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Existence of a non-inductive linear form on certain solvable Lie algebras

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Introduction.

Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{g}^*$ its dual vector space. Given $f \in \mathfrak{g}^*$, we set

$$\mathfrak{g}(f) = \{x \in \mathfrak{g}\,;\ f([x, y]) = 0 \quad \text{for all } y \in \mathfrak{g}\}.$$

Then, $\mathfrak{g}(f)$ acts naturally on $\mathfrak{g}/\mathfrak{g}(f)$. We say that f is *inductive* if for every $x \in \mathfrak{g}(f)$, the operator $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}(f)} x$ is nilpotent. So, if f is inductive, the linear Lie algebra $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}(f)}\mathfrak{g}(f)$ is nilpotent by Engel's theorem. We refer the reader to the papers Poguntke [**6**, Lemma 2] and Tauvel [**10**, Lemme 3.1] for various equivalent conditions for the inductivity of linear forms (we note that although the base field k is assumed to be algebraically closed throughout [**10**], the proof of Lemme 3.1 in that paper still works for $k=\boldsymbol{R}$).

Now let $\mathfrak{g}$ be exponential and $G=\exp\mathfrak{g}$ the corresponding connected and simply connected Lie group. Denote by $\hat{G}$ the equivalence classes of irreducible unitary representations of G with the Fell topology. We equip the finite dimensional vector space $\mathfrak{g}^*$ with the natural topology and the coadjoint orbit space $\mathfrak{g}^*/G$ with the quotient topology. Then, one knows that the Kirillov-Bernat mapping $\rho: \mathfrak{g}^*/G \to \hat{G}$ is a continuous bijection and it is a long-standing conjecture that ρ is a homeomorphism. Among several works toward this conjecture (cf. Fujiwara [**2**] and its Introduction), Boidol [**1**] made inductive linear forms play a significant role as follows.

THEOREM (Boidol). *Let $G=\exp\mathfrak{g}$ be an exponential Lie group. If every linear form on $\mathfrak{g}$ is inductive, ρ is a homeomorphism.*

Thus there arises a natural question: to what extent does the above Boidol's theorem cover the exponential Lie groups? This motivated the present work and the purpose of this note is to provide a class of completely solvable (hence exponential) Lie algebras $\mathfrak{s}$ on which there is always a non-inductive linear form. So, the Boidol's theorem is not applicable for the solvable Lie groups $S=\exp\mathfrak{s}$. Furthermore, by a theorem of Poguntke [**6**, Theorem 10] (for our S, Theorem 3 in [**5**] suffices), the involutory Banach algebra $L^1(S)$ is not symmetric (we refer the reader to e. g. [5] for the definition of the symmetry of L^1-algebras).

We organize this note as follows. In § 1 are given sufficient conditions for the existence of a non-inductive linear form on certain completely solvable Lie algebras. § 2 is devoted to the study of the case of Iwasawa subalgebras of semisimple Lie algebras. Our result (Theorem 2.7) says that there is a non-inductive linear form on an Iwasawa subalgebra $\mathfrak{s}$ of a semisimple Lie algebra $\mathfrak{g}$ if and only if the real rank of some simple component of $\mathfrak{g}$ is at least two. A similar result for normal j-algebras introduced by Pyatetskii-Shapiro [7] will be established in § 3. In § 4, we give an example common to §§ 2 and 3.

This work grew out of discussions with Professor Hidénori Fujiwara, to whom I wish to express my thanks for evoking an interest in the present topic. I also thank Professors Pierre Eymard, Michel Duflo and Michihiko Hashizume for instructive conversations.

§ 1. Sufficient conditions for the existence of a non-inductive linear form.

