) of Proposition 2.2. This completes the proof of Proposition 2.2.

§ 3. A function-theoretic lemma.

The purpose of this section is to prove the following function-theoretic lemma.

LEMMA 3.1. *Let g be a meromorphic function on $\Delta(R)$ which omits q distinct values $\alpha_1, \cdots, \alpha_{q-1}$ and $\alpha_q = \infty$, where $q \geqq 3$. For $0 < (q-1)\varepsilon' < \varepsilon$, there exists a constant B depending only on $\varepsilon, \varepsilon', \alpha_1, \cdots, \alpha_q$ such that*

$$\frac{(1+|g(z)|^2)^{(q-2-\varepsilon)/2}|g'(z)|}{(\prod_{i=1}^{q-1}|g(z)-\alpha_i|)^{1-\varepsilon'}} \leqq B\left(\frac{2R}{R^2-|z|^2}\right).$$

For the proof, we set

$$B(w) = \frac{(1+|w|^2)^{(q-2)/2}}{\sum_{i=1}^{q-1}|(w-\alpha_1)\cdots(w-\alpha_{i-1})(w-\alpha_{i+1})\cdots(w-\alpha_{q-1})|}.$$

Then, $B(w)$ is bounded by a constant B_1 because it is continuous on $\boldsymbol{C} \setminus \{\alpha_1, \cdots, \alpha_{q-1}\}$ and the limits $\lim_{|w| \to \infty} B(w)$ and $\lim_{w \to \alpha_i} B(w)$ $(1 \leqq i \leqq q-1)$ exist. Therefore, we have the following

(3.2) *In the situation of Lemma 3.1, there exists a constant B_1 depending only on $\alpha_1, \cdots, \alpha_q$ such that*

$$\frac{(1+|g|^2)^{(q-2)/2}|g'|}{\prod_{i=1}^{q-1}|g-\alpha_i|} \leqq B_1\left(\sum_{i=1}^{q-1} \frac{|g'|}{|g-\alpha_i|}\right).$$

We shall prove next the following

(3.3) *Let $g, \alpha_1, \cdots, \alpha_q$ be as in Lemma 3.1 and $\eta > 0$. Then, there exist some constants $C_i > 0$ $(1 \leqq i \leqq q-1)$ depending only on $\alpha_1, \cdots, \alpha_q, \eta$ such that*

$$\frac{|g'|}{(1+|g|^2)^{\eta/2}|g-\alpha_i|\left(1+\log^+\frac{1}{|g-\alpha_i|}\right)} \leqq C_i\left(\frac{2R}{R^2-|z|^2}\right). \tag{5}$$

To this end, we take a constant $K_0 > \max(1, |\alpha_1|, \cdots, |\alpha_{q-1}|)$ and set

$$\Delta_1 := \{z \in \Delta(R)\,;\, |g(z)| < K_0\}$$
$$\Delta_2 := \{z \in \Delta(R)\,;\, |g(z)| \geqq K_0\}.$$

Then, by Proposition 2.2,

$$\lambda_{\alpha_1, \cdots, \alpha_{q-1}}(g(z)) \geqq \frac{A_i}{|g(z)-\alpha_i|\left(1+\log^+\left|\frac{1}{g(z)-\alpha_i}\right|\right)}$$

for $z\in\Delta_1$ and

$$\lambda_{\alpha_1,\cdots,\alpha_{q-1}}(g(z)) \geqq \frac{A_0}{|g(z)|\log|g(z)|}$$

for $z\in\Delta_2$. On the other hand, since $\Delta\log\lambda_{\alpha_1,\cdots,\alpha_{q-1}}=\lambda^2_{\alpha_1,\cdots,\alpha_{q-1}}$, Theorem 2.1 implies that

$$|g'(z)|\lambda_{\alpha_1,\cdots,\alpha_{q-1}}(g(z)) \leqq \frac{2R}{R^2-|z|^2}.$$

