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Divergent formal solutions to Fuchsian partial differential equations

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§ 1. Introduction.

Let $x=(x_1, \cdots, x_d)\in \boldsymbol{R}^d$ be the variable in $\boldsymbol{R}^d$ and let us put $\partial=(\partial_1, \cdots, \partial_d)$ where $\partial_j=\partial/\partial x_j$, $j=1, \cdots, d$. For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_d)\in \boldsymbol{N}^d$, $\boldsymbol{N}=\{0, 1, 2, \cdots\}$ we set $(x\cdot\partial)^\alpha=(x_1\partial_1)^{\alpha_1}\cdots(x_d\partial_d)^{\alpha_d}$. Let $\lambda\in \boldsymbol{C}^d$ be given and fixed. Then we shall study the characterization of divergent formal solutions $u(x)$ of the form $u(x)=x^\lambda\sum_{\eta\in \boldsymbol{N}^d} v_\eta x^\eta/\eta!$ of the equation

$$(1.1)\qquad P(x\,;x\cdot\partial)u \equiv \Big(\sum_{|\alpha|=m} a_\alpha(x\cdot\partial)^\alpha+\sum_{|\beta|\leqq m-\sigma} b_\beta(x)(x\cdot\partial)^\beta\Big)u(x) = f(x)x^\lambda$$

where $\sigma\geqq 1$ is an integer, $m\in \boldsymbol{N}$ and a_α's are complex constants. We assume that the function $b_\beta(x)$ is analytic at the origin and that $f(x)$ is a given analytic function.

For ordinary differential equations of Fuchs type (i.e. $d=1$ in (1.1)) we know that all formal solutions of Equation (1.1) converge. Nevertheless, in the case $d\geqq 2$ we often get divergent formal solutions of Equation (1.1) if the coefficients satisfy certain conditions (cf. [**3**], [**9**]). In fact there exist equations with infinite-dimensional kernel and those with small denominators. Typical examples are the equations $(x_1\partial_1-\tau x_2\partial_2)u=f(x)$ where τ is a positive rational and irrational number respectively. By using elementary facts of diophantine analysis we can show that there exists an irrational $\tau>0$ and an entire function $f(x)$ such that the equation for this τ and $f(x)$ has a formal solution $u(x)=\sum u_\eta x^\eta$ with the estimate $|\eta|!^s/|u_\eta|\to 0$ as $|\eta|\to\infty$ for $s=1, 2, \cdots$. In this case the formal solution has bad behavior. Even for these simple examples the criterion which distinguishes such bad equations from good ones can only be expressed by the number-theoretical properties of τ, and is not simple. Hence if we are to study formal solutions of more general equations in the analytic category we need very delicate and complicated arguments (cf. [8], [**9**]). It is an interesting problem to give a meaning to such divergent solutions and to study whether this phenomenon is peculiar to analytic solutions or also occurs for C^∞-solutions.

Instead of considering formal solutions in the analytic category we consider then in the C^∞ one. Then the situation is very simple. Namely we can show that every formal solution is the asymptotic expansion of some smooth solution. Moreover, if the formal solution converges then the corresponding smooth solution turns to be analytic. We remark that the results here are independent of the diophantine properties of parameters in the individual equation. We also remark that we can extend this result for equations with some irregularity in the lower order terms.

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§ 2. Notations and results.

Let us expand the functions $b_\beta(x)$ in (1.1) into Taylor series, $b_\beta(x)=\sum_\gamma b_{\beta,\gamma}x^\gamma/\gamma!$ and denote by Γ_0 the smallest closed convex cone with vertex at the origin which contains all γ's such that $b_{\beta,\gamma}\neq 0$ for some β. We define $p(\eta)$ by

$$p(\eta)=\sum_{|\alpha|=m} a_\alpha \eta^\alpha+\sum_{|\beta|\leqq m-\sigma} b_\beta(0)\eta^\beta, \tag{2.1}$$

and denote the m-th homogeneous part of $p(\eta)$ by $p_m(\eta)$.

We assume

(A.1) p_m is hyperbolic with respect to all $\omega\in\Gamma_0$, that is, $p_m(\xi+i\omega)\neq 0$ for all $\xi+i\omega\in \boldsymbol{R}^d+i(\Gamma_0\setminus\{0\})$.

Here, if Γ_0 is empty we assume that $p_m(\eta)$ is hyperbolic with respect to some $\omega\in\boldsymbol{R}^d$, $\omega>0$.

Let us take $\omega(\in\Gamma_0)$ such that p_m is hyperbolic with respect to ω and let $\xi\in\boldsymbol{R}^d$, $|\xi|=1$. We expand $p_m(\tau\omega+\xi)$ in the ascending power of t,

$$p_m(\tau\omega+\xi)=\tau^\sigma L_\xi(\omega)+O(\tau^{\sigma+1})$$

where $L_\xi(\omega)\not\equiv 0$ and $\sigma\equiv\sigma(\xi)$ is the multiplicity of the localization of p_m at ξ. Then we assume

(A.2) $\sigma(\xi)\leqq\sigma$ for all $\xi\in\boldsymbol{R}^d$, $|\xi|=1$.

