J. Math. Soc. Japan
Vol. 40, No. 1, 1988

A note on the Kahn-Priddy map

Dedicated to Professor Hirosi Toda on his 60th birthday

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(Received June 12, 1986)

§ 0. Introduction.

L_p denotes the infinite dimensional lens space mod a prime p. L_p^k stands for its k-skeleton with the usual cellular decomposition $L_p=S^1\cup e^2\cup\cdots\cup e^{2n-1}\cup e^{2n}\cup\cdots$. In particular L_2^k is the real projective space P^k. Let $\lambda_k: E^{2m+1}L_p^k\to S^{2m+1}$ for $m\geqq 1$ be a mapping. Then we adopt the following definition ([7], [11], [14]): λ_k for $2p-3\leqq k\leqq 2(m+1)(p-1)-2$ is called a Kahn-Priddy map if the functional $\mathfrak{P}^1(Sq^2)$-operation of λ_k is nontrivial (resp.). From the definition, the t-fold suspension $E^t\lambda_k$ for $t\geqq 0$ is also a Kahn-Priddy map. By abuse of notation, a mapping $E^t\lambda_k$ is regarded as an element of the cohomotopy group $\pi^c(E^cL_p^k)$ for $c=t+2m+1$. λ_k' stands for the restriction $\lambda_k|E^{2m+1}L_p^{k-1}$. A stable map $E^\infty\lambda_k$ is often written $\lambda_k: L_p^k\to S^0$.

The main purpose of the present note is to determine the orders $\#(E^t\lambda_{2n})$ and $\#(E^t\lambda_{2n}')$ completely. The problem determining the order of the Kahn-Priddy map was first posed by Nishida who obtained $\#(E^\infty\lambda_{2n})=\#(E^\infty\lambda_{2n}')=p^{[n/(p-1)]}$ for an odd prime p [15]. Here $[x]$ denotes the integral part of x. In the case $p=2$, the author [12] obtained $\#(E^t\lambda_{2n})=2^{\phi(2n)}$. Here $\phi(n)$ is the number of integers in the interval $[1, n]$ congruent to 0, 1, 2 or 4 mod 8.

Nishida's method is to use the algebraic K-group of L_p^k. Our method is to follow that of [12] of which the classical KO-group of P^k [1] is used. In the case of an odd prime p, it suffices to use the K-group of L_p^k [8]. To determine the infimum of the order of a Kahn-Priddy map, we shall use the d- or e-invariant [2]. To determine the supremum, we shall use the suspension order of the stunted space L_p^{2n}/L_p^{2p-4} [4].

Let $\rho: L_p^{2n-1}\to L_p^{2n-1}/L_p^{2n-2}=S^{2n-1}$ be the canonical map. Let $\alpha_s\in\pi_{2s(p-1)-1}(S^0)$ for an odd prime p be Adams-Toda's element such that $\#\alpha_s=p$ and $e_C(\alpha_s)\equiv -1/p \bmod 1$ [2]. Then we have the following

THEOREM 1. *Let p be an odd prime, $m\geqq 1$ and $p-1\leqq n\leqq (m+1)(p-1)-1$.*

This research was partially supported by Grant-in-Aid for Scientific Research (No. 61540026), Ministry of Education, Science and Culture.

Let $\lambda_{2n}: E^{2m+1}L_p^{2n}\to S^{2m+1}$ *be a Kahn-Priddy map and* $\lambda'_{2n}=\lambda_{2n}|E^{2m+1}L_p^{2n-1}$. *Then, for* $t\geqq 0$,

$$\#(E^t\lambda_{2n}) = \#(E^t\lambda'_{2n}) = p^{[n/(p-1)]}.$$

In particular the following relation holds if $n=s(p-1)$:

$$p^{s-1}\lambda'_{2n} \equiv x\alpha_s\rho \mod \mathrm{Ker}\, e_C$$

for some $x\not\equiv 0 \mod p$.