Therefore, we have

$$\frac{|g'|}{(1+|g|^2)^{\eta/2}|g-\alpha_i|\left(1+\log^+\frac{1}{|g-\alpha_i|}\right)}$$

$$\leqq \frac{|g'|}{|g-\alpha_i|\left(1+\log^+\frac{1}{|g-\alpha_i|}\right)}$$

$$\leqq \frac{1}{A_i}\frac{2R}{R^2-|z|^2}$$

for $z\in\Delta_1$ and

$$\frac{|g'|}{(1+|g|^2)^{\eta/2}|g-\alpha_i|\left(1+\log^+\frac{1}{|g-\alpha_i|}\right)}$$

$$\leqq \frac{\log|g|}{(1+|g|^2)^{\eta/2}(1-|\alpha_i|/K_0)}\frac{|g'|}{|g|\log|g|}$$

$$\leqq \frac{B_3}{A_0}\left(\frac{2R}{R^2-|z|^2}\right)$$

for $z\in\Delta_2$, where $B_3:=\sup_{|w|\geq K_0}(1-|\alpha_i|/K_0)^{-1}(1+|w|^2)^{-\eta/2}\log|w|<+\infty$. The constant $C_i:=\max(1/A_i, B_3/A_0)$ satisfies the inequality (5).

PROOF OF LEMMA 3.1. Since

$$\frac{(1+|g|^2)^{(q-2-\varepsilon)/2}|g'|}{(\prod_{i=1}^{q-1}|g-\alpha_i|)^{1-\varepsilon'}} = \frac{(1+|g|^2)^{(q-2)/2}|g'|}{\prod_{i=1}^{q-1}|g-\alpha_i|}\frac{(\prod_{i=1}^{q-1}|g-\alpha_i|)^{\varepsilon'}}{(1+|g|^2)^{\varepsilon/2}},$$

we have only to show by virtue of (3.2) that there exists a constant D_i such that

$$k_i(z):=\frac{(\prod_{i=1}^{q-1}|g(z)-\alpha_i|)^{\varepsilon'}}{(1+|g(z)|^2)^{\varepsilon/2}}\frac{|g'(z)|}{|g(z)-\alpha_i|} \leqq D_i\left(\frac{2R}{R^2-|z|^2}\right) \tag{6}$$

for each i $(1\leqq i\leqq q-1)$.

Take ε'' with $0<\varepsilon'<\varepsilon''$ and $\varepsilon-(q-1)\varepsilon''>0$, and set

$$H(w):=\frac{|(w-\alpha_1)\cdots(w-\alpha_{q-1})|^{\varepsilon'}\left(1+\log^+\frac{1}{|w-\alpha_i|}\right)}{(1+|w|^2)^{(q-1)\varepsilon''/2}}.$$

The function $H(w)$ on $C\setminus\{\alpha_1, \cdots, \alpha_{q-1}\}$ is obviously continuous and $\lim_{w\to\alpha_i}H(w)=0$ $(1\leqq i\leqq q)$. Therefore, $H(w)$ is bounded by a constant depending only on $\alpha_1, \cdots, \alpha_q, \varepsilon', \varepsilon''$. On the other hand, for $\eta:=\varepsilon-(q-1)\varepsilon''>0$,

$$k_i(z)=\frac{|g'(z)|H(g(z))}{(1+|g(z)|^2)^{\eta/2}|g(z)-\alpha_i|\left(1+\log^+\frac{1}{|g(z)-\alpha_i|}\right)}.$$

By the use of (3.3) we have the desired inequality (6).

§4. Minimal surfaces in $\boldsymbol{R}^3$.

Let $x=(x_1, x_2, x_3): M\to\boldsymbol{R}^3$ be a (connected oriented) minimal surface in $\boldsymbol{R}^3$. With each positive isothermal local coordinates (u, v) associating a holomorphic local coordinate $z=u+\sqrt{-1}\,v$, we may regard M as a Riemann surface. Let $G: M\to S^2$ be the Gauss map of M. By definition, G maps each point p of M to the unit vector $G(p)\in S^2$ which is normal to M at p. Instead of G, we study the map $g: M\to\bar{\boldsymbol{C}}:=\boldsymbol{C}\cup\{\infty\}$ which is the conjugate of the composite of G and the stereographic projection from S^2 onto $\bar{\boldsymbol{C}}$. By the assumption of minimality of M, g is a meromorphic function on M.