Note that the condition (A.2) corresponds to Levi's condition.

(A.3) $p_m(\omega)\neq 0$ for any $\omega=(\omega_1,\cdots,\omega_d)$, $|\omega|=1$ such that $\omega_j\geqq 0$, $j=1,\cdots,d$ and that $\omega_\nu=0$ for some ν, $1\leqq\nu\leqq d$.

Now our main result is

THEOREM 2.1. *Suppose that the conditions* (A.1)~(A.3) *are satisfied. Moreover suppose that Equation* (1.1) *has a formal solution* $\tilde{u}(x)=x^\lambda\sum_{\eta\in N^d}\tilde{v}_\eta x^\eta/\eta!$.

Then there exists a $v(x) \in C^{\infty}(\boldsymbol{R}^d)$ *such that* $u(x) \equiv v(x)x^{\lambda}$ *satisfies Equation* (1.1) *in some neighborhood of the origin and that, for* $k=0, 1, 2, \cdots$,

$$v(x) - \sum_{|\eta| \leq k} \tilde{v}_{\eta} x^{\eta}/\eta! = O(|x|^{k+1}) \quad as \ |x| \to 0. \tag{2.2}$$

REMARK 2.1. The proof of Theorem 2.1 and Lemma 3.1 also yields that if the formal sum $\tilde{u}(x)x^{-\lambda}$ converges in some neighborhood of the origin then the function $v(x)$ is equal to this sum in some neighborhood of the origin, hence, is analytic.

REMARK 2.2. We remark that the condition (A.3) does not follow from the conditions (A.1) and (A.2) by a linear change of variables in general. In order to show this, let us consider the case $d=2$. First we note that the differentiation in Equation (1.1) has the form $x_j \partial_j$. It is not difficult to see that linear transforms which preserve $x_j \partial_j$ are nothing but the products of two transforms, a scaling of the coordinates, $x_j = r_j y_j$, $j=1, 2$ for some $r_j \neq 0$ and the permutation in the variables. Now let us consider the operator

$$p(x \cdot \partial) = x_1 \partial_1 x_2 \partial_2 (x_1 \partial_1 - 2x_2 \partial_2)(2x_1 \partial_1 - x_2 \partial_2) + x_1 x_2 ((x_1 \partial_1)^3 + (x_2 \partial_2)^3).$$

We can easily see that $\Gamma_0 = \{t(1, 1); t > 0\}$. This operator satisfies (A.1) and (A.2) and does not satisfy (A.3) (cf. Corollary 2.2 which follows). The same properties hold for operators obtained from the above one by the linear change of variables preserving $x_j \partial_j$, $j=1, 2$. Hence we have the assertion.

REMARK 2.3. We can extend Theorem 2.1 to the case where the formal solution $\tilde{u}(x)$ contains a polynomial of $\log x_j$, $j=1, \cdots, d$. Such formal solutions really appear if we apply the well-known Frobenius' method in the theory of ordinary differential equations to (1.1) in case $p(\eta+\lambda)$ vanishes for some $\eta \in \boldsymbol{N}^d$. We can show that the terms in the formal solution not containing logarithmic factors are characterized as in Theorem 2.1 and the terms containing a logarithmic factor of x_j $(j=1, \cdots, d)$ are of the form $\prod_j (\log x_j)^{\nu_j} \times$(analytic functions of x). Hence Theorem 2.1 can be extended to this case.

It is possible to consider more general formal solutions $\tilde{u}(x)$ admitting all negative powers of x; $\tilde{u}(x) = x^{\lambda} \sum_{\eta \in Z^d} v_{\eta} x^{\eta}/\eta!$. In order to avoid the terms containing logarithmic factors, and for the sake of simplicity we assume that $p(k+\lambda) \neq 0$ for all $k \in \boldsymbol{Z}^d$. Then we can show that if $\tilde{u}(x)$ is a formal solution of Equation (1.1) it follows that $v_{\eta}=0$ for all $\eta \in \boldsymbol{Z}^d \setminus \boldsymbol{N}^d$. Hence studying these formal solutions is reduced to the former case. This fact is proved by the straightforward computations using the method of indeterminate coefficients. So we omit the proof.

REMARK 2.4. Suppose that $\lambda=0$. Then Theorem 2.1 implies that if Equation (1.1) has a divergent formal solution it has a smooth but non-analytic solution. This implies; if the operator $P(x\,;\,x\cdot\partial)$ in (1.1) is analytic-hypoelliptic then all formal solutions of (1.1) converge. The converse is not true since the function $v(x)$ in Theorem 2.1 is not unique in general. For example the operator $P(x\cdot\partial)=\prod_j(x_1\partial_1-\tau_j x_2\partial_2+c_j)$ has a solution $v(x)\in C_0^\infty(\boldsymbol{R}^2)$, $v\neq 0$, $P(x\cdot\partial)v=0$ such that $v(x)$ is flat on $x_1=0$ and $x_2=0$ for appropriately chosen constants c_j and $\tau_j>0$.