Let $\mu_s\in\pi_{8s+1}(S^0)$ be an element such that $\#\mu_s=2$ and $d_R(\mu_s)\equiv 1 \mod 2$ [**2**]. μ_0 is the Hopf map η. Notice that the suffix of μ_s is different from Adams' one. Let $\alpha_s\in\pi_{8s-1}(S^0)$ be an element such that $\#\alpha_s=2$ and $e'_R(\alpha_s)\equiv 1/2 \mod 1$ [**2**]. Then we have the following

THEOREM 2. *Let* $\lambda_{2n}: E^{2n+1}P^{2n}\to S^{2n+1}$ *be a Kahn-Priddy map and* $\lambda'_{2n}=\lambda_{2n}|E^{2n+1}P^{2n-1}$. *Then*

i) $\#\lambda_{2n} = \#(E\lambda_{2n}) = 2^{\phi(2n)}$.

ii) $\#\lambda'_{2n} = \begin{cases} 2^{\phi(2n-1)} & \text{if } n \text{ is odd,} \\ 2^{\phi(2n)} & \text{if } n \text{ is even.}\end{cases}$

In particular the following relations hold:

a) $2^{\phi(2n-2)}\lambda'_{2n} \equiv \mu_s\rho \mod \mathrm{Ker}\, d_R$ *if* $n=4s+1\geqq 1$.

b) $2^{\phi(2n-1)}\lambda'_{2n} \equiv \mu_s\eta^2\rho \mod \mathrm{Ker}\, e'_R$ *if* $n=4s+2\geqq 2$.

c) $2^{\phi(2n-1)}\lambda'_{2n} \equiv \alpha_s\rho \mod \mathrm{Ker}\, e'_R$ *if* $n=4s\geqq 4$.

We remark that the periodic family of elements of $\pi_{2n-1}(S^0)$ are recovered as a byproduct of our proof of the theorems.

This note consists of four sections and one appendix. § 1-§ 3 are devoted to proving the theorems. In § 4 we shall characterize a Kahn-Priddy map as a mapping of mod p Hopf invariant one. In the appendix we shall give a short proof of Toda's result about the stable order of P^{2n} [**16**].

§ 1. A K-theoretic characterization of the Kahn-Priddy map.

Let $i: L_p^{2n}\to L_p^{2n+1}$ be the inclusion. Let $\sigma\in\tilde{K}(L_p^{2n+1})$ be an element induced from the canonical complex line bundle over the complex projective space CP^n. Then, by Theorem 1, Lemmas 2.4 and 2.5 of [8], we have the following

PROPOSITION 1.1. *Let* $n=s(p-1)+r$, $0\leqq r<p-1$. *Then*

i) $\tilde{K}^{-1}(L_p^{2n}) = 0$.

ii) $i^*: \tilde{K}(L_p^{2n+1}) \to \tilde{K}(L_p^{2n})$ *is an isomorphism.*

iii) $\tilde{K}(L_p^{2n+1}) \approx (\boldsymbol{Z}_{p^{s+1}})^r + (\boldsymbol{Z}_{p^s})^{p-r-1}$,

where the first r *summands are generated by* $\sigma^1, \cdots, \sigma^r$ *and the rest are generated by* $\sigma^{r+1}, \cdots, \sigma^{p-1}$. *The ring structure is given by* $\sigma^p=-\sum_{i=1}^{p-1}\binom{p}{i}\sigma^i$, $\sigma^{n+1}=0$.

We denote by $M_p^n = S^{n-1} \cup_p e^n$ a $\boldsymbol{Z}_p$-Moore space and by $\rho' : L_p^{2n} \to L_p^{2n}/L_p^{2n-2} = M_p^{2n}$ the canonical map. Then, by Proposition 1.1, we have the following

LEMMA 1.2. *Let* $n = s(p-1)$ *for* $s \geqq 1$. *Then*

$$\mathrm{Im}\{\rho'^* : \tilde{K}(M_p^{2n}) \longrightarrow \tilde{K}(L_p^{2n})\} = \{p^{s-1}\sigma^{p-1}\} \approx \boldsymbol{Z}_p .$$

The p-component of $\pi_i(X)$ is written ${}^p\pi_i(X)$. If $p\alpha = 0$ for $\alpha \in \pi_{n-1}(S^k)$, we denote by $\bar{\alpha} \in \pi^k(M_p^n)$ an extension of α.

Hereafter we assume that p is an odd prime, unless otherwise stated. The following is well known: For $n \geqq 3$, ${}^p\pi_{i+n}(S^n)$ is 0 if $1 \leqq i < 2p-3$ or $i = 2p-2$ and ${}^p\pi_{n+2p-3}(S^n) = \{E^{n-3}\alpha_1\} \approx \boldsymbol{Z}_p$. So we have the following

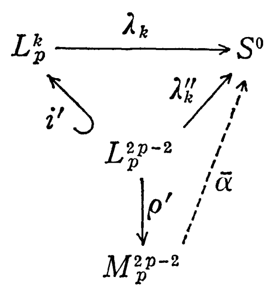
LEMMA 1.3. *Let* $n \geqq 3$. *Then*
i) $\pi^n(E^n L_p^{2p-4}) = 0$.
ii) $\pi^n(M_p^{n+2p-2}) = \{E^{n-3}\bar{\alpha}_1\} \approx \boldsymbol{Z}_p$.