For the proof of Theorem I, we may replace M by the universal covering of M. On the other hand, there is no compact minimal surface in $\boldsymbol{R}^3$, and any meromorphic function on $\boldsymbol{C}$ which omits three distinct values is a constant because of Picard's theorem. Therefore, by Koebe's uniformization theorem we assume that M is the unit disc Δ.

Set $\phi_i:=\partial x_i/\partial z=(\partial x_i/\partial u-\sqrt{-1}\,\partial x_i/\partial v)/2$ for each $i=1, 2, 3$. By elementary calculation, we see

$$g=\frac{\phi_3}{\phi_1-\sqrt{-1}\,\phi_2}$$

(see [**10**]). On the other hand, the metric on M induced from $\boldsymbol{R}^3$ is given by $ds^2=\lambda^2|dz|^2=2(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2)|dz|^2$. If we set $f:=\phi_1-\sqrt{-1}\,\phi_2$, it is easily shown that

$$\lambda^2=|f|^2(1+|g|^2)^2,$$

where f has no zero in case that g has no pole. The curvature K of M is given by

$$K=-\frac{\Delta\log\lambda}{\lambda^2}=-\frac{4|g'|^2}{|f|^2(1+|g|^2)^4}. \tag{7}$$

Now, suppose that $\bar{C}\setminus g(M)$ contains five distinct points $\alpha_1, \cdots, \alpha_5$ as in Theorem I. By a suitable coordinate change we may assume that $\alpha_5=\infty$. Let z_0 be an arbitrary point of M. Our purpose is to prove that

$$|K(z_0)| \leqq \frac{C}{d(z_0)^2}$$

for a suitable positive constant C depending only on $\alpha_1, \cdots, \alpha_5$, where $d(z_0)$ is the largest lower bound of the lengths of all piecewise smooth curves going from z_0 to the boundary of M. Without loss of generality, we assume that $z_0=o$ and $K(o)\neq 0$. Take real numbers ε, ε' with $0<4\varepsilon'<\varepsilon<1$. Set $p:=2/(3-\varepsilon)$. We consider a many-valued analytic function

$$\psi := \frac{f^{1/(1-p)}(\prod_{i=1}^{4}(g-\alpha_i))^{p(1-\varepsilon')/(1-p)}}{(g')^{p/(1-p)}} \tag{8}$$

on an open set $M':=\{z\in M\,;\, g'(z)\neq 0\}$. Take an arbitrary single-valued branch ψ_0 of ψ in a neighborhood of the origin. Then ψ_0 has an analytic continuation ψ_γ along any continuous curve $\gamma:[0, 1]\to M'$ with $\gamma(0)=o$. Let $\pi:\tilde{M}'\to M'$ be the universal covering of M'. Each point $\tilde{z}$ of $\tilde{M}'$ corresponds to the homotopy class of a continuous curve $\gamma:[0, 1]\to M'$ with $\gamma(0)=o$ and $\gamma(1)=\pi(\tilde{z})$. Define

$$w = F(\tilde{z}) := \int_\gamma \psi_\gamma(z)dz\,. \tag{9}$$

Obviously, F is a single-valued holomorphic function on $\tilde{M}'$ and satisfies the condition that $F(\tilde{o})=0$ and $dF(\tilde{z})\neq 0$ for any $\tilde{z}\in\tilde{M}'$, where $\tilde{o}$ denotes the point of $\tilde{M}'$ corresponding to the constant curve o. Then, we can find a positive constant R such that F maps a connected open neighborhood U of $\tilde{o}$ bijectively onto $\Delta(R):=\{w\in \boldsymbol{C}\,;\,|w|<R\}$. Choose the largest R with this property and consider a map $\Phi:=\pi\cdot(F|U)^{-1}:\Delta(R)\to M$. Here, we shall give the following estimate of R.