In the case $d=2$ the assumptions in Theorem 2.1 are very simple. We state it as a corollary.

COROLLARY 2.2. *Suppose that $d=2$ and that the following conditions are satisfied.*

(2.3) $$p_m(\omega)\neq 0 \quad \text{for } \omega=(1, 0) \text{ and } (0, 1).$$

(2.4) $p_m(\omega)\neq 0$ *for all* $\omega\in\Gamma_0$ *and all roots of the equation* $p_m(z, 1)=0$ *are real with multiplicities smaller than* σ.

Then we have the same assertion as in Theorem 2.1.

PROOF. It is clear that the condition (2.3) is equivalent to (A.3) and that (2.4) to (A.1) and (A.2). q.e.d.

REMARK 2.5. We can extend Corollary 2.2 for equations with some irregularity in the lower terms. Namely we consider the equation

(2.5) $$\Big(\sum_{|\alpha|=m} a_\alpha(x\cdot\partial)^\alpha+\sum_{|\beta|\leqq m-\sigma} b_\beta(x)\partial^\beta\Big)u(x)=f(x)x^\lambda$$

where $b_\beta(x)$ is analytic at the origin and $d=2$.

In order to state our result we introduce some notations. Let us define $b_{\beta,\gamma}$ by $b_{\beta,\gamma}=(\partial_x^\gamma b_\beta)(0)$ for $|\beta|\leqq m-1$, and set $b_{\alpha,\alpha}=a_\alpha$ for $|\alpha|=m$. We define the set M_P by $M_P=\{\gamma-\beta\,;\,b_{\beta,\gamma}\neq 0$ for some β and $\gamma\}$, and denote by Γ_0 the smallest closed convex cone with vertex at the origin which contains M_P. We note that, for Equation (1.1), this definition of Γ_0 agrees with the former one. Now we assume

(C.1) Γ_0 is a proper cone, i.e. contains no straight line and there exists a vector $\theta=(\theta_1, \theta_2)\in\Gamma_0$ such that $\theta_1>0$, $\theta_2>0$.

We note that Equation (1.1) satisfies (C.1). Moreover, if we freeze some variable and consider it as the equation of $(d-1)$ independent variables it still satisfies (C.1). Nevertheless Equation (2.5) satisfying (C.1) does not possess this property. Namely it may appear an irregular-singular type equation. This simple difference makes the situation difficult.

We determine the integers n_1 and $\tilde{n}_1$ respectively by $n_1=-\min\{\eta_1; \eta\in M_P\}$ and $\tilde{n}_1=\min\{\eta_2; \eta\in M_P, \eta_1=-n_1\}$ where $\eta=(\eta_1, \eta_2)$, and n_2 and $\tilde{n}_2$ by exchanging the parts of η_1 and η_2 in the definitions of n_1 and $\tilde{n}_1$ respectively. Next we determine the integer $\tilde{m}_1$ by $\tilde{m}_1=\max\{|\beta|; \gamma-\beta\in M_P, \gamma_1-\beta_1=-n_1, b_{\beta,\gamma}\neq 0$ for some β and $\gamma\}$, and $\tilde{m}_2$ by replacing β_1, γ_1 and n_1 respectively by β_2, γ_2 and n_2. Then we assume

$$b_{\beta,\gamma}\neq 0 \quad \text{for} \quad (\beta, \gamma)=((n_1, \tilde{m}_1-n_1), (0, \tilde{m}_1-n_1+\tilde{n}_1)) \text{ and } ((\tilde{m}_2-n_2, n_2), (\tilde{m}_2-n_2+\tilde{n}_2, 0)). \tag{C.2}$$

Note that for Equation (1.1) with $d=2$ we may suppose that $n_1=\tilde{n}_1=\tilde{n}_2=n_2=0$, $\tilde{m}_1=\tilde{m}_2=m$. Hence (C.2) is equivalent to (2.3). Then we have

THEOREM 2.3. *Suppose that $d=2$ and that the conditions* (C.1), (C.2) *and* (2.4) *are satisfied. Then we have the same assertion as in Theorem* 2.1 *for the equation* (2.5).

This theorem is proved, in principle, by the same method as for the proof of Theorem 2.1. But, technically, there are some difficulties which are caused by the effect of the lower terms of Equation (2.5). The main point is the modification of the proof of Lemma 3.1 which is possible in the case $d=2$ (cf. Remark 3.1).

§ 3. Proof of the main theorem.

We prepare two lemmas.

LEMMA 3.1. *Assume that Equation* (1.1) *has a formal solution $\tilde{u}(x)$ given in Theorem* 2.1 *and that the condition* (A.3) *is satisfied. Then there exists a $w(x)\in C_0^\infty(\boldsymbol{R}^d)$ and a neighborhood V of the origin such that the function $P(x; x\cdot\partial)(x^\lambda w(x))-x^\lambda f(x)$ is flat on $V\cap\{x\in\boldsymbol{R}^d; x_j=0\}$ for $j=1, \cdots, d$.*

PROOF. We divide the proof into two steps.

