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A construction of certain 3-manifolds with orientation reversing involution

By Masako KOBAYASHI

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1. Introduction.

In his paper [4], Kawauchi proved that if a closed orientable 3-manifold M admits an orientation reversing involution, then the torsion part of the first integral homology group, $\mathrm{Tor}\, H_1(M; Z)$, is isomorphic to $A\oplus A$ or $Z_2\oplus A\oplus A$ where A is an abelian group of finite order. Moreover, for any given abelian group G with $\mathrm{Tor}\, G\cong A\oplus A$, there exists a closed orientable irreducible 3-manifold M admitting an orientation reversing involution with $H_1(M; Z)\cong G$. And if M is a closed orientable 3-manifold admitting an orientation reversing involution with $H_1(M; Z)\cong Z_2\oplus A\oplus A$ where A is an abelian group of odd order, then M must be a connected sum of P^3 and a certain manifold.

In this paper, for the remaining cases, we will prove the following theorems.

THEOREM 1. *For any abelian group G with* $\mathrm{Tor}\, G\cong Z_2\oplus A\oplus A$ *(possibly, $A=0$) and $G/\mathrm{Tor}\, G\neq 0$, there exists a closed orientable irreducible 3-manifold M admitting an orientation reversing involution with $H_1(M; Z)\cong G$.*

THEOREM 2. *For any abelian group $G\cong Z_2\oplus A\oplus A$ where A is an abelian group of non zero even order, there exists a closed orientable irreducible 3-manifold M admitting an orientation reversing involution with $H_1(M; Z)\cong G$.*

We refer to [2] and [3] for general definitions and terminology.

2. Proof of Theorem 1.

We identify a 3-sphere S^3 with $R^3\cup\{\infty\}$, and consider the antipodal map $\tau: S^3\to S^3$ by $\tau(x, y, z)=(-x, -y, -z)$ $\tau(\infty)=(\infty)$.

LEMMA 3. *There exists a closed orientable irreducible 3-manifold M admitting an orientation reversing involution with $H_1(M; Z)\cong Z\oplus Z_2$.*

PROOF. Consider a graph T in S^3 as in Figure 1. We choose the graph T so that T contains the origin $0=(0, 0, 0)$ of S^3 and T is invariant by τ, the

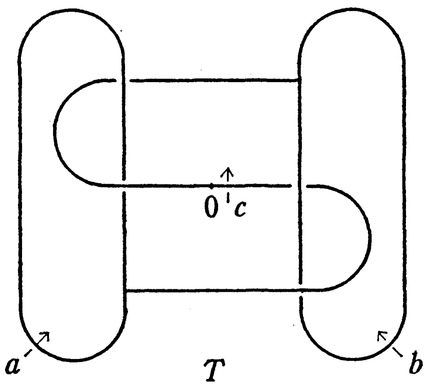


Figure 1.

antipodal map of S^3. Let $N(T)$ be a τ-invariant regular neighborhood of T and $M_1=\overline{S^3-N(T)}$. Note that $F=\partial M_1$ is a closed orientable surface of genus two. Let M_2 be a quotient space of $F\times I$ by an identification map of $F\times\{1\}$; $(x, 1)\sim(\tau'(x), 1)$, where I denotes the unit interval $[0, 1]$ and $\tau'=\tau|_F$. Then M_2 is a twisted I-bundle over a closed non orientable surface, and M_2 has a canonical involution induced by τ'. Let $M=M_1\cup_h M_2$, where h is the identity map of $F=\partial M_2$ onto $F=\partial M_1$. Then M has an orientation reversing involution.

By the ordinary cut and paste argument (cf. [2]), if M_1 and M_2 are irreducible and ∂-irreducible, then M is irreducible.

Since M_2 is a twisted I-bundle over a closed surface, M_2 is irreducible and ∂-irreducible.

For M_1, suppose S is an embedded 2-sphere in $M_1=\overline{S^3-N(T)}$. Then, we can regard S as an embedded 2-sphere in S^3 which does not meet T. By the Schönflies theorem, S bounds two 3-balls in S^3 and T is contained in one of these 3-balls. Hence S bounds another 3-ball in M_1. Hence, M_1 is irreducible.