(4.1) *There exists a positive constant E_1 depending only on $\alpha_1, \cdots, \alpha_5$ and ε, ε' such that*

$$R^{1-p} \leqq E_1|K(o)|^{-1/2}\,.$$

To see this, we set $h(w)=g(\Phi(w))$. Since

$$\left|\frac{dw}{dz}\right| = \frac{|f|(\prod_{i=1}^{4}|g-\alpha_i|)^{p(1-\varepsilon')}}{|g'|^p}\left|\frac{dw}{dz}\right|^p$$

by (8) and (9), we have

$$\Phi^*ds^2 = \lambda(\Phi(w))^2\left|\frac{dz}{dw}\right|^2|dw|^2$$

$$= |f\cdot\Phi|^2(1+|g\cdot\Phi|^2)^2\cdot\frac{|g'(\Phi(w))|^{2p}|dz/dw|^{2p}}{|f\cdot\Phi|^2(\prod_{i=1}^{4}|g\cdot\Phi-\alpha_i|)^{2p(1-\varepsilon')}}|dw|^2$$

$$= \frac{(1+|h|^2)^2|h'|^{2p}}{(\prod_{i=1}^4|h-\alpha_i|)^{2p(1-\varepsilon')}}|dw|^2.$$

On the other hand, since $d\Phi(\mathrm{o})\neq 0$ for the map $z=\Phi(w)$, we can take w as a holomorphic local coordinate around the origin. The curvature $K(\mathrm{o})$ of M at the origin is given by

$$K(\mathrm{o}) = -\frac{4|h'(\mathrm{o})|^2}{(1+|h(\mathrm{o})|^2)^2}\frac{(\prod_{i=1}^4|h(\mathrm{o})-\alpha_i|)^{2p(1-\varepsilon')}}{(1+|h(\mathrm{o})|^2)^2|h'(\mathrm{o})|^{2p}}$$

$$= -\frac{4|h'(\mathrm{o})|^{2(1-p)}(\prod_{i=1}^4|h(\mathrm{o})-\alpha_i|)^{2p(1-\varepsilon')}}{(1+|h(\mathrm{o})|^2)^4}.$$

Now, apply Lemma 3.1 to the function h. Then, we see

$$\frac{(1+|h(\mathrm{o})|^2)^{(3-\varepsilon)/2}|h'(\mathrm{o})|}{(\prod_{i=1}^4|h(\mathrm{o})-\alpha_i|)^{1-\varepsilon'}} \leqq \frac{2B}{R}.$$

Consequently,

$$R^{1-p} \leqq \frac{(2B)^{1-p}(\prod_{i=1}^4|h(\mathrm{o})-\alpha_i|)^{(1-\varepsilon')(1-p)}}{(1+|h(\mathrm{o})|^2)^{(1-p)/p}|h'(\mathrm{o})|^{1-p}}$$

$$\leqq 2|K(\mathrm{o})|^{-1/2}\frac{(2B)^{1-p}(\prod_{i=1}^4|h(\mathrm{o})-\alpha_i|)^{1-\varepsilon'}}{(1+|h(\mathrm{o})|^2)^{(p+1)/p}}.$$

For sufficiently small ε, ε',

$$E_1 := 2\sup_{w\in C}\frac{(2B)^{1-p}(\prod_{i=1}^4|w-\alpha_i|)^{1-\varepsilon'}}{(1+|w|^2)^{(p+1)/p}} < \infty.$$

The constant E_1 satisfies the inequality (9). Thus, we conclude (4.1).

Now, for each point a with $|a|=R$ we consider a line segment

$$L_a:\ w = ta, \qquad 0\leqq t<1$$

in $\Delta(R)$ and a curve

$$\Gamma_a:\ z = \Phi(ta), \qquad 0\leqq t<1$$

in M'. We shall prove that there exists a point a_0 with $|a_0|=R$ such that Γ_{a_0} tends to the boundary of M, namely, for each compact set C in M we can find some t_0 with $0<t_0<1$ satisfying the condition that $\Phi(ta_0)\notin C$ for $t_0<t<1$. Assume that there is no point with such property. Then, for each point a with $|a|=R$ there exists a sequence $\{t_\nu;\nu=1, 2, \cdots\}$ which tends to 1 as ν tends to $+\infty$ such that $\{\Phi(t_\nu a);\nu=1, 2, \cdots\}$ converges to a point $z_0\in M$. Then, $g'(z_0)\neq 0$. In fact, if $g'(z_0)=0$, then we can find a positive constant E_2 such that