Step 1. We set $\tilde{v}(x)=x^{-\lambda}\tilde{u}(x)$ and rewrite $\tilde{v}(x)$ as follows.

$$\tilde{v}(x)=\sum_{j=0}^{\infty}(x_1\cdots x_d)^j\Big\{\sum_{k=1}^{d}\tilde{v}_{j,k}(x)\Big\} \tag{3.1}$$

where $\tilde{v}_{j,k}(x)$ does not contain the variable x_k. In fact we define $\tilde{v}_{0,1}(x)$ as all powers of x in $\tilde{v}(x)$ which do not contain x_1. Next we define $\tilde{v}_{0,2}(x)$ as those in $\tilde{v}(x)-\tilde{v}_{0,1}(x)$ not containing x_2. It is easy to see that $\tilde{v}-\tilde{v}_{0,1}=O(x_1)$ and that $\tilde{v}-\tilde{v}_{0,1}-\tilde{v}_{0,2}=O(x_1x_2)$. Hence, by the same way we can define $\tilde{v}_{0,k}(x)$ not containing x_k $(k=1, \cdots, d)$ such that $\tilde{v}(x)-\sum_{k=0}^{d}\tilde{v}_{0,k}(x)=O(x_1\cdots x_d)$. By repeating this procedure we get (3.1).

Next we shall show that the formal sums $\tilde{v}_{j,k}(x)$ $(j=0, 1, \cdots;\ k=1, \cdots, d)$ converge and represent analytic functions $v_{j,k}(x)$ of x in some neighborhood of the origin independent of j and k, and that, for $\nu=0, 1, \cdots,$

$$f(x)-P_\lambda\Big(\sum_{j=0}^{\nu}(x_1\cdots x_d)^j\sum_{k=1}^{d}v_{j,k}(x)\Big)=O((x_1\cdots x_d)^{\nu+1}) \tag{3.2}$$

where $P_\lambda=x^{-\lambda}Px^\lambda$. Note that $P_\lambda(x\,;x\cdot\partial)=P(x\,;x\cdot\partial+\lambda)$.

For the proof let us calculate the equations which the $\tilde{v}_{j,k}$ satisfy. We note that $P(x^\lambda\tilde{v})=x^\lambda P_\lambda\tilde{v}$. Hence $\tilde{v}(x)$ is the formal solution of the following equation

$$P_\lambda\tilde{v}(x)=f(x). \tag{3.3}$$

Putting $x_1=0$ in (3.3) we get, from (3.1) and the definition of P

$$P_\lambda(0, x'\,;x'\cdot\partial')\tilde{v}_{0,1}(x')=f(0, x') \tag{3.4}$$

where $x=(x_1, x')$ and $\partial'=(\partial_2, \cdots, \partial_d)$. It follows from (A.3) and the result of Kashiwara-Kawai-Sjöstrand [7] that the formal solution $\tilde{v}_{0,1}(x')$ converges in some neighborhood of the origin. Moreover, by the results of Zerner [**11**] it is defined and analytic in some neighborhood of the origin depending only on $f(x)$ and the equation P_λ. We can easily see that $f(x)-P_\lambda v_{0,1}(x)=O(x_1)$.

Next we consider the equation $P_\lambda w=f(x)-P_\lambda v_{0,1}(x)$. If we set $x_2=0$ in this equation we get a similar equation to (3.4). We can easily see that this reduced equation has the formal solution $\tilde{v}_{0,2}(x)$. Hence, by the same argument as above $\tilde{v}_{0,2}(x)$ converges in some neighborhood of the origin depending only on $f(x)$ and P_λ. Denoting this function by $v_{0,2}(x)$ we can easily verify that $f(x)-P_\lambda(v_{0,1}+v_{0,2})=O(x_1x_2)$. By the same way we can easily see that the formal sum $\tilde{v}_{0,k}(x)$ $(1\leqq k\leqq d)$ converges in some neighborhood of the origin depending only on $f(x)$ and P_λ. Denoting these functions by $v_{0,k}(x)$ $(1\leqq k\leqq d)$, we can easily verify (3.2) with $\nu=0$.

Now suppose that the formal sums $\tilde{v}_{k,j}$ $(j=1, \cdots, d\,;\ k\leqq\nu)$ converge and represent analytic functions $v_{k,j}(x)$ in some neighborhood of the origin independent of k and j, and that the condition (3.2) is satisfied for some $\nu\geqq 0$. We write the left-hand side of (3.2) in the form $(x_1\cdots x_d)^{\nu+1}g(x)$. It follows from (3.1), (3.2) and (3.3) that the formal sum $\tilde{v}_{\nu+1,1}$ satisfies the equation

$$P_\lambda(0, x'\,;x\cdot\partial+\vec{e}(\nu+1))\tilde{v}_{\nu+1,1}(x')=g(0, x') \tag{3.5}$$

where $\vec{e}=(1, \cdots, 1)$. We note that this equation has the same form as the equation (3.4). Hence, by the same arguments as above the formal sum $\tilde{v}_{\nu+1,1}(x')$

converges and is analytically continued to some $v_{\nu+1,1}(x')$. By the same arguments as for $\tilde{v}_{0,k}$ we can easily see that the formal sums $\tilde{v}_{\nu+1,k}$ ($k=1, \cdots, d$) converge and satisfy (3.2) with ν replaced by $\nu+1$.