If $p\alpha = 0$ for $\alpha \in \pi_i(S^0)$, $d_C(\bar{\alpha}) \equiv -pe_C(\alpha) \mod p$ by Proposition 12.3 of [**2**]. Then the following is well known.

LEMMA 1.4. *The following are equivalent if* $x \not\equiv 0 \mod p$:
i) $\alpha = x\alpha_1$,
ii) *The functional* $\mathfrak{P}^1$*-operation of* α *is nontrivial,*
iii) $d_C(\bar{\alpha}) \equiv x \mod p$, *namely,* $\bar{\alpha}^* : \tilde{K}(S^0) \to \tilde{K}(M_p^{2p-2}) \approx \boldsymbol{Z}_p$ *is onto.*

Let $\lambda_k : L_p^k \to S^0$ for $k \geqq 2p-2$ be a mapping and $\lambda_k'' = \lambda_k | L_p^{2p-2}$. Then, by Lemma 1.3, there exists an element $\alpha \in {}^p\pi_{2p-3}(S^0)$ such that the following diagram commutes:

(1.1)



By Lemma 1.4 and (1.1), λ_k is a Kahn-Priddy map if and only if $\alpha = x\alpha_1$. The following is a key to our approach.

LEMMA 1.5. *Let* $n = s(p-1) + r$, $0 \leqq r < p-1$ *and* $s \geqq 1$. *Let* $k = 2n$ *or* $2n+1$. *Then* $\lambda_k : L_p^k \to S^0$ *is a Kahn-Priddy map if and only if*

$$\mathrm{Im}\{\lambda_k^* : \tilde{K}(S^0) \longrightarrow \tilde{K}(L_p^k)\} = \{\sigma^{p-1} \mod p\tilde{K}(L_p^k)\} \approx \boldsymbol{Z}_{p^s} .$$

PROOF. Suppose that λ_k is a Kahn-Priddy map. Then, by (1.1), Lemmas

1.2 and 1.4, $\operatorname{Im}\lambda_k''^*=\operatorname{Im}\rho'^*=\{\sigma^{p-1}\}\approx \boldsymbol{Z}_p$. By Proposition 1.1, $i'^*:\tilde{K}(L_p^k)\to\tilde{K}(L_p^{2p-2})$ is onto. So we have $\operatorname{Im}\lambda_k^*=(i'^*)^{-1}(\operatorname{Im}\lambda_k''^*)=\{\sigma^{p-1} \bmod p\tilde{K}(L_p^k)\}$. Obviously the converse is true. This completes the proof.

§ 2. Proof of Theorem 1.

ι_X stands for the identity class of a space X. $\#\iota_{EX}$ is called the suspension order of X [16] and denoted by $|EX|$. The stable order of X is written $\|X\|=|E^\infty X|$.

By Proposition 4.2 of [19] and Lemma 2.7 of [4], we have the following

LEMMA 2.1. $|E(L_p^{2n}/L_p^{2k})| = |EL_p^{2(n-k)}| = p^{1+[(n-k-1)/(p-1)]}$.

By Lemma 2.10 of [8] and by Theorem 3.1 of [9], we have the following

LEMMA 2.2. $\tilde{K}(L_p^{2n}/L_p^{2k}) \approx \tilde{K}(L_p^{2(n-k)})$.

By Proposition 1.1, Lemmas 2.1 and 2.2, we have the following

LEMMA 2.3.

$$|E^{t+1}(L_p^{2n}/L_p^{2k})| = |E^{t+1}L_p^{2(n-k)}| = p^{1+[(n-k-1)/(p-1)]} \quad \textit{for} \quad t\geqq 0.$$

REMARK. The 2-primary version of Lemma 2.3 is not valid. By use of Proposition 5.1 and Theorem 5.6 of [13], we have $\|P^8/P^2\|=16$. In general, by Corollary 2.2 of [10] and by Corollary 3 to Theorem 4.3 of [16], we have $\|P^{2n}/P^{2k}\| \,\big|\, 2\|P^{2(n-k)}\|$.