$$|\phi(z)| \geqq \frac{E_2}{|z-z_0|^{mp/(1-p)}}$$

in a neighborhood V of z_0, where m denotes the zero multiplicity of g' at z_0. Therefore, we have

$$\begin{aligned} R &= \int_{L_a} |dw| = \int_{\Gamma_a} \left| \frac{dw}{dz} \right| |dz| \\ &= \int_{\Gamma_a} |\phi(z)| \, |dz| \\ &\geqq E_2 \int_{L_a \cap V} \frac{|dz|}{|z-z_0|^{mp/(1-p)}} = \infty , \end{aligned}$$

because $mp/(1-p)=2m/(1-\varepsilon)>1$. This contradicts (4.1). Thus, we have $z_0 \in M'$. Take a relatively compact, simply connected open neighborhood V' of z_0 with $\bar{V}' \subset M'$. Since $|\phi|$ is a nowhere zero continuous function on M', there exists a positive constant E_3 such that $|\phi(z)| \geqq E_3$ on $\bar{V}'$. If there exists a sequence $\{t'_\nu ; \nu=1, 2, \cdots\}$ which tends to 1 as ν tends to $+\infty$ such that $\Phi(t'_\nu a) \notin V'$, then we have easily an absurd conclusion

$$R = \int_{\Gamma} |dw| \geqq E_3 \int_{\Gamma} |dz| = \infty .$$

Therefore, $\Phi(ta) \in V'$ $(t_0<t<1)$ for some t_0. Moreover, by the same argument as above, we can easily conclude

$$\lim_{t \to 1} \Phi(ta) = z_0 .$$

Take a connected component $\tilde{V}$ of $\pi^{-1}(V')$ which includes $\{(F|U)^{-1}(ta) : t_0<t<1\}$. Since $\pi|\tilde{V} : \tilde{V} \to V'$ is a homeomorphism, $(F|U)^{-1}(ta)$ tends to a point $\tilde{z}_0 \in \tilde{M}$ as t tends to 1. On the other hand, F maps an open neighborhood of $\tilde{z}_0$ biholomorphically onto an open neighborhood of a. This shows that $(F|U)^{-1}$ can be extended holomorphically to a neighborhood of each point a with $|a|=R$ as a map into $\tilde{M}'$. Since $\{w ; |w|=R\}$ is compact, we can easily find a constant R' with $R<R'$ such that there exists a holomorphic map $H(w) : \Delta(R') \to M'$ with the property that $H(w)=(F|U)^{-1}(w)$ for $w \in \Delta(R)$ and $(F \cdot H)(w)=w$ for $w \in \Delta(R')$. Then, F maps an open set $H(\Delta(R'))$ biholomorphically onto $\Delta(R')$. This contradicts the property of R. Accordingly, we can choose a point a_0 with $|a_0|=R$ such that Γ_{a_0} tends to the boundary of M. Therefore, $d(\text{o})$ is not larger than the length of Γ_{a_0}.

Now, we apply Lemma 3.1 to the function h to see

$$\frac{(1+|h|^2)|h'|^p}{(\prod_{i=1}^4 |h-\alpha_i|)^{p(1-\varepsilon')}} \leqq B^p \Big(\frac{2R}{R^2-|w|^2} \Big)^p ,$$

where $0<p=2/(3-\varepsilon)<1$. This implies that

$$
\begin{aligned}
d(o) &\leqq \int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} \Phi^* ds \\
&= \int_{L_{a_0}} \frac{(1+|h|^2)|h'|^p}{(\prod_{i=1}^4 |h-\alpha_i|)^{p(1-\varepsilon')}} |dw| \\
&\leqq B^p \int_{L_{a_0}} \left(\frac{2R}{R^2-|w|^2} \right)^p |dw| \\
&= B^p \int_0^R \left(\frac{2R}{R^2-t^2} \right)^p dt \\
&= 2^p B^p R^{1-p} \int_0^1 \frac{dt}{(1-t^2)^p} .
\end{aligned}
$$

By the help of (4.1) we complete the proof of Theorem I.