Step 2. We shall give a meaning to the formal sum (3.1). For simplicity we set $t=x_1 \cdots x_d$. Let B_r ($r>0$) be a ball centered at the origin such that all $v_{j,k}$'s are defined and bounded on B_r. Let $\phi(s) \in C_0^\infty(\boldsymbol{R}^1)$ be such that $\phi \equiv 1$ in a neighborhood of the origin and $\phi \equiv 0$ for $|s| \geqq 1$. For $j=0, 1, 2, \cdots$, we define ρ_j by

$$\rho_j = \max_{x \in B_r} \sum_{k=1}^{d} |v_{j,k}(x)|\,, \tag{3.6}$$

and define $\phi_j(t) \equiv \phi_j(x_1 \cdots x_d)$ by $\phi_j(t) = \phi(te\rho_j^{2/j})$ in case $\rho_j^{1/j}$ is not bounded as $j \to \infty$ and, by $\phi_j(t) \equiv 1$ if otherwise. Now we define $w(x) \in C_0^\infty(\boldsymbol{R}^d)$ by

$$w(x) = \sum_{j=0}^{\infty} t^j \phi_j(t) \sum_{k=1}^{d} v_{j,k}(x)\,. \tag{3.7}$$

We note that if $\rho_j^{1/j}$ is bounded when $j \to \infty$ the right-hand side of (3.6) converges and is analytic in some neighborhood of the origin. It is equal to the original sum. If otherwise, noting that $|t| \leqq e^{-1}\rho_j^{-2/j}$ on the support of $\phi_j(t)$ we get, from (3.7),

$$\Big|\sum_{k,j} t^j \phi_j(t) v_{j,k}(x)\Big| \leqq \sum_j e^{-j} \rho_j^{-2} \Big|\sum_k v_{j,k}(x)\Big| \leqq \sum_j e^{-j} \rho_j^{-1} < \infty\,.$$

Hence the right-hand side of (3.7) converges on B_r. Moreover, by using the Cauchy's formula for $v_{j,k}$ we can easily show that it converges with respect to the C^∞ topology.

In order to prove that the function $P(x^\lambda w) - f(x)x^\lambda$ is flat, let us consider $f - P_\lambda w$ where $P_\lambda = x^{-\lambda} P x^\lambda$. Let ν be an integer. Then it follows from (3.7) that

$$\begin{aligned} f - P_\lambda w = \Big\{ f - P_\lambda \Big(\sum_{j=0}^{\nu} \sum_k t^j v_{j,k} \Big) \Big\} + P_\lambda \Big(\sum_{j=0}^{\nu} t^j (\phi_j(t) - 1) \sum_k v_{j,k} \Big) \\ + P_\lambda \Big(\sum_{j=\nu+1}^{\infty} t^j \phi_j(t) \sum_k v_{j,k} \Big). \end{aligned} \tag{3.8}$$

The first term in the right-hand side of (3.8) is $O(t^{\nu+1})$ by (3.2). The second term vanishes identically since $\phi_j(t) \equiv 1$ if $|t|$ is sufficiently small. The function in the parenthesis of the third term is $O(t^{\nu+1})$. Since the operator P_λ preserves the power of x in view of the definition of P we see that the third term is $O(t^{\nu+1})$. Therefore we have that $f - P_\lambda w = O(t^{\nu+1})$. Since ν is arbitrary we have proved Lemma 3.1.

REMARK 3.1. If we assume (C.2) instead of (A.3) then the assertion of Lemma 3.1 is true for the equation (2.5). The proof is done as follows.

We first construct a $u_0(x)\in C_0^\infty(\boldsymbol{R}^2)$ such that the Taylor expansion of $u_0(x)$ at the origin is equal to the formal solution $\tilde{u}$. This implies that the function $P_\lambda u_0-f$ is flat at the origin. Next we determine smooth functions $\tilde{v}_{j,1}(x_1)$ and $\tilde{v}_{j,2}(x_2)$ $(j=0, 1, \cdots)$ such that they are flat at the origin and that the formal sum $\tilde{v}(x)$ defined by (3.1) with $d=2$ satisfies that $P_\lambda\tilde{v}+P_\lambda u_0-f(x)=O((x_1x_2)^\nu)$ for $\nu=1, 2, \cdots$. Under the assumption (C.2) we can show that this yields a system of ordinary differential equations of Fuchs type for $\tilde{v}_{j,1}$ and $\tilde{v}_{j,2}$ $(j=0, 1, \cdots)$ with the inhomogeneous part, C^∞ and flat at the origin. Hence we can determine $\tilde{v}_{j,1}$ and $\tilde{v}_{j,2}$ as desired. Once we can determine $\tilde{v}(x)$ the proof of the remaining part is the same to that of Lemma 3.1.