Now we shall prove Theorem 1. Let $n=s(p-1)+r$, $0\leqq r<p-1$ and $s\geqq 1$. Let $c=2m+1$ for $m\geqq 1$ and $\rho'': L_p^{2n}\to L_p^{2n}/L_p^{2p-4}$ be the canonical map. Then, by Lemma 1.3, there exists a mapping $\tilde{\lambda}_{2n}: E^c(L_p^{2n}/L_p^{2p-4})\to S^c$ such that the following diagram commutes:

$$\begin{array}{ccc} E^cL_p^{2n} & \xrightarrow{\lambda_{2n}} & S^c \\ & {\scriptstyle E^c\rho''}\searrow \quad \nearrow{\scriptstyle \tilde{\lambda}_{2n}} & \\ & E^c(L_p^{2n}/L_p^{2p-4}). & \end{array}$$

So, by Lemma 2.3,

$$\#(E^t\lambda_{2n}) \,\Big|\, \#(E^t\tilde{\lambda}_{2n}) \,\Big|\, |E^{t+c}(L_p^{2n}/L_p^{2p-4})| = p^{1+[(n-p+1)/(p-1)]} = p^s.$$

By Lemma 1.5, $p^s \,\big|\, \#(E^\infty\lambda_{2n}) \,\big|\, \#(E^t\lambda_{2n})$. Hence $\#(E^t\lambda_{2n})=p^s$.

Next we shall determine the order of $E^t\lambda'_{2n}$. By Lemma 1.5 and the above result,

$$p^{[(n-1)/(p-1)]}\,\Big|\,\#(E^{\infty}\lambda'_{2n})\,\Big|\,\#(E^{t}\lambda'_{2n})\,\Big|\,\#(E^{t}\lambda_{2n})=p^{s}.$$

Therefore $\#(E^t\lambda'_{2n})=p^s$ if $r>0$ and $\#(E^t\lambda'_{2n})=p^{s-1}$ or p^s if $r=0$.

It remains to prove $\#\lambda'_{2n}=p^s$ for $\lambda'_{2n}=\lambda_{2n}\Big| L_p^{2n-1}$ with $n=s(p-1)$. We consider the natural map between the cofibre sequences

$$\text{(2.1)}\qquad \begin{array}{ccccc} L_p^{2n-2} & \xrightarrow{\ i_0\ } & L_p^{2n-1} & \xrightarrow{\ \rho\ } & S^{2n-1} \\ \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_1} \\ L_p^{2n-2} & \xrightarrow{\ i'\ } & L_p^{2n} & \xrightarrow{\ \rho'\ } & M_p^{2n}. \end{array}$$

LEMMA 2.4. *Let $n=s(p-1)$ for $s\geqq 1$. Then there exists an element $\bar{\alpha}'_s\in \pi^0(M_p^{2n})$ such that $p^{s-1}\lambda_{2n}=\bar{\alpha}'_s\rho'$, $d_C(\bar{\alpha}'_s)\equiv x\not\equiv 0 \bmod p$ and $\#\bar{\alpha}'_s=p$.*

PROOF. Let $\lambda''_{2n}=\lambda_{2n}\,|\,L_p^{2n-2}$. Then $\#\lambda_{2n}=p^s$ and $\#\lambda''_{2n}=p^{s-1}$. So, by the lower sequence of (2.1), there exists an element $\bar{\alpha}'_s$ satisfying the first relation. By Lemmas 1.2 and 1.5, $\operatorname{Im}\rho'^*=\operatorname{Im}(p^{s-1}\lambda_{2n})^*\approx \boldsymbol{Z}_p$. Therefore we have the second. The third is obvious. This completes the proof.

We define an element $\alpha'_s\in\pi_{2n-1}(S^0)$ by $\alpha'_s=\bar{\alpha}'_s\circ i_1$. Then, by Proposition 12.3 of [2], $e_C(\alpha'_s)\equiv-(1/p)d_C(\bar{\alpha}'_s)\equiv -x/p \bmod 1$, and so $\#\alpha'_s=p$.

PROPOSITION 2.5. *Let $n=s(p-1)$ for $s\geqq 1$. Then*

$$p^{s-1}\lambda'_{2n}=\alpha'_s\rho \quad \textit{and} \quad \#\lambda'_{2n}=p^s.$$

PROOF. By (2.1) and Lemma 2.4, $p^{s-1}\lambda'_{2n}=(\bar{\alpha}'_s\rho')\circ i=\alpha'_s\rho$. $e_C(p^{s-1}\lambda'_{2n})$ is well-defined, and by Proposition 3.2. (c) of [2],

$$e_C(p^{s-1}\lambda'_{2n})=e_C(\alpha'_s\rho)=d_C(E\rho)e_C(\alpha'_s)\equiv \pm x/p \bmod 1.$$

For $(E\rho)^*:\tilde{K}^{-1}(S^{2n-1})\to\tilde{K}^{-1}(L_p^{2n-1})\approx\boldsymbol{Z}$ is an isomorphism by Proposition 1.1. This completes the proof.