Let $\mathfrak{s}$ be a completely solvable Lie algebra. We assume that $\mathfrak{s}$ can be decomposed as a semidirect product $\mathfrak{s}=\mathfrak{n}\rtimes\mathfrak{a}$, where $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{s}$, $\mathfrak{a}$ an abelian subalgebra of $\mathfrak{s}$ and $\mathfrak{a}$ acts on $\mathfrak{n}$ by diagonalizable derivations. For $\alpha\in\mathfrak{a}^*$, we set

$$\mathfrak{n}_\alpha = \{x\in\mathfrak{n};\ [a, x]=\alpha(a)x \quad \text{for all } a\in\mathfrak{a}\}. \tag{1.1}$$

Then, there is a finite subset Δ in $\mathfrak{a}^*$ such that $\mathfrak{n}_\alpha\neq\{0\}$ for $\alpha\in\Delta$ and $\mathfrak{n}=\sum_{\alpha\in\Delta}\mathfrak{n}_\alpha$. By the decomposition $\mathfrak{s}=\mathfrak{a}+\sum_{\alpha\in\Delta}\mathfrak{n}_\alpha$, $\mathfrak{s}^*$ is naturally identified with $\mathfrak{a}^*+\sum_{\alpha\in\Delta}\mathfrak{n}_\alpha^*$.

THEOREM 1.1. *Suppose that there exist α, $\beta\in\Delta$ such that the following three conditions are satisfied:*

(i) $\alpha+\beta\in\Delta$.

(ii) $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta]=\mathfrak{n}_{\alpha+\beta}$.

(iii) *There is $a_0\in\mathfrak{a}$ such that $\alpha(a_0)\neq 0$, $(\alpha+\beta)(a_0)=0$.*

Then, any non-zero linear form $f\in\mathfrak{n}_{\alpha+\beta}^*\hookrightarrow\mathfrak{s}^*$ is non-inductive.

EXAMPLE 1.2. Let $\mathfrak{s}_4$ denote the four dimensional Lie algebra usually called the split oscillator. $\mathfrak{s}_4$ has the basis a, x, y, z such that

$$[a, x]=-x,\quad [a, y]=y,\quad [x, y]=z\,,$$

other brackets being zero or deduced by skew-symmetry. Then, with $\mathfrak{a}=\boldsymbol{R}a$ and $\mathfrak{n}=\boldsymbol{R}x+\boldsymbol{R}y+\boldsymbol{R}z$, we obtain $\mathfrak{s}_4=\mathfrak{n}\rtimes\mathfrak{a}$. Define $\alpha\in\mathfrak{a}^*$ by $\alpha(a)=1$. Clearly we have $\Delta=\{-\alpha, \alpha, 0\}$. The three conditions (i)$\sim$(iii) in Theorem 1.1 are satisfied with $\beta=-\alpha$. So, denoting by a^*, x^*, y^*, z^* the dual basis, we see that z^* is non-inductive.

COROLLARY 1.3. *Suppose that there exist linearly independent* $\alpha, \beta \in \Delta$ *such that the conditions* (i), (ii) *in Theorem* 1.1 *are satisfied. Then, any non-zero linear form* $f \in \mathfrak{n}_{\alpha+\beta}^*$ *is non-inductive.*

PROOF OF THEOREM 1.1. Let $y \in \mathfrak{s}$ be arbitrary and express y as $y = a + \sum_{\gamma \in \Delta} y_\gamma$, where $a \in \mathfrak{a}$ and $y_\gamma \in \mathfrak{n}_\gamma$. Let $a_0 \in \mathfrak{a}$ be as in (iii) of the statement of the theorem. Then, $[a_0, y] = \sum \gamma(a_0) y_\gamma$, so that

$$f([a_0, y]) = (\alpha+\beta)(a_0) f(y_{\alpha+\beta}) = 0 .$$

This implies $a_0 \in \mathfrak{s}(f)$. Furthermore, if $\mathfrak{n}_\alpha \subset \mathfrak{s}(f)$, then we would have $f([x, z]) = 0$ for all $x \in \mathfrak{n}_\alpha$ and $z \in \mathfrak{n}_\beta$. By (ii), this contradicts the assumption that $f \neq 0$ on $\mathfrak{n}_{\alpha+\beta}$. Hence $\mathfrak{n}_\alpha \not\subset \mathfrak{s}(f)$. Therefore, the operator $\mathrm{ad}_{\mathfrak{s}/\mathfrak{s}(f)} a_0$ is not nilpotent, because $\alpha(a_0) \neq 0$. Q.E.D.

§2. The case of Iwasawa subalgebras of semisimple Lie algebras.