Next we shall construct a fundamental solution $K(x;\zeta)$ for the operator P_λ in the Mellin's sense where ζ is a parameter. Let $\tilde{\Gamma}_0$ be the component of the set $\{\omega\in\boldsymbol{R}^d; p_m(\omega)\neq 0\}$ containing Γ_0. Then we consider the equation

$$(3.9)\qquad P_\lambda(x;x\cdot\partial)(x^{-\zeta}K(x;\zeta))\equiv P(x;x\cdot\partial+\lambda)(x^{-\zeta}K(x;\zeta))=x^{-\zeta}$$

where $-\zeta=\omega+i\xi\in\tilde{\Gamma}_0\setminus\{0\}+i\boldsymbol{R}^d$. We then have

LEMMA 3.2. *Suppose that the conditions* (A.1) *and* (A.2) *are satisfied. Then there exist* $\omega_0>0$ *and a neighborhood* V_0 *of the origin such that if* $(x,\zeta)\in X_0\equiv\{(x,\zeta); x\in V_0, \zeta\in-\tilde{\Gamma}_0\setminus\{0\}+i\boldsymbol{R}^d, |\mathrm{Re}\,\zeta|\geqq\omega_0\}$ *there exists a solution* $K(x;\zeta)$ *of Equation* (3.9). *Moreover, for each* $\alpha\in\boldsymbol{N}^d$ *the function* $\partial_x^\alpha K(x;\zeta)$ *is holomorphic and bounded when* $(x,\zeta)\in X_0$ *and* $\mathrm{Re}\,\zeta\in-\Gamma_2^\circ$ *where* $\Gamma_2\subset\tilde{\Gamma}_0$ *is any closed convex cone whose interior is not empty.*

PROOF OF LEMMA 3.2. We divide the proof into two steps.

Step 1. First we note that under the hyperbolicity (A.1) the cone $\tilde{\Gamma}_0$ is convex open cone. Let $p(\eta)$ be given by (2.1) and let Γ_1 $(\Gamma_2^\circ\supset\Gamma_1\supsetneqq\Gamma_0)$ be a closed convex cone in $\tilde{\Gamma}_0$ whose interior is not empty. Then we shall show that there exist constants $C_0>0$ and $C_1>0$ independent of η such that

$$(3.10)\qquad |p(\eta+\lambda)|\geqq C_0(|\eta+\lambda|+1)^{m-\sigma}$$

for all $\eta\in\Gamma_1\setminus\{0\}+i\boldsymbol{R}^d$ such that $|\mathrm{Re}\,\eta|\geqq C_1$. In order to prove this we write

$$(3.11)\qquad p(\eta+\lambda)=|\eta+\lambda|^m p_m(\eta+\lambda/|\eta+\lambda|)+p_\sigma(\eta+\lambda)$$

where $p_\sigma(\eta+\lambda)$ is the polynomial of $\eta+\lambda$ with degree$\leqq m-\sigma$ $(\leqq m-1)$. We set $t=|\mathrm{Re}\,\eta+\lambda|/|\eta+\lambda|$, $\omega=(\mathrm{Re}\,\eta+\lambda)/|\mathrm{Re}\,\eta+\lambda|$, $\xi=(\mathrm{Im}\,\eta+\lambda)/|\eta+\lambda|$. Then we have that $(\eta+\lambda)/|\eta+\lambda|=t\omega+i\xi$ and that $\omega\in\Gamma_2\setminus\{0\}$ for some sufficiently large $C_1>0$ since $\mathrm{Re}\,\eta\in\Gamma_1$ and $\Gamma_1\subset\Gamma_2^\circ$.

Recalling the definition of $\tilde{\Gamma}_0$ in the introduction of Lemma 3.2 and by using [Corollary 12.4.5; 4] we have that p_m is hyperbolic with respect to $\tilde{\Gamma}_0$. Hence we have (A.1) with Γ_0 replaced by $\tilde{\Gamma}_0$. Noting that p_m is homogeneous this implies that $p_m(\eta+\lambda/|\eta+\lambda|)\neq 0$ if η satisfies that $|t|\geqq t_0>0$ for some $t_0>0$. Hence we get (3.10) from (3.11) by taking C_1 sufficiently large. Therefore we assume that $|t|<t_0$ for small t_0. Similarly we may assume that ξ is in a small neighborhood of some ξ_0 such that $p_m(\xi_0)=0$.