§ 5. Minimal surfaces in $\boldsymbol{R}^4$.

Let $x=(x_1, x_2, x_3, x_4): M \to \boldsymbol{R}^4$ be a complete minimal surface in $\boldsymbol{R}^4$. As in the case of minimal surfaces in $\boldsymbol{R}^3$, for the proof of Theorem II we may assume that M is biholomorphic to the unit disc Δ. As is well-known, the set of all oriented 2-planes in $\boldsymbol{R}^4$ is canonically identified with the quadric

$$Q_2(\boldsymbol{C}) := \{(w_1 : \cdots : w_4) ; w_1^2 + \cdots + w_4^2 = 0\}$$

in $P^3(\boldsymbol{C})$. By definition, the Gauss map $G: M \to Q_2(\boldsymbol{C})$ is the map which maps each point z of M to the point of $Q_2(\boldsymbol{C})$ corresponding to the oriented tangent plane of M at z. The quadric $Q_2(\boldsymbol{C})$ is biholomorphic to $\bar{\boldsymbol{C}} \times \bar{\boldsymbol{C}}$. By suitable identifications we may regard G as a pair of meromorphic functions $g=(g_1, g_2)$ on M. Set $\phi_i := \partial x_i / \partial z$ for $i=1, \cdots, 4$. Then, g_1 and g_2 are given by

$$g_1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}, \qquad g_2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}$$

and the metric on M induced from $\boldsymbol{R}^4$ is given by

$$ds^2 = |f|^2 (1+|g_1|^2)(1+|g_2|^2)|dz|^2 ,$$

where $f := \phi_1 - \sqrt{-1}\,\phi_2$.

We first study the case where $g_i \not\equiv$ const. for $i=1, 2$. Suppose that g_1 and g_2 omit q_1 distinct values $\alpha_1, \cdots, \alpha_{q_1}=\infty$ and q_2 distinct values $\beta_1, \cdots, \beta_{q_2}=\infty$ respectively. Moreover, we assume that $g_1'(o) \neq 0$, $g_2'(o) \neq 0$ and

$$q_1 > 2, \qquad q_2 > 2, \qquad \frac{1}{q_1-2} + \frac{1}{q_2-2} < 1 . \tag{10}$$

Take real numbers ε, ε' such that $0 < (q_i-1)\varepsilon' < \varepsilon < q_i - 2$ for $i=1, 2$ and

$$\frac{1}{q_1-2-\varepsilon}+\frac{1}{q_2-2-\varepsilon}<1\,.$$

Set $p_i:=1/(q_i-2-\varepsilon)$ for $i=1, 2$. By the assumption (10), we see $q_i\geqq 4$ $(i=1, 2)$. Moreover, we have $q_2\geqq 5$ in the case $q_1=4$, and $q_2\geqq 4$ in the case $q_1\geqq 5$. It suffices to consider the cases $(q_1, q_2)=(4, 5)$ and $(q_1, q_2)=(5, 4)$. In each case, $p_i/(1-p_1-p_2)>1$ $(i=1, 2)$ for a sufficiently small ε. We now consider a many-valued function

$$\phi:=\frac{f^{1/(1-p_1-p_2)}(\prod_{i=1}^{q_1-1}(g_1-\alpha_i))^{p_1(1-\varepsilon')/(1-p_1-p_2)}(\prod_{j=1}^{q_2-1}(g_2-\beta_j))^{p_2(1-\varepsilon')/(1-p_1-p_2)}}{(g_1')^{p_1/(1-p_1-p_2)}(g_2')^{p_2/(1-p_1-p_2)}} \tag{11}$$

on a set $M':=\{z\in M;\, g_1'(z)\neq 0$ and $g_2'(z)\neq 0\}$. Let ϕ_0 be a single-valued branch of ϕ in a neighborhood of the origin and $\pi: \tilde{M}'\to M'$ be the universal covering of M'. As in the previous section, for each $\tilde{z}\in\tilde{M}'$ taking a continuous curve γ whose homotopy class corresponds to $\tilde{z}$ and an analytic continuation ϕ_γ of ϕ_0 along γ, we define