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition with the associated Cartan involution θ. Here $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$ (resp. -1)-eigenspace of θ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We denote by Λ the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. For every $\alpha \in \Lambda$, the restricted root subspace corresponding to α is written as $\mathfrak{g}_\alpha$. Fix an order in Λ and let Λ^+ be the positive system. Let $\mathfrak{n} = \sum_{\alpha \in \Lambda^+} \mathfrak{g}_\alpha$. Then, we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. Put $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$. By an *Iwasawa subalgebra* of $\mathfrak{g}$, we mean this subalgebra $\mathfrak{s}$, which is completely solvable. In the case of Iwasawa subalgebras, the condition for the existence of a non-inductive linear form is simplified by the following lemma.

LEMMA 2.1. *Let* $\alpha, \beta \in \Lambda^+$ *be linearly independent. Suppose* $\alpha + \beta \in \Lambda^+$. *Then, one has* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

For a proof, see Keene [**3**, Theorem 2]. Thus we have the following proposition by virtue of Corollary 1.3.

PROPOSITION 2.2. *Let* $\mathfrak{s}$ *be an Iwasawa subalgebra of a semisimple Lie algebra* $\mathfrak{g}$. *If there are linearly independent* $\alpha, \beta \in \Lambda^+$ *such that* $\alpha + \beta \in \Lambda^+$, *then one can find a non-inductive linear form on* $\mathfrak{s}$.

COROLLARY 2.3. *Suppose that* $\mathfrak{g}$ *is a real simple Lie algebra of real rank strictly greater than one. Then, there is a non-inductive linear form on* $\mathfrak{s}$.

PROOF. A glance at the table given in the book [**11**, p. 30~p. 32] convinces us that if $\mathfrak{g}$ is real simple and of real rank strictly greater than one, there are always two simple restricted roots α, β such that $\alpha+\beta\in\Lambda^{+}$. Q.E.D.

On the other hand, if the real rank of a semisimple Lie algebra $\mathfrak{g}$ is equal to one, the matters are quite different. Let us begin with the following well-known lemma.

LEMMA 2.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\alpha\in\Lambda^{+}$ and assume that $\alpha/2\in\Lambda^{+}$. Then, for any non-zero $u\in\mathfrak{g}_{\alpha/2}$, $\operatorname{ad}u$ maps $\mathfrak{g}_{\alpha/2}$ onto $\mathfrak{g}_{\alpha}$.*

PROOF. We include here a proof for reader's convenience. Let $H_{\alpha}\in\mathfrak{a}$ be the element such that $\alpha(H)=B(H_{\alpha}, H)$ holds for any $H\in\mathfrak{a}$, where B is the Killing form of $\mathfrak{g}$. Put $y=\theta u$. Then, $y\in\mathfrak{g}_{-\alpha/2}$. Let $h=[u, y]$. Then, h belongs to the centralizer of $\mathfrak{a}$ as well as to $\mathfrak{p}$. Hence $h\in\mathfrak{a}$. Now, for any $H\in\mathfrak{a}$, we have

$$\begin{aligned} B(h, H) &= B([u, y], H) = B(y, [H, u]) \\ &= \frac{1}{2}\alpha(H)B(\theta u, u) = \frac{1}{2}B(B(u, \theta u)H_{\alpha}, H). \end{aligned}$$

Hence we get $h=B(u, \theta u)H_{\alpha}/2$, so that for any $z\in\mathfrak{g}_{\alpha}$,

$$\begin{aligned} [u, [z, y]] &= [z, [u, y]] \qquad (\text{because } [u, z]=0) \\ &= -\alpha(h)z = -\frac{1}{2}B(u, \theta u)\alpha(H_{\alpha})z. \end{aligned}$$

Since $B(u, \theta u)\neq 0$ and $\alpha(H_{\alpha})\neq 0$, we conclude that $z\in(\operatorname{ad}u)(\mathfrak{g}_{\alpha/2})$, because $[z, y]\in\mathfrak{g}_{\alpha/2}$. Q.E.D.

COROLLARY 2.5. *Under the same assumption as Lemma 2.4, $\dim\mathfrak{g}_{\alpha/2}$ is even.*

PROOF. Let $f\in\mathfrak{g}_{\alpha}^{*}$ be non-zero. Then, $x, y\to f([x, y])$ is a non-degenerate skew-symmetric bilinear form on $\mathfrak{g}_{\alpha/2}\times\mathfrak{g}_{\alpha/2}$ by virtue of Lemma 2.4. Hence $\dim\mathfrak{g}_{\alpha/2}$ is even. Q.E.D.