Let $\omega\in\Gamma_2$. Then we factor $p_m(\tau\omega+\xi)$ as a polynomial of τ

$$p_m(\tau\omega+\xi) = p_m(\omega)\prod_1^m(\tau+\lambda_k(\omega;\xi)). \tag{3.12}$$

We note that $\lambda_k(\omega;\xi)$ is real-valued and continuous in ω and ξ in case $\omega\in\tilde{\Gamma}_0$ and ξ is real. In view of the definition of ω and ξ we shall estimate the term $|p_m(it\omega+\xi)|$ when ω and ξ range over the set $\{\omega;|\omega|=1, \omega\in\Gamma_2\}\times\{\xi;\varepsilon_1\leqq|\xi|\leqq 1\}$ where ε_1 is some positive constant. Let $\sigma=\sigma(\xi_0)$ be the multiplicity of the localization of p_m at ξ_0 and let $L_{\xi_0}(\omega)$ be the localization polynomial (cf. (A.2)). By the assumption (A.1) and by the well-known fact for hyperbolic polynomials the polynomial $L_{\xi_0}(\omega)$ is hyperbolic with respect to $\tilde{\Gamma}_0$ (cf. [Lemma 3.42; **1**]). Especially we have that $L_{\xi_0}(\omega)\neq 0$ for $\omega\in\tilde{\Gamma}_0$. Comparing this with the expression (3.12) we see that there exist at least $m-\sigma$ of $\lambda_k(\omega;\xi_0)$'s which do not vanish for $\omega\in\tilde{\Gamma}_0$. For the sake of simplicity let us assume this holds for $k=1,\cdots,m-\sigma$. By the continuity of $\lambda_k(\omega;\xi)$ the quantities $|\lambda_k(\omega;\xi)|$, $k=1,\cdots,m-\sigma$ are bounded from below by some constant $C(\xi_0)$ when ω ranges over the compact set $\omega\in\Gamma_2$, $|\omega|=1$ and ξ in some neighborhood $U(\xi_0)$ of ξ_0. On the other hand since $|p_m(\omega)|\geqq C_2$ for $\omega\in\Gamma_2$ with some $C_2>0$ we obtain, from (3.12) with $\tau=it$,

$$|p_m(it\omega+\xi)| \geqq C_2|t|^\sigma C(\xi_0)^{m-\sigma} \qquad \text{for all } \xi\in U(\xi_0) \text{ and all } \omega\in\Gamma_2,\ |\omega|=1.$$

Next we move ξ_0 in the compact set $\varepsilon_1\leqq|\xi_0|\leqq 1$ and make the same estimates. Since this set is compact we can cover it by the finite number of $U(\xi^j)$'s where $\varepsilon_1\leqq|\xi^j|\leqq 1$. Let C_3 be the smallest of all $C(\xi^j)$ and take t_0 so small that $t_0\leqq C_3$. Then we have

$$|p_m(it\omega+\xi)| \geqq C_2C_3^{m-\sigma}|t|^\sigma, \qquad \omega\in\Gamma_2,\ |\omega|=1,\ \varepsilon_1\leqq|\xi|\leqq 1$$

where σ is the largest multiplicity of $\sigma(\xi)$. In view of the definition of t, ω and ξ this implies (3.10).

Step 2. We wish to determine $K(x;\zeta)$ in the form

$$K(x;\zeta) = \sum_{\gamma\in\Gamma_0\cap N^d} K_\gamma x^\gamma/\gamma!. \tag{3.13}$$

Now we substitute the expansions of $K(x;\zeta)$ and $b_\beta(x)$, $b_\beta(x)=\sum_\nu b_{\beta,\nu}x^\nu/\nu!$ into (3.9) and compare the coefficients of $x^{\gamma-\zeta}$. This yields the recurrence formula

$$(3.14)\qquad p(\gamma-\zeta+\lambda)K_\gamma+\sum_{\substack{\gamma=\nu+\delta,\gamma\neq\delta\\|\beta|<m}} b_{\beta,\nu}K_\delta(\delta-\zeta+\lambda)^\beta\gamma!/(\nu!\,\delta!)=\delta_{\gamma,0}$$

where $\delta_{\gamma,0}$ denotes the Kronecker's delta. Since $\mathrm{Re}(\gamma-\zeta)\in\Gamma_1\setminus\{0\}$ and $\mathrm{Re}(\gamma-\zeta)\geqq\mathrm{Re}(-\zeta)$ by definition it follows from (3.10) that $p(\gamma-\zeta+\lambda)$ does not vanish if we take $|\mathrm{Re}\,\zeta|$ sufficiently large. Hence we can determine K_γ inductively by (3.14).

In order to show that the formal sum (3.13) converges we note that $|(\delta-\zeta+\lambda)^\beta|\leqq(|\gamma|+|\zeta|+|\lambda|)^{m-1}$. On the other hand, since $\gamma\geqq0$, $-\mathrm{Re}\,\zeta\geqq0$ we have that $|\gamma-\zeta|\geqq C_4(|\gamma|+|\zeta|)$ where $C_4>0$ is independent of γ and ζ. Hence, by (3.10) with $\eta=\gamma-\zeta$ and (3.14) we can easily show that the formal sum converges in some neighborhood V of the origin. Moreover we can easily see that the convergence is uniform with respect to ζ such that $\mathrm{Re}\,\zeta\in-\Gamma_1\setminus\{0\}$, $|\mathrm{Re}\,\zeta|\geqq\omega_0$ if we take $\omega_0>0$ sufficiently large. q.e.d.