Suppose D is a properly embedded essential disk in M_1. Remove $D\times[-1, 1]$, the regular neighborhood of D, from M_1, and we denote its closure by M_1'. If both M_1' and $\partial M_1'=(D\times\{-1, 1\})\cup(F-\partial D\times(-1, 1))$ are connected, then M_1' is a submanifold of S^3 and its boundary is a torus. Hence we may assume that M_1' is a non trivial knot exterior or a solid torus. Since we obtain M_1 by attaching a 1-handle to M_1', we have $\pi_1(M_1)\cong H*Z$ (a free product), where H is a knot group or Z. If M_1' is connected but $\partial M_1'$ is not, $\partial M_1'$ consists of two tori, since ∂D is essential on F. Then $\operatorname{rank} H_1(\partial M_1'; Z)=4$, and $\operatorname{rank} H_1(M_1'; Z)\geqq\operatorname{rank} H_1(\partial M_1'; Z)/2=2$. Since we obtain M_1 by attaching a 1-handle to M_1', we must have $\operatorname{rank} H_1(M_1; Z)\geqq 3$. But, we can see from Figure 1 that $\operatorname{rank} H_1(M_1; Z)=2$. It is impossible. If M_1' is disconnected, let N_1 and N_2 be the connected component of M_1', then N_i is a submanifold of S^3 and ∂N_i is a torus. Hence N_i is a non trivial knot exterior or a solid torus. Since M_1 is a boundary sum of N_1 and N_2, we may assume $\pi_1(M_1)\cong H_1*H_2$, where H_i is a knot group or Z $(i=1, 2)$.

Hence, if there exists an essential disk in M_1, we must have $\pi_1(M_1)\cong H_1*H_2$, where H_i is a knot group. (We may regard Z as the fundamental group of a trivial knot exterior.) We will see Alexander matrices of H_1*H_2 (cf. [**1**], [**5**]). Consider any epimorphism ϕ from H_1*H_2 to an infinite cyclic group $\langle t:\ \rangle$. Then $\phi|H_i$ $(i=1, 2)$ is a homomorphism from H_i onto a subgroup $\langle t^{\alpha_i}:\ \rangle$ of $\langle t:\ \rangle$, where at least one of α_i is non zero. An Alexander matrix of H_1*H_2 must be the block sum of Alexander matrices of H_1 and H_2. Hence the k-th Alexander polynomials Δ_k $(k=0, 1, 2)$ of H_1*H_2 must satisfy the conditions: $\Delta_0=0$, $\Delta_1=0$ and $\Delta_2=\Delta_1^1\times\Delta_1^2$, where Δ_1^i is the first Alexander polynomial of H_i. Note that, since H_i is a knot group, Δ_1^i is a polynomial in the group ring of $\langle t^{\alpha_i}:\ \rangle$ such that $\Delta_1^i(t^{\alpha_i})\doteq\Delta_1^i((t^{\alpha_i})^{-1})$ (i.e. $\Delta_1^i(t^{\alpha_i})=t^{u_i}\Delta_1^i((t^{\alpha_i})^{-1})$ for some $u_i\in Z$). Hence we must have $\Delta_2(t)\doteq\Delta_2(t^{-1})$.

We may choose the generators of $\pi_1(M_1)$ as indicated in Figure 1, then we have

$$\pi_1(M_1)\cong\langle a, b, c: b^{-1}aca^{-1}[ca]b[ac]=1\rangle\,.$$

Let ϕ be an epimorphism from $\pi_1(M_1)$ to $\langle t:\ \rangle$ defined by

$$\phi(a)=t^2\,,\quad \phi(b)=t\quad\text{and}\quad \phi(c)=1\,.$$

By the Fox calculus ([**1**], [**5**]), we have an Alexander matrix of $\pi_1(M_1)$;

$$(\ 0\quad 0\quad t^2-1+t^{-1})\,,$$

and the Alexander polynomials;

$$\Delta_0=0,\quad \Delta_1=0\quad\text{and}\quad \Delta_2=t^2-1+t^{-1}\,.$$

It contradicts $\Delta_2(t)\doteq\Delta_2(t^{-1})$. Hence, M_1 is ∂-irreducible.