From the definition, α'_s coincides with $x\alpha_s$ up to $\operatorname{Ker} e_C$. This completes the proof of Theorem 1.

§3. Proof of Theorem 2.

The argument in this section is based on the following theorems owing to Adams [**1**] and Toda [**16**] respectively.

THEOREM A. *$\widetilde{KO}(P^n)\approx\boldsymbol{Z}_{2^{\phi(n)}}$ and it is generated by the stable canonical line bundle ξ over P^n.*

THEOREM B. $\|P^{2n}\| = 2^{\phi(2n)}$.

A 2-primary version of Lemma 1.5 is the following (Lemma 2.1 of [**12**])

LEMMA 3.1. *$\lambda_n: P^n \to S^0$ is a Kahn-Priddy map if and only if $\lambda_n^*: \widetilde{KO}(S^0) \to \widetilde{KO}(P^n)$ is onto.*

By Lemma 3.1, $\#\lambda_{2n} = 2^{\phi(2n)}$ and $\#\lambda'_{2n} = 2^{\phi(2n-1)}$ or $2^{\phi(2n)}$. If $n \equiv 3 \bmod 4$, $\phi(2n) = \phi(2n-1)$. So we have $\#\lambda'_{2n} = 2^{\phi(2n-1)}$ in this case. If $n \equiv 0$, 1 or 2 mod 4, $\phi(2n) = \phi(2n-1)+1$. Furthermore, $\phi(2n-1) = \phi(2n-2)+1$ if $n \equiv 1 \bmod 4$ and $\phi(2n-1) = \phi(2n-2)$ if n is even. Therefore, by use of (2.1) for $p=2$, we have the following

LEMMA 3.2. i) *Let $n \equiv 1 \bmod 4$. Then*

$$\mathrm{Im}\{\rho'^*: \widetilde{KO}(M_2^{2n}) \longrightarrow \widetilde{KO}(P^{2n})\} = \{2^{\phi(2n-2)}\xi\} \approx \boldsymbol{Z}_4$$

and

$$\mathrm{Im}\{\rho^*: \widetilde{KO}(S^{2n-1}) \longrightarrow \widetilde{KO}(P^{2n-1})\} = \{2^{\phi(2n-2)}\xi\} \approx \boldsymbol{Z}_2.$$

ii) *Let n be even. Then*

$$\mathrm{Im}\{\rho'^*: \widetilde{KO}(M_2^{2n}) \longrightarrow \widetilde{KO}(P^{2n})\} = \{2^{\phi(2n-2)}\xi\} \approx \boldsymbol{Z}_2.$$

By Lemmas 3.1, 3.2 and by the parallel argument to the one in the proof of Lemma 2.4, we have the following lemmas.

LEMMA 3.3. *Let $n=4s+1$ for $s \geqq 0$. Then there exists an element $\bar{\mu}'_s \in \pi^0(M_2^{2n})$ such that $2^{\phi(2n-2)}\lambda_{2n} = \bar{\mu}'_s\rho'$, $d_R(\bar{\mu}'_s) \equiv \pm 1 \bmod 4$ and $\#\bar{\mu}'_s = 4$.*

LEMMA 3.4. *Let $n=4s$ ($n=4s-2$, resp.) for $s \geqq 1$. Then there exists an element $\bar{\alpha}'_s$ ($\bar{\beta}'_s$) of $\pi^0(M_2^{2n})$ such that $2^{\phi(2n-2)}\lambda_{2n} = \bar{\alpha}'_s\rho'$ ($2^{\phi(2n-2)}\lambda_{2n} = \bar{\beta}'_s\rho'$) and $d_R(\bar{\alpha}'_s) \equiv 1 \bmod 2$ ($d_R(\bar{\beta}'_s) \equiv 1 \bmod 2$, resp.).*

REMARK 1. Lemma 3.4 for $s=1$ is obtained from Proposition 2.1 and Theorem 4.6 of [**13**].

We define an element $\mu'_s \in \pi_{2n-1}(S^0)$ by $\mu'_s = \bar{\mu}'_s \circ i_1$. Then $\#\mu'_s = 2$ and $2^{\phi(2n-2)}\lambda'_{2n} = \mu'_s\rho$. By Lemmas 3.1 and 3.2, $d_R(\mu'_s) \equiv 1 \bmod 2$. This leads us to the following

PROPOSITION 3.5. *Let $n=4s+1$ for $s \geqq 0$. Then*

$$2^{\phi(2n-2)}\lambda'_{2n} = \mu'_s\rho \quad \text{and} \quad \#\lambda'_{2n} = 2^{\phi(2n-1)}.$$

We define an element α'_s (β'_s) by $\alpha'_s = \bar{\alpha}'_s \circ i_1$ ($\beta'_s = \bar{\beta}'_s \circ i_1$, resp.). Then $e'_R(\alpha'_s) \equiv e'_R(\beta'_s) \equiv 1/2 \bmod 1$ and $\#\alpha'_s = \#\beta'_s = 2$.