PROPOSITION 2.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra of real rank one and $\mathfrak{s}=\mathfrak{a}+\mathfrak{n}$ an Iwasawa subalgebra of $\mathfrak{g}$. Then, every linear form on $\mathfrak{s}$ is inductive.*

PROOF. First of all, we note that $f\in\mathfrak{s}^{*}$ is inductive if and only if so is $(\operatorname{ad}^{*}s)f$ for any $s\in S:=\exp\mathfrak{s}$. Therefore, it suffices to show that each representative of the coadjoint orbits in $\mathfrak{s}^{*}$ is inductive.

Since $\mathfrak{g}$ is of real rank one, $\mathfrak{n}$ is written as $\mathfrak{n}=\mathfrak{g}_{\alpha/2}+\mathfrak{g}_{\alpha}$ for some positive restricted root α, where $\mathfrak{g}_{\alpha/2}=\{0\}$ possibly. Let us see the coadjoint action of S on $\mathfrak{s}^{*}$. Express every element $s\in S$ as $s=\exp H\exp u\exp x$ with $H\in\mathfrak{a}$, $u\in\mathfrak{g}_{\alpha/2}$ and $x\in\mathfrak{g}_{\alpha}$, then a simple computation yields

$$(\mathrm{ad}\, s)^{-1}(a+b+c) = a+\left(e^{-\alpha(H)/2}b+\frac{1}{2}\alpha(a)u\right)$$
$$+(e^{-\alpha(H)}c-e^{-\alpha(H)/2}[u, b]+\alpha(a)x),$$

where $a\in\mathfrak{a}$, $b\in\mathfrak{g}_{\alpha/2}$ and $c\in\mathfrak{g}_\alpha$. Let $f\in\mathfrak{s}^*$ and set $f=f_0+f_{1/2}+f_1$, $(\mathrm{ad}^* s)f=g_0+g_{1/2}+g_1$ with f_k, $g_k\in\mathfrak{g}_{k\alpha}^*$ (understanding $\mathfrak{g}_{0\alpha}=\mathfrak{a}$). Then, we get from the above

$$\begin{aligned} g_0(a) &= f_0(a)+\frac{1}{2}f_{1/2}(u)\alpha(a)+f_1(x)\alpha(a), \\ g_{1/2}(b) &= e^{-\alpha(H)/2}f_{1/2}(b)-e^{-\alpha(H)/2}f_1([u, b]), \\ g_1(c) &= e^{-\alpha(H)}f_1(c). \end{aligned} \tag{2.1}$$

When $\mathfrak{g}_{\alpha/2}=\{0\}$, it should be understood that the middle formula in (2.1) is missing and that $f_{1/2}=0$ in the first of (2.1). Fixing a norm on $\mathfrak{s}^*$, we let $\mathfrak{S}_k$ $(k=1/2, 1)$ be the unit sphere in $\mathfrak{g}_{k\alpha}^*$. The coadjoint orbits in $\mathfrak{s}^*$ are described as follows:

Case 1, $\mathfrak{g}_{\alpha/2}=\{0\}$.
- (i) $\mathfrak{a}^*+\{r\sigma\,;\, r>0\}$ $(\sigma\in\mathfrak{S}_1)$,
- (ii) singleton $\{\gamma\}$ $(\gamma\in\mathfrak{a}^*)$,

Case 2, $\mathfrak{g}_{\alpha/2}\neq\{0\}$.
- (iii) $\mathfrak{a}^*+\mathfrak{g}_{\alpha/2}^*+\{r\sigma\,;\, r>0\}$ $(\sigma\in\mathfrak{S}_1)$,
- (iv) $\mathfrak{a}^*+\{r\tau\,;\, r>0\}$ $(\tau\in\mathfrak{S}_{1/2})$,
- (v) singleton $\{\gamma\}$ $(\gamma\in\mathfrak{a}^*)$,