We will see $H_1(M; Z)$. We choose the generators for $H_1(M_1; Z)$, $H_1(M_2; Z)$ and $H_1(F; Z)$ represented by curves indicated in Figure 2. Then we have

$$H_1(M_1; Z)\cong\langle a_1, a_2:\ \rangle,\quad H_1(M_2; Z)\cong\langle x, y, z: 2z=0\rangle$$

$$\text{and}\quad H_1(F; Z)\cong\langle m_1, m_2, l_1, l_2:\ \rangle$$

as abelian group presentations. By the homomorphism i_1 (or i_2) induced by the inclusion map from F to M_1 (or M_2, respectively), the generators of $H_1(F; Z)$ are mapped as follows;

$$i_1(m_1)=a_1,\quad i_1(m_2)=a_2,\quad i_1(l_1)=0,\quad i_1(l_2)=0\,,$$

$$i_2(m_1)=x,\quad i_2(m_2)=-x,\quad i_2(l_1)=y\quad\text{and}\quad i_2(l_2)=y\,.$$

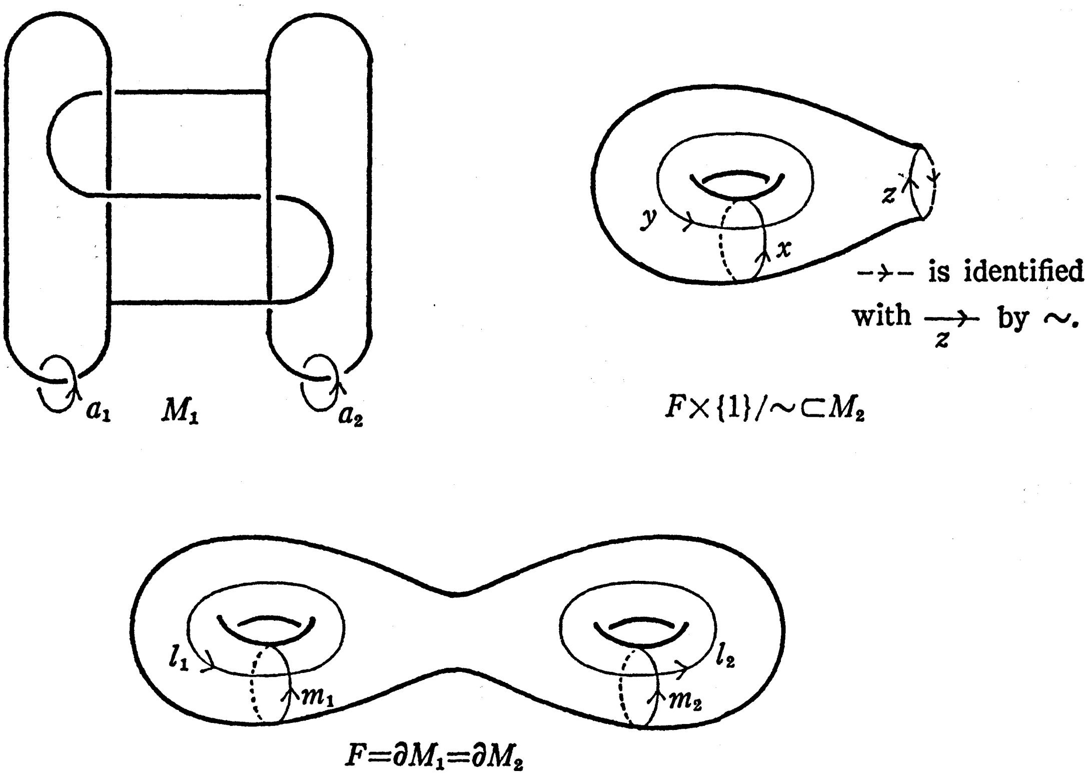


Figure 2.

Hence we have

$$\begin{aligned} H_1(M; Z) &\cong \langle a_1, a_2, x, y, z: 2z=0, a_1=x, a_2=-x, y=0\rangle \\ &\cong \langle x, z: 2z=0\rangle \\ &\cong Z\oplus Z_2 . \end{aligned}$$

This completes the proof.

Let $J, J'\subset S^3$ be τ-invariant non trivial knots such that J contains the fixed points of τ, and J' does not contain them. Let $M_3=\overline{S^3-N(J)}$ and $M_4=\overline{S^3-N(J')}$, where $N(J)$ and $N(J')$ are τ-invariant regular neighborhoods of J and J'. We may assume that $N(J')$ does not contain the fixed points of τ.