PROPOSITION 3.6. *Let $n=4s$ ($n=4s-2$, resp.) for $s \geqq 1$. Then*

$$2^{\phi(2n-2)}\lambda'_{2n} = \alpha'_s\rho \qquad (2^{\phi(2n-2)}\lambda'_{2n} = \beta'_s\rho,\ resp.)$$

and

$$\#\lambda'_{2n} = 2^{\phi(2n)}.$$

PROOF. The first is a direct consequence of Lemma 3.4.

By the proof of Theorem 1. i) of [3], we have a split monomorphism:

$$\begin{array}{ccc} \widetilde{KO}^{-1}(S^{2n-1}) & \xrightarrow{(E\rho)^*} & \widetilde{KO}^{-1}(P^{2n-1}) . \\ \wr\wr & & \wr\wr \\ \boldsymbol{Z} & & \boldsymbol{Z}+\boldsymbol{Z}_2 \end{array}$$

Therefore $d_R(E\rho)=\pm 1$ and $e'_R(\alpha'_s\rho)=d_R(E\rho)e'_R(\alpha'_s)\equiv 1/2 \mod 1$ ($e'_R(\beta'_s\rho)\equiv 1/2 \mod 1$, resp.). This completes the proof.

From the definition, $\mu'_s\equiv\mu_s \mod \operatorname{Ker} d_R$, $\alpha'_s\equiv\alpha_s \mod \operatorname{Ker} e'_R$ and $\beta'_s\equiv\mu_{s-1}\eta^2 \mod \operatorname{Ker} e'_R$ (cf. Proposition 12.17 of [2]). This completes the proof of Theorem 2.

REMARK 2. Let $n=4s-2$ or $4s$ for $s\geqq 1$. Then, by Theorems 1.5 and 1.6 of [2] and by the Adams conjecture, $\operatorname{Im} J=J\pi_{2n-1}(SO)=\{j_{2n-1}\}\approx\boldsymbol{Z}_{m(n)}$ and $e'_R:\pi_{2n-1}(S^0)\rightarrow\boldsymbol{Z}_{m(n)}$ is a split epimorphism such that $\operatorname{Im}(e'_R J)\approx\boldsymbol{Z}_{m(n)}$. So β'_s and α'_s can be chosen as $(m(n)/2)j_{2n-1}$ respectively.

EXAMPLE. i) Define a mapping $g_n: E^nP^{n-1}\rightarrow S^n$ as the adjoint of a composition of natural maps ([7], [12])

$$P^{n-1} \longrightarrow SO(n) \longrightarrow \Omega^n S^n .$$

Then $g'_n=g_n|E^nP^{n-2}=Eg_{n-1}$. So, by Theorem 2, $\#(Eg_{2n})=2^{\phi(2n-1)}$ or $2^{\phi(2n)}$ according as n is odd or even. This improves the result of the last example of [12].

ii) Let $f_n: E^nP^{n-1}\rightarrow S^n$ be an attaching map in the symmetric square of S^n ([6], [12]). Then we have $f'_n=f_n|E^nP^{n-2}\simeq\pm Ef_{n-1}\circ E^2\varepsilon$, where ε denotes a self homotopy equivalence of $E^{n-2}P^{n-2}$. So, by i), $\#(Ef_{2n})=\#(Eg_{2n})$.

iii) Let $h_m: E^cL_p^{c(p-1)-2}\rightarrow S^c$ for $c=2m+1\geqq 3$ be the mapping in Lemma 8.2 of [18]. Then it is a Kahn-Priddy map, and by Theorem 1, $\#h_m=p^m$.

§4. The mod p Hopf invariant of the Kahn-Priddy map.

In this section we shall consider an odd primary version of Theorem 1.2 of [12]. For a cohomology operation θ, we denote by θ_λ the functional θ-operation of a mapping λ. By the ring structure of $H^*(L_p^{2n};\boldsymbol{Z}_p)$, by the properties of the reduced power operation $\mathfrak{P}^i$ and the Bockstein operation Δ and by the Adem relation, we have the following (cf. Proof of Theorem 3 of [11]).