where we have used Lemma 2.4 to derive (iii). We pick, as representatives, $\sigma\in\mathfrak{S}_1$ in (i), $\gamma\in\mathfrak{a}^*$ in (ii), $\sigma\in\mathfrak{S}_1$ in (iii), $\tau\in\mathfrak{S}_{1/2}$ in (iv) and $\gamma\in\mathfrak{a}^*$ in (v). If $f=\gamma$ (cases (ii) and (v)), then $\mathfrak{s}(f)=\mathfrak{s}$. Hence f is inductive. If $f=\sigma\in\mathfrak{S}_1$ (cases (i) and (iii)), then $\mathfrak{s}(f)=\mathrm{Ker}\,\sigma\subset\mathfrak{g}_\alpha$, which says that f is inductive. For the case (iv), $f=\tau$, we have $\mathfrak{s}(f)=\mathrm{Ker}\,\tau+\mathfrak{g}_\alpha$. Hence f is inductive. Consequently, any linear form on $\mathfrak{s}$ is inductive. Q. E. D.

THEOREM 2.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{s}$ an Iwasawa subalgebra of $\mathfrak{g}$. Then, there is a non-inductive linear form on $\mathfrak{s}$ if and only if some simple component of $\mathfrak{g}$ is of real rank at least two.*

PROOF. The if part is clear from the proof of Corollary 2.3 combined with Proposition 2.2. So, suppose that every simple component $\mathfrak{g}_k$ $(1\leq k\leq l)$ of $\mathfrak{g}$ is of real rank one. Let $\mathfrak{s}_k$ be an Iwasawa subalgebra of $\mathfrak{g}_k$. Then, $\mathfrak{s}=\mathfrak{s}_1\times\cdots\times\mathfrak{s}_l$ (direct product of Lie algebras) is an Iwasawa subalgebra of $\mathfrak{g}$. By Proposition 2.5, every linear form on $\mathfrak{s}_k$ $(1\leq k\leq l)$ is inductive. From this it is easy to see that every linear form on $\mathfrak{s}$ is inductive. Q. E. D.

§ 3. The case of normal j-algebras.

For normal j-algebras, we have a similar result as in the case of Iwasawa subalgebras of semisimple Lie algebras. Let us start with the definition of normal j-algebras.

Let $\mathfrak{s}$ be a Lie algebra, j a linear operator on $\mathfrak{s}$ such that $j^2=-1_{\mathfrak{s}}$ and $\omega\in\mathfrak{s}^*$. Then, the triplet $(\mathfrak{s}, j, \omega)$ is termed a *normal j-algebra* if the following (i)$\sim$(iii) are satisfied:

(i) $\mathfrak{s}$ is completely solvable.

(ii) Extend j to $\mathfrak{s}_C$ by complex linearity and let $\mathfrak{s}^-$ be the $-i$-eigenspace of j in $\mathfrak{s}_C$. Then, $\mathfrak{s}^-$ is a complex subalgebra of $\mathfrak{s}_C$.

(iii) (a) $\omega([x, jx])>0$ for all non-zero $x\in\mathfrak{s}$.

(b) $\omega([jx, jy])=\omega([x, y])$ for all $x, y\in\mathfrak{s}$.

We summarize here a requisite fundamental structure of $\mathfrak{s}$ following Rossi and Vergne [**9**, Theorem 4.3]. For proofs, we refer the reader to Pyatetskii-Shapiro [**7**, Theorem 2, p. 61] or Rossi [**8**, Theorem 5.13]. Given a normal j-algebra $(\mathfrak{s}, j, \omega)$, we define a real inner product $\langle\cdot, \cdot\rangle$ on $\mathfrak{s}$ by $\langle x, y\rangle=\omega([x, jy])$. Let $\mathfrak{n}=[\mathfrak{s}, \mathfrak{s}]$. Denote by $\mathfrak{a}$ the orthogonal complement to $\mathfrak{n}$. Then, $\mathfrak{s}=\mathfrak{a}+\mathfrak{n}$. We know that $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{s}$ and the representation of $\mathfrak{a}$ on $\mathfrak{n}$ by the adjoint action is diagonalizable. Define $\mathfrak{n}_\alpha$ as in (1.1) and take all $\alpha\in\mathfrak{a}^*$ such that $\mathfrak{n}_\alpha\neq\{0\}$ and $j\mathfrak{n}_\alpha\subset\mathfrak{a}$. Number these α as $\alpha_1, \cdots, \alpha_l$. We have $l=\dim\mathfrak{a}$ and $\dim\mathfrak{n}_{\alpha_k}=1$ $(1\leqq k\leqq l)$. The number l is called the *rank* of the normal j-algebra $(\mathfrak{s}, j, \omega)$. If we order $\alpha_1, \cdots, \alpha_l$ in a suitable way, then all $\alpha\in\mathfrak{a}^*$ such that $\mathfrak{n}_\alpha\neq\{0\}$ are of the following form (not all possibilities need occur):