Note that we can construct a homology 3-sphere M_5 with $\pi_1(M_5)$ infinite, by $M_5=M_3\cup_h M_4$, where h is a homeomorphism of ∂M_4 onto ∂M_3 which carries a preferred longitude of $\partial N(J')$ to a meridian of $\partial N(J)$. Then, M_5 admits an orientation reversing involution induced by τ on M_3 and M_4.

PROOF OF THEOREM 1. Let $G\cong(\overset{s}{\oplus}Z)\oplus Z_2\oplus Z_{p_1}\oplus Z_{p_2}\oplus\cdots\oplus Z_{p_r}\oplus Z_{p_1}\oplus Z_{p_2}\oplus\cdots\oplus Z_{p_r}$ ($s\geqq 1$, $r\geqq 0$, $p_1, p_2, \cdots, p_r\in Z$). Let $K_1, K_2, \cdots, K_{s-1}, L_1, L_2, \cdots, L_r$

$\subset M_4 \subset M_6$ be $r+s-1$ knots and $T \subset M_4 \subset M_6$ the graph same as in the proof of Lemma 3 which satisfy the following conditions;

(1) $K_1, \cdots, K_{s-1}$ and T are τ-invariant,

(2) $K_1, \cdots, K_{s-1}, L_1, \cdots, L_r, \tau(L_1), \cdots, \tau(L_r)$ and T are mutually disjoint,

(3) $[K_i] \neq 1$, $[L_i] \neq 1$, $[T] \neq 1$ in $\pi_1(M_6)$,

(4) each two of $K_1, \cdots, K_{s-1}, L_1, \cdots, L_r, \tau(L_1), \cdots, \tau(L_r)$ and T have the linking number 0 in M_6, and

(5) none of knots contains the fixed point of τ.

For example we can choose such knots and graph as Figure 3. Remove a small τ-invariant regular open neighborhood of $\bigcup_{i=1}^{s-1} K_i \cup \bigcup_{j=1}^{r} (L_j \cup \tau(L_j)) \cup T$ from M_6, and attach $s-1$ copies of $M_4 = \overline{S^3 - N(J')}$, $2r$ copies of a non trivial knot exterior

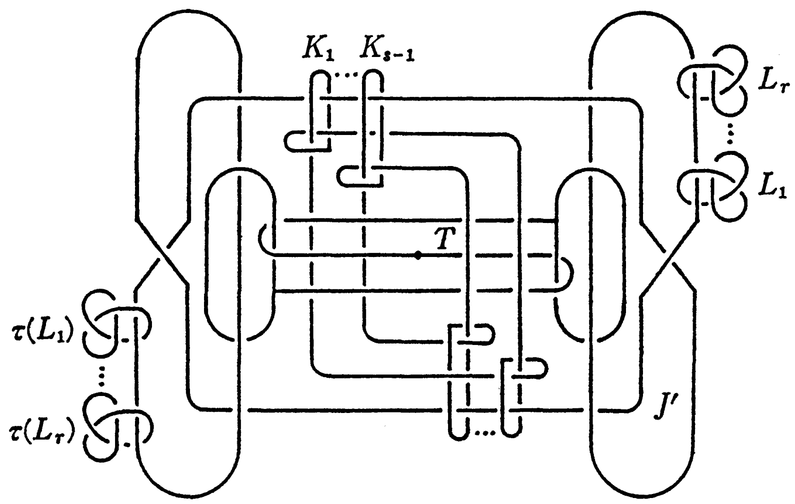


Figure 3.

$\overline{S^3 - N(L)}$ and a twisted I-bundle as follows;

(1) $\partial N(T)$ is identified with the boundary of a twisted I-bundle as in the proof of Lemma 3,

(2) $\partial N(K_i)$ $(i=1, \cdots, s-1)$ is identified with a copy of $\partial M_4 = \partial N(J')$ so that a preferred longitude is a preferred longitude of $\partial N(J')$,

(3) $\partial N(L_i)$ $(i=1, \cdots, r)$ is identified with a copy of $\partial(\overline{S^3 - N(L)}) = \partial N(L)$ so that a preferred longitude of $\partial N(L_i)$ is a curve linking with L p_i-times in S^3, and

(4) $\partial N(\tau(L_i))$ $(i=1, \cdots, r)$ is identified with a copy of $\partial(\overline{S^3 - N(L)}) = \partial N(L)$ so that the attaching homeomorphism commutes with τ.