LEMMA 4.1. *Let $\lambda: E^{2m+1}L_p^{2n}\to S^{2m+1}$ for $p-1\leqq n\leqq(m+1)(p-1)-1$ be a mapping and i a positive integer with $i(p-1)\leqq n$. Then the nontriviality of $\mathfrak{P}_\lambda^i(Sq_\lambda^{2i})$ or $(\varDelta\mathfrak{P}^i)_\lambda(Sq_\lambda^{2i+1})$ for some i implies that for all i (resp.).*

From the definition and Lemma 4.1, $\lambda: E^{2m+1}L_p^{2n}\to S^{2m+1}$ is a Kahn-Priddy map if and only if $\mathfrak{P}_\lambda^m\neq 0$.

We denote by $Q_2^{2m-1}=\Omega(\Omega^2S^{2m+1}, S^{2m-1})$ the homotopy fiber of the inclusion $S^{2m-1}\to\Omega^2S^{2m+1}$. For a finite CW-complex K, we consider the following exact sequence [**17**]:

$$(4.1)\qquad \cdots\longrightarrow\pi^{2m-1}(EK)\xrightarrow{E^2}\pi^{2m+1}(E^3K)\xrightarrow{H^{(2)}}[K, Q_2^{2m-1}]\longrightarrow\cdots.$$

LEMMA 4.2. *Suppose $\mathfrak{P}_\lambda^m\neq 0$ for $\lambda\in\pi^{2m+1}(E^3K)$. Then*

$$H^{(2)}(\lambda)\neq 0.$$

PROOF. We assume $H^{(2)}(\lambda)=0$. Then, by (4.1), there exists an element $\theta\in\pi^{2m-1}(EK)$ such that $\lambda=E^2\theta$. $\mathfrak{P}^m$ commutes with the cohomology suspension isomorphism and $\mathfrak{P}^mH^{2m-1}(\ ;\boldsymbol{Z}_p)=0$. Therefore we have $\mathfrak{P}_\lambda^m=0$. This completes the proof.

LEMMA 4.3. *Let $K=E^{2m-2}L_p^{2m(p-1)}$. Suppose $H^{(2)}(\lambda)\neq 0$ for $\lambda\in\pi^{2m+1}(E^3K)$. Then $\mathfrak{P}_\lambda^m\neq 0$.*

PROOF. Let $\lambda': E^3K\to S^{2m+1}$ be a Kahn-Priddy map. Then, by Lemmas 4.1 and 4.2, $H^{(2)}(\lambda')\neq 0$. Since Q_2^{2m-1} is p-equivalent to M_p^{2mp-2} in $\dim\leqq 4mp-6$ [**17**], $[K, Q_2^{2m-1}]\approx[M_p^{2mp-2}, M_p^{2mp-2}]\approx\boldsymbol{Z}_p$. So there exists an integer $a\not\equiv 0 \bmod p$ such that $H^{(2)}(\lambda)=aH^{(2)}(\lambda')$. By (4.1), there exists an element $\theta\in\pi^{2m-1}(EK)$ such that $\lambda=a\lambda'+E^2\theta$. Therefore $\mathfrak{P}_\lambda^m=a\mathfrak{P}_{\lambda'}^m+\mathfrak{P}_{E^2\theta}^m=a\mathfrak{P}_{\lambda'}^m\neq 0$. This completes the proof.

A mod p Hopf homomorphism $H_p: {}^p\pi^{2m+1}(E^3K)\to{}^p\pi^{2mp+1}(E^3K)$ is defined as a composition [**17**]

$${}^p\pi^{2m+1}(E^3K)\xrightarrow{H^{(2)}}{}^p[K, Q_2^{2m-1}]\xrightarrow{I}{}^p\pi^{2mp+1}(E^3K).$$

If $K=E^{2m-2}L_p^{2m(p-1)}$, $\pi^{2mp+1}(E^3K)\approx\boldsymbol{Z}_p$ and I is an isomorphism by (2.5) and (2.7) of [**17**]. Hence, by Lemmas 4.2 and 4.3, we have the following

PROPOSITION 4.4. *$\lambda: E^{2m+1}L_p^{2m(p-1)}\to S^{2m+1}$ for $m\geqq 1$ is a Kahn-Priddy map if and only if $H_p(\lambda)\neq 0$.*

REMARK. By the parallel argument, precisely, by use of the EHP-sequence and Sq_λ^{2m+1} for $\lambda: E^{2m+1}P^{2m}\to S^{2m+1}$, we can directly prove Theorem 1.2 of [**12**].

Appendix.