$$\tag{3.1} \begin{gathered} \frac{1}{2}(\alpha_m+\alpha_k) \quad (1\leqq k<m\leqq l), \qquad \frac{1}{2}(\alpha_m-\alpha_k) \quad (1\leqq k<m\leqq l), \\ \frac{1}{2}\alpha_k \quad (1\leqq k\leqq l), \qquad \alpha_k \quad (1\leqq k\leqq l). \end{gathered}$$

Moreover, we have

$$\tag{3.2} j\mathfrak{n}_{(\alpha_m-\alpha_k)/2}=\mathfrak{n}_{(\alpha_m+\alpha_k)/2} \qquad (m>k),$$

$$\tag{3.3} j\mathfrak{n}_{\alpha_m/2}=\mathfrak{n}_{\alpha_m/2} \qquad (1\leqq m\leqq l).$$

Finally, we can choose non-zero $u_i\in\mathfrak{n}_{\alpha_i}$ such that $[ju_i, u_i]=u_i$. Then we have $\alpha_k(ju_m)=\delta_{km}$ (the Kronecker's symbol). The following lemma is a counterpart of Lemma 2.1 in the case of normal j-algebras.

LEMMA 3.1. *If* $\mathfrak{n}_{(\alpha_m-\alpha_k)/2}\neq\{0\}$ $(m>k)$, *then one has*

$$[\mathfrak{n}_{(\alpha_m-\alpha_k)/2}, \mathfrak{n}_{(\alpha_m+\alpha_k)/2}]=\mathfrak{n}_{\alpha_m}.$$

PROOF. Clearly it suffices to prove $\mathfrak{n}_{\alpha_m}\subset[\mathfrak{n}_{(\alpha_m-\alpha_k)/2}, \mathfrak{n}_{(\alpha_m+\alpha_k)/2}]$. We fix a non-zero $x\in\mathfrak{n}_{(\alpha_m-\alpha_k)/2}$. Then, by (3.2), we have $jx\in\mathfrak{n}_{(\alpha_m+\alpha_k)/2}$ and $[x, jx]\in\mathfrak{n}_{\alpha_m}$. Since $\dim\mathfrak{n}_{\alpha_m}=1$, we have only to prove $[x, jx]\neq 0$. Suppose then $[x, jx]=0$. This implies $\omega([x, jx])=0$, so that $x=0$, because $(\mathfrak{s}, j, \omega)$ is a normal j-algebra. Thus we arrive at a contradiction. Q.E.D.

If the normal j-algebra $(\mathfrak{s}, j, \omega)$ is of rank one, then $\mathfrak{s}$ is of the form $\mathfrak{s}=\mathfrak{a}+\mathfrak{n}_{\alpha/2}+\mathfrak{n}_\alpha$ for some non-zero $\alpha\in\mathfrak{a}^*$ with $\mathfrak{n}_{\alpha/2}=\{0\}$ possibly. Note that (3.3) implies that $\dim\mathfrak{n}_{\alpha/2}$ is even. We put $2n=\dim\mathfrak{n}_{\alpha/2}$. Then, since $\dim\mathfrak{n}_\alpha=1$, it is easy to see that $\mathfrak{s}$ is isomorphic to an Iwasawa subalgebra of $\mathfrak{su}(n+1, 1)$ $(n=0, 1, 2, \cdots)$. So, Proposition 2.6 leads us to the following proposition.

PROPOSITION 3.2. *Let $(\mathfrak{s}, j, \omega)$ be a normal j-algebra of rank one. Then, every linear form on $\mathfrak{s}$ is inductive.*

THEOREM 3.3. *Let $(\mathfrak{s}, j, \omega)$ be a normal j-algebra. Then, there is a non-inductive linear form on $\mathfrak{s}$ if and only if $\mathfrak{n}_{(\alpha_m-\alpha_k)/2}\neq\{0\}$ for some m, k $(m>k)$.*

PROOF. Suppose first that $\mathfrak{n}_{(\alpha_m-\alpha_k)/2}\neq\{0\}$ for some m, k $(m>k)$. We note that (3.2) implies $\mathfrak{n}_{(\alpha_m+\alpha_k)/2}\neq\{0\}$. Put $\alpha=(\alpha_m-\alpha_k)/2$ and $\beta=(\alpha_m+\alpha_k)/2$. Clearly α, β are linearly independent and the condition (i) in Theorem 1.1 is satisfied. Condition (ii) is guaranteed by Lemma 3.1, so there is a non-inductive linear form on $\mathfrak{s}$ by Corollary 1.3.