We call the resulting manifold M. We can see that M has the required first integral homology group. The irreducibility of M follows from the irreducibility and ∂-irreducibility of each part of M. Note that every non trivial knot exterior is irreducible and ∂-irreducible.

This completes the proof.

3. Proof of Theorem 2.

LEMMA 4. *There exists a closed orientable irreducible 3-manifold M admitting an orientation reversing involution with $H_1(M; Z) \cong Z_2 \oplus Z_{2n} \oplus Z_{2n}$ $(n \in Z)$.*

PROOF. Let B_i $(i=1, 2, 3)$ be a 3-ball and τ_i an orientation reversing involution of B_i with one fixed point. Let $D_i \subset \partial B_i$ be a 2-disk such that $D_i \cap \tau_i(D_i) = \emptyset$ $(i=1, 2, 3)$, and $D'_2 \subset \partial B_2$ a 2-disk such that D_2, $\tau_2(D_2)$, D'_2 and $\tau(D'_2)$ are mutually disjoint. We will attach four 1-handles to them, one from D_1 to D_2, one from $\tau_1(D_1)$ to $\tau_2(D_2)$, one from D'_2 to D_3, and one from $\tau_2(D'_2)$ to $\tau_3(D_3)$. We call the resulting manifold M_6. M_6 is topologically a handlebody of genus two and admitting an orientation reversing involution τ which extends τ_1, τ_2 and τ_3. Let α and β be generators of $H_1(M_6; Z)$ as in Figure 4. We choose knots K_1 and K_2 which satisfy the following conditions;

(1) K_1 is τ-invariant and contains two of fixed points,

(2) K_2 does not contain any fixed point,

(3) $[K_1] = \alpha \in H_1(M_6; Z)$ and $[K_2] = \beta \in H_1(M_6; Z)$, and

(4) K_1, K_2 and $\tau(K_2)$ are mutually disjoint

(see Figure 4). Note that $[\tau(K_2)] = -\beta \in H_1(M_6; Z)$.

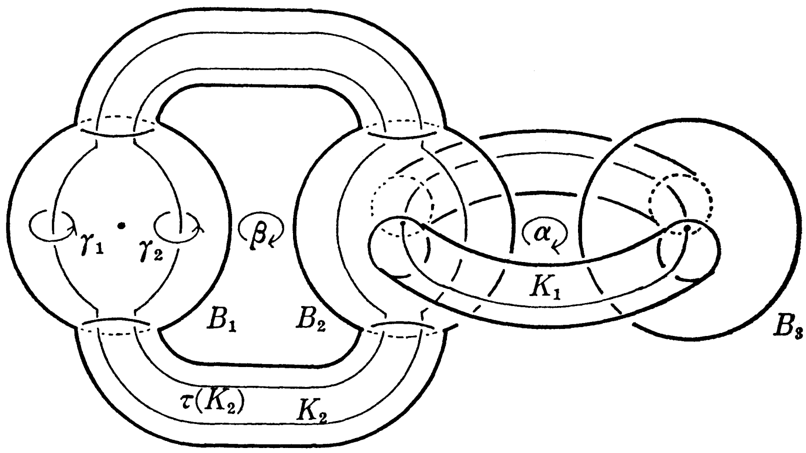


Figure 4.

Remove a small τ-invariant regular open neighborhood of $K_1 \cup K_2 \cup \tau(K_2)$ from M_6. For K_1, consider $M_4 = \overline{S^3 - N(J')}$ (the same M_4 as in the section 2) and identify $\partial N(K_1)$ with $\partial N(J')$ so that a preferred longitude of $\partial N(J')$ is a meridian of $\partial N(K_1)$. For K_2 and $\tau(K_2)$, consider two copies of a non trivial knot exterior $\overline{S^3-N(L)}$. Identify $\partial N(K_2)$ and $\partial N(\tau(K_2))$ with two copies of $\partial N(L)$ so that a preferred longitude of one copy of $\partial N(L)$ is a curve C on $\partial N(K_2)$ with $[C]=n\gamma_1+\beta\in H_1(\overline{M_6-N(K_2)\cup N(\tau(K_2))}; Z)$, and a preferred longitude of another copy is a curve C' on $\partial N(\tau(K_2))$ with $[C']=n\gamma_2-\beta\in H_1(\overline{M_6-N(K_2)\cup N(\tau(K_2))}; Z)$, where γ_1 and γ_2 are new generators created by removing $N(K_2)$ and $N(\tau(K_2))$ from M_6 (see Figure 4). We call the resulting manifold M_7. Then we have