REMARK 3.2. We cannot drop the assumptions (A.1) and (A.2) in Lemma 3.2 in general. In fact, if there exists a solution $K(x;\zeta)$ of (3.9) we have (3.14). By setting $\gamma=0$ in (3.14) we have that $p(-\zeta+\lambda)K_0=1$, which implies that $p(-\zeta+\lambda)\neq0$ for $\mathrm{Re}\,\zeta\in-\tilde{\Gamma}_0$, $|\mathrm{Re}\,\zeta|$ large, which is nothing but the hyperbolicity (A.1). Similarly it is not difficult to construct an operator P not satisfying (A.2) such that Equation (3.9) has a solution $K(x;\zeta)$ which is not uniformly bounded in ζ when $\mathrm{Re}\,\zeta\in-\tilde{\Gamma}_0$, $|\mathrm{Re}\,\zeta|$ large.

PROOF OF THEOREM 2.1. Let us take the neighborhood U of the origin so small that Lemmas 3.1 and 3.2 are valid, and let $w(x)$ be the function given by Lemma 3.1. We take a $\phi_0(x)\in C_0^\infty(\boldsymbol{R}^d)$ such that $\mathrm{supp}\,\phi_0\subset U$ and $\phi_0\equiv1$ in a neighborhood of the origin, and set $g(x)=\phi_0(x)(x^\lambda f(x)-P(x,x\cdot\partial)(x^\lambda w))$. Then the function $g(x)$ is flat on the hyperplanes $x_j=0$ $(j=1,\cdots,d)$, and has compact support.

Let $\hat{g}(\zeta)$ be the Mellin transform of $g(x)$;

$$(3.15)\qquad \hat{g}(\zeta)=\int_{\boldsymbol{R}_+^d}g(x)x^{\zeta-e}dx,\qquad e=(1,\cdots,1).$$

We can easily see that $\hat{g}(\zeta)$ is an entire function of ζ and rapidly decreasing as $\mathrm{Im}\,\zeta$ tends to infinity while $\mathrm{Re}\,\zeta$ remains bounded. The inversion formula is given by

$$(3.16)\qquad g(x)=(2\pi i)^{-d}\int_{\omega+i\boldsymbol{R}^d}\hat{g}(\zeta)x^{-\zeta}d\zeta,\qquad x_1>0,\cdots,x_d>0$$

where $\zeta=\omega+i\xi$ and ω is a fixed vector. These formulas follow from the well-known inversion formulas of the Fourier-Laplace transform by the change of variables $x\rightarrow e^{t}$.

Now it follows from the definition of $\tilde{\Gamma}_0$ that the cone $-\tilde{\Gamma}_0$ contains an open cone Γ_1 such that $\theta_j>0$, $j=1, \cdots, d$ for every $\theta=(\theta_1, \cdots, \theta_d)\in\Gamma_1$. Let $K(x;\zeta)$ be the function given by Lemma 3.2, and set

$$w_1(x)=(2\pi i)^{-d}\int_{\omega+iR^d}K(x;\zeta)\hat{g}(\zeta)x^{-\zeta}d\zeta \tag{3.17}$$

where $\zeta=\omega+i\xi$ and ω is a vector in $\Gamma_1\setminus\{0\}$ such that $|\omega|\geqq\omega_0$. Then it follows from Lemma 3.2 and (3.16) that $w_1(x)$ is smooth in the domain $\{x=(x_1, \cdots, x_d);$ $0<x_j<\varepsilon_0,\ j=1, \cdots, d\}$ for some $\varepsilon_0>0$, and satisfies the equation $Pw_1=g$ there. Moreover, by the Cauchy's integral formula we see that the integral (3.17) is independent of ω in $\Gamma_1\setminus\{0\}$, $|\omega|\geqq\omega_0$. Hence, by taking ω so large in Γ_1 we may assume that the factor $x^{-\zeta}$ in the integrand of (3.17) can be divided by arbitrarily large power of x. This implies that $w_1(x)$ is flat on $x_j=0$, $j=1, \cdots, d$. Hence the function $w_1(x)$ is smooth and satisfies that $Pw_1=g$ in $x_j\geqq0$, $j=1, \cdots, d$.

In order to construct the solution of the equation $Pw_1=g$ in the sector $\{x;\varepsilon(k)x_k\geqq0,\ k=1, \cdots, d\}$ where $\varepsilon(k)=1$ or -1, we make the linear change of variables $x_k\rightarrow-x_k$ for k such that $\varepsilon(k)=-1$. Since such transformation preserves the monomial $x_k\partial_k$ we can reduce the problem to the above case, and construct the solution in each sector. We split the neighborhood of the origin into the sum of such sectors, and construct the solution in each sector. Then, by patching up these solutions we obtain a smooth solution of the equation $Pw_1=g$ in some neighborhood of the origin because all these solutions are flat on $x_j=0$, $j=1, \cdots, d$. We denote this solution by $w_1(x)$, and set $v(x)=w(x)+x^{-\lambda}w_1(x)$. Then, in view of the definition of $g(x)$ and Lemma 3.1 we can easily see that the function $v(x)$ has the property as desired. This ends the proof of Theorem 2.1. q.e.d.

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