Conversely, suppose $\mathfrak{n}_{(\alpha_m-\alpha_k)/2}=\{0\}$ for all m, k $(m>k)$. By (3.2), we have also $\mathfrak{n}_{(\alpha_m+\alpha_k)/2}=\{0\}$ for all m, k $(m>k)$. Therefore (3.1) implies that $\mathfrak{s}$ is decomposed as $\mathfrak{s}=\sum_{1\le k\le l}\mathfrak{s}_k$, where $\mathfrak{s}_k=\boldsymbol{R}ju_k+\mathfrak{n}_{\alpha_k/2}+\mathfrak{n}_{\alpha_k}$ with $\mathfrak{n}_{\alpha_k/2}=\{0\}$ possibly. Thus $\mathfrak{s}$ is a direct product (as Lie algebra) of $\mathfrak{s}_k$'s. By Proposition 3.2, every linear form on $\mathfrak{s}_k$ is inductive. From this, we can conclude easily that each linear form on $\mathfrak{s}$ is inductive. Q.E.D.

§4. An example.

A typical example of solvable Lie algebra treated in this note is supplied by an Iwasawa subalgebra $\mathfrak{s}$ of $\mathfrak{sp}(2, \boldsymbol{R})$. We realize $\mathfrak{s}$ by a Lie algebra of 4×4 real matrices (with the usual bracket operation of matrices) as follows:

$$\mathfrak{s}=\left\{[a, b, c; x, y, z]:=\begin{pmatrix} a & c & x & z \\ 0 & b & z & y \\ 0 & 0 & -a & 0 \\ 0 & 0 & -c & -b \end{pmatrix};\ \begin{matrix} a, b, c\in\boldsymbol{R} \\ x, y, z\in\boldsymbol{R} \end{matrix}\right\}.$$

$Sp(2, \boldsymbol{R})$ has a maximal compact subgroup K isomorphic to the unitary group $U(2)$. Since $Sp(2, \boldsymbol{R})/K$ is a hermitian symmetric space, one can introduce in $\mathfrak{s}$ a structure of normal j-algebra (cf. Rossi and Vergne [**9**, p. 372]). Let $\mathfrak{n}_1=$

$\boldsymbol{R}[0, 0, 1; 0, 0, 0]$ and $\mathfrak{n}_2=\boldsymbol{R}[0, 0, 0; 0, 0, 1]$. We see easily that $[\mathfrak{n}_1, \mathfrak{n}_2]=\mathfrak{n}_3:=\boldsymbol{R}[0, 0, 0; 1, 0, 0]$. Then, by Theorem 2.7 or Theorem 3.3, there is a non-inductive linear form on $\mathfrak{s}$.

On the other hand, we say after Poguntke [**4**] that a Lie algebra $\mathfrak{g}$ is *symmetric* if the involutory Banach algebra $L^1(G)$, where G is the corresponding connected and simply connected Lie group, is symmetric. Poguntke gave a list [**4**, p. 162] of non-symmetric solvable Lie algebras of dimension at most six. By Satz 2 of that paper, if $\mathfrak{g}$ is a six dimensional non-symmetric Lie algebra, then either $\mathfrak{g}$ is contained in that list or some proper quotient of $\mathfrak{g}$ is isomorphic to one of the Lie algebras in the list.

Let us return to our example $\mathfrak{s}$ above. We have seen that there is a non-inductive linear form on $\mathfrak{s}$. Then, by [**6**, Theorem 10] or [**5**, Theorem 3], $\mathfrak{s}$ is not symmetric. Clearly the dimension of $\mathfrak{s}$ is six, but it is not isomorphic to any six dimensional Lie algebra in the Poguntke's list. So, some proper quotient of $\mathfrak{s}