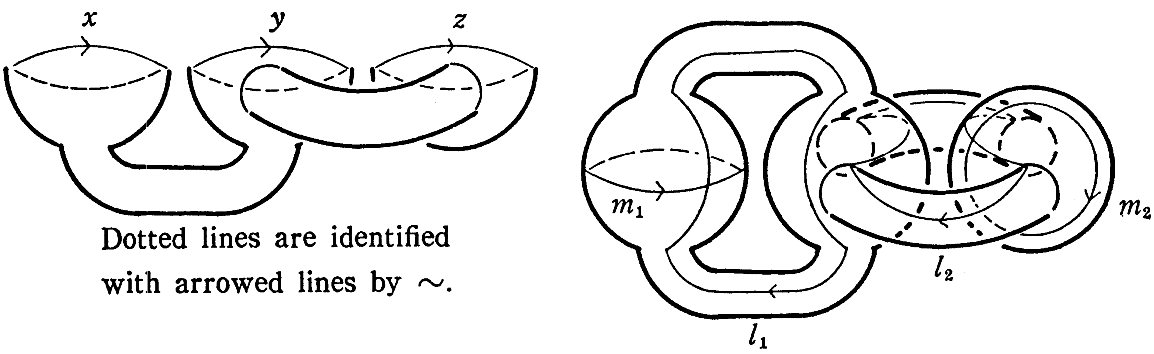
$$H_1(M_7; Z)\cong\langle\alpha,\ \beta,\ \gamma_1,\ \gamma_2:\ n\gamma_1+\beta=0,\ n\gamma_2-\beta=0\rangle\,.$$

By this construction, we can see that M_7 has an orientation reversing involution which is an extension of τ on M_6 and S^3.

Let $F=\partial M_7$ (an orientable closed surface of genus two) and M_8 a quotient space of $F\times I$ by an identification map of $F\times\{1\}$; $(x, 1)\sim(\tau'(x), 1)$, where $\tau'=\tau|_F$. Then M_8 is a twisted I-bundle over a non orientable closed surface, and M_8 has a canonical involution induced by τ'.

Let $M=M_7\cup_h M_8$ where h is the identity map of the boundary F, then M has an orientation reversing involution.

We can see $H_1(M; Z)$ by using $\partial M_7=\partial M_8=F$ and the inclusion maps i_1 and i_2 as in the proof of Lemma 3. We choose the generators for $H_1(M_8; Z)$ and $H_1(F; Z)$ represented by curves indicated in Figure 5.



$F\times\{1\}/\sim\subset M_8$ $\qquad\qquad$ $F=\partial M_7=\partial M_8$

Figure 5.

Then we have

$$H_1(M_8; Z)\cong\langle x,\ y,\ z:\ 2x+2y+2z=0\rangle\quad\text{and}$$

$$H_1(F; Z)\cong\langle m_1,\ m_2,\ l_1,\ l_2:\ \ \rangle\,.$$

It is easy to check that

$$i_1(m_1)=\gamma_1+\gamma_2\,,\quad i_1(m_2)=0\,,\quad i_1(l_1)=\beta\,,\quad i_1(l_2)=\alpha\,,\quad i_2(m_1)=2x\,,$$

$$i_2(m_2)=2z\,,\quad i_2(l_1)=x+y+2z \quad\text{and}\quad i_2(l_2)=2x+y+z\,.$$

Hence,

$$\begin{aligned} H_1(M;Z) &\cong \langle \alpha, \beta, \gamma_1, \gamma_2, x, y, z:\quad n\gamma_1+\beta=0,\ n\gamma_2-\beta=0,\ 2x+2y+2z=0, \\ &\qquad \gamma_1+\gamma_2=2x,\ 2z=0,\ \beta=x+y+2z,\ \alpha=2x+y+z\rangle \\ &\cong \langle \gamma_2, x, z:\quad 2n\gamma_2=0,\ 2nx=0,\ 2z=0\rangle \\ &\cong Z_2\oplus Z_{2n}\oplus Z_{2n}\,. \end{aligned}$$

For the irreducibility of M, as in the proof of Lemma 3, we only prove the irreducibility and ∂-irreducibility of each part of M. A non trivial knot exterior and a twisted I-bundle over a closed surface clearly have these properties. Hence we shall prove it for $\overline{M_6-N(K_1)\cup N(K_2)\cup N(\tau(K_2))}$, denote by M_6'.