In this appendix we shall give a short proof of Theorem B in §3. For a CW-pair (Y, B), we consider the canonical cofibre sequence

(A1) $$B \xrightarrow{i} Y \xrightarrow{p} Y/B \xrightarrow{\gamma} EB \xrightarrow{Ei} \cdots.$$

By Theorem 1.2 of [**16**] and its proof, we have the following

LEMMA A1. i) $|EY| \Big| \#(Ei)\#(Ep) \Big| |EB| \#(Ep) \Big| |EB| |E(Y/B)|$.

ii) $$\|Y\| \Big| \#i\#p \Big| \|B\| \#p \Big| \|B\| \|Y/B\|.$$

We denote by $\iota'_n = \iota_X$ for $X = P^n$. Let $i: S^1 \to P^2$ and $p: P^2 \to S^2$ be the canonical maps. Then the following is Theorem 2.3 of [**16**].

LEMMA A2. $\|P^2\| = 4$ *and* $2\iota'_2 = i\eta p$.

We consider (A1) for a pair (P^{2n+2}, P^{2n}):

(A1)′ $$P^{2n} \xrightarrow{i'} P^{2n+2} \xrightarrow{p'} E^{2n}P^2 \xrightarrow{\gamma'} EP^{2n} \longrightarrow \cdots.$$

Let $p_{2n+2} = E^{2n}p \circ p' : P^{2n+2} \to S^{2n+2}$. Then it is easy to show the following: $\pi^{2n+2}(P^{2n+2}) = \{p_{2n+2}\} \approx \boldsymbol{Z}_2$ and $\pi^{2n+1}(P^{2n+2}) = \{\eta p_{2n+2}\} \approx 0$ or $\boldsymbol{Z}_2$ according as n is odd or even. So, by (A1)′ and Lemma A2, we have the following

LEMMA A3. i) *If n is odd,* $\#p' = 2$.
ii) *If n is even,* $\#p' = 4$ *and* $2p' = i\eta p_{2n+2}$.

By Lemmas A1, A2, A3, we have

(A2) $$\|P^4\| \Big| 8.$$

By Theorem 2.5 of [**16**], we have

(A3) $$\|P^6\| \Big| 8.$$

REMARK 1. By inspecting the computations of §2, 3 of [**13**], we can also show that $\|P^6\| = 8$.

The following is Corollary to Theorem 2.8 of [**16**].

THEOREM A4. $$\|P^{2n}\| \Big| 2^{\phi(2n)}.$$

PROOF. By Lemma A2, (A2) and (A3), the assertion is true for $n \leqq 3$. We inductively assume that the assertion is true for $n \geqq 3$.

i) The case $n \equiv 1 \bmod 2$ or $n \equiv 0 \bmod 4$. Let $a = 0$ or 1 according as n is odd or even. Then, by Lemmas A1 and A3, $\|P^{2n+2}\| \big| \|P^{2n}\| \# p' = 2^{\phi(2n)} 2^{1+a} = 2^{\phi(2n+2)}$.

ii) The case $n \equiv 2 \bmod 4$. We consider (A1) for a pair (P^{2n+2}, P^{2n-4}):

$$P^{2n-4} \xrightarrow{i''} P^{2n+2} \xrightarrow{p''} P^{2n+2}/P^{2n-4}.$$

By the James periodicity [5], $P^{2n+2}/P^{2n-5} = E^{2n-4}(P^6 \vee S^0)$ if $n \equiv 2 \bmod 4$. So we have $P^{2n+2}/P^{2n-4} = E^{2n-4}P^6$. Therefore, by Lemma A1 and (A3), $\|P^{2n+2}\| \big| \|P^{2n-4}\| \|P^6\| = 2^{\phi(2n-4)+3} = 2^{\phi(2n+2)}$. Consequently the induction is complete.

By Theorem A of § 3, $2^{\phi(2n)} \big| \|P^{2n}\|$. So, by Theorem A4, $\|P^{2n}\| = 2^{\phi(2n)}$. This completes the proof of Theorem B.

REMARK 2. i) By the similar method to the above, we can prove Lemma 2.3 for $t=\infty$.

ii) The argument in this appendix still holds in the unstable case. We have $|EP^2| \big| 4$ and $|EP^4| \big| 8$. So we have $|E^kP^{2n}| \big| 2^{\phi(2n)}$ for all n if $|E^kP^6| \big| 8$ for some $k \geqq 1$. Since $|E^6P^6| \big| 8$ by (A3), we have $|E^6P^{2n}| \big| 2^{\phi(2n)}$.

CONJECTURE. $|EP^{2n}| = 2^{\phi(2n)}$ *for all* n.

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