$$F(\tilde{z}):=\int_\gamma \phi_\gamma(\zeta)d\zeta\,.$$

Then, $F(\tilde{0})=0$ and $dF(\tilde{z})\neq 0$ for all $\tilde{z}\in\tilde{M}'$. We choose the largest R such that F maps a connected neighborhood of $\tilde{0}$ bijectively onto $\Delta(R)$, where $R<+\infty$ by virtue of Liouville's theorem. Set $h_i(w):=g_i(\Phi(w))$ on $\Delta(R)$ for $i=1, 2$, where $\Phi=\pi\cdot(F|U)^{-1}$. The metric on $\Delta(R)$ induced from M by Φ is given by

$$\Phi^*ds^2=|f\cdot\Phi|^2(1+|h_1|^2)(1+|h_2|^2)\left|\frac{dz}{dw}\right|^2|dw|^2\,.$$

On the other hand, by (11) and the definition of F, we have

$$\left|\frac{dw}{dz}\right|=\frac{|f|(\prod_{i=1}^{q_1-1}|g_1-\alpha_i|)^{p_1(1-\varepsilon')}(\prod_{j=1}^{q_2-1}|g_2-\beta_j|)^{p_2(1-\varepsilon')}}{|g_1'|^{p_1}|g_2'|^{p_2}}\left|\frac{dw}{dz}\right|^{p_1+p_2}.$$

It follows that

$$\left|\frac{dz}{dw}\right|=\frac{|h_1'|^{p_1}|h_2'|^{p_2}}{|f|(\prod_{i=1}^{q_1-1}|h_1-\alpha_i|)^{p_1(1-\varepsilon')}(\prod_{j=1}^{q_2-1}|h_2-\beta_j|)^{p_2(1-\varepsilon')}},$$

because $h_i'(w)=g_i'(\Phi(w))\Phi'(w)$ $(i=1, 2)$. Therefore, we obtain

$$\Phi^*ds^2=\frac{(1+|h_1|^2)(1+|h_2|^2)|h_1'|^{2p_1}|h_2'|^{2p_2}}{(\prod_{i=1}^{q_1-1}|h_1-\alpha_i|)^{2p_1(1-\varepsilon')}(\prod_{j=1}^{q_2-1}|h_2-\beta_j|)^{2p_2(1-\varepsilon')}}|dw|^2\,.$$

By the same reason as in the previous section, we can find a point a_0 with $|a_0|=R$ such that for the line segment L from 0 to a_0 in $\Delta(R)$ the curve $\Gamma=\Phi(L)$ tends to the boundary of M. By the assumption of the completeness of M the length d of Γ is infinite. On the other hand, we obtain by the help of Lemma 3.1

$$d \leqq \int_L \frac{(1+|h_1|^2)^{1/2}(1+|h_2|^2)^{1/2}|h_1'|^{p_1}|h_2'|^{p_2}}{(\Pi_i|h_1-\alpha_i|)^{p_1(1-\varepsilon')}(\Pi_j|h_2-\beta_j|)^{p_2(1-\varepsilon')}}|dw|$$

$$\leqq B'\int_L\left(\frac{2R}{R^2-|w|^2}\right)^{p_1+p_2}|dw| = B''R^{1-(p_1+p_2)} < \infty ,$$

which is absurd. This completes the proof of Theorem II, (i).

We next consider the case $g_1 \not\equiv$const. and $g_2 \equiv$const. Suppose that g_1 omits four distinct values $\alpha_1, \cdots, \alpha_4$, where we assume $\alpha_4=\infty$. In this case, we use a many-valued function

$$\phi := \frac{f^{1/(1-p)}(\prod_{i=1}^4(g_1-\alpha_i))^{p(1-\varepsilon')/(1-p)}}{(g')^{1/(1-p)}}$$

instead of (11), where $0<3\varepsilon'<\varepsilon<1$ and $p:=1/(2-\varepsilon)$. By the same method as above, we can construct a continuous curve of finite length which goes from the origin to the boundary of M. This contradicts the assumption that M is complete. Therefore, we conclude Theorem II, (ii).

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