Suppose S is an essential 2-sphere in M_6', then S is also a 2-sphere in the handlebody M_6. Since a handlebody is irreducible, S bounds a 3-ball B in M_6. Hence B contains at least one of K_1, K_2 or $\tau(K_2)$. Since K_1, K_2 and $\tau(K_2)$ are not contractible in the handlebody, it is impossible. Hence M_6' is irreducible.

Suppose D is an essential 2-disk in M_6'. Since K_1, K_2 and $\tau(K_2)$ are not contractible in the handlebody M_6, ∂D is not on either $\partial N(K_1)$, $\partial N(K_2)$ or $\partial N(\tau(K_2))$. Hence ∂D is on ∂M_6, and we may regard that D is a proper 2-disk in M_6. If D did not separate M_6, then D must cut a curve representing the generators of $\pi_1(M_6)$. But we choose K_1, K_2 and $\tau(K_2)$ to be such curves. Hence it is impossible. If ∂D was trivial in $\pi_1(\partial M_6)$, then D with a disk on ∂M_6 bounds a 3-ball, and this 3-ball must contain K_1, K_2 or $\tau(K_2)$. But it is impossible, because K_1, K_2 and $\tau(K_2)$ are not contractible in M_6. The remaining possibility is the case when D separates M_6 into M' and M'', and ∂D is non trivial in $\pi_1(\partial M_6)$. In this case, D represents the amalgamating subgroup of $\pi_1(M_6)\cong Z*Z$, hence $\pi_1(M')\cong\pi_1(M'')\cong Z$. Note that the knots K_1 and K_2 are chosen to be generators of $\pi_1(M_6)\cong Z*Z\cong\langle\alpha, \beta:\ \rangle$ and $[\tau(K_2)]=\beta^{-1}\in\pi_1(M_6)$ (now, we consider α and β in Figure 4 are the generators of $\pi_1(M_6)$, ignoring the base point). Hence K_2 and $\tau(K_2)$ are homotopic without meeting D, so without meeting K_1. But it is impossible. Hence M is ∂-irreducible.

This completes the proof.

PROOF OF THEOREM 2. Let $G\cong Z_2\oplus Z_{2n}\oplus Z_{2n}\oplus Z_{p_1}\oplus Z_{p_2}\oplus\cdots\oplus Z_{p_r}\oplus Z_{p_1}\oplus Z_{p_2}\oplus\cdots\oplus Z_{p_r}$ ($r\geqq 0$, n, p_1, p_2, $\cdots$, $p_r\in Z$). We consider knots L_1, L_2, $\cdots$, L_r and L' in M_5 (M_5 is the homology 3-sphere in the section 2), such that L' is τ-invariant, L_1, L_2, $\cdots$, L_r, $\tau(L_1)$, $\tau(L_2)$, $\cdots$, $\tau(L_r)$ and L' are mutually disjoint, and $[L_i]\neq 1$, $[L]\neq 1$ in $\pi_1(M_5)$ ($i=1, 2, \cdots, r$). We will do like as in the proof of Theorem 1. Remove a small τ-invariant regular neighborhood of $\bigcup_{i=1}^{r}(L_i\cup\tau(L_i))\cup L'$ from M_5, and attach $2r$ copies of a non trivial knot exterior to $\partial N(L_i)$ and $\partial N(\tau(L_i))$ ($i=1, 2, \cdots, r$) for the required torsion of G. We call the resulting manifold M_9.

We will construct the same manifold as in the proof of Lemma 4, but for $\partial N(K_1)$ ($\subset M_6$), we will attach M_9 so that a preferred longitude of $\partial N(L')=\partial M_9$ is a meridian of $\partial N(K_1)$.

By this construction, the resulting manifold has the required first integral homology group. And the irreducibility of the manifold follows from the irreducibility and the ∂-irreducibility of each part.

This completes the proof.

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Masako KOBAYASHI
Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku,
Osaka 558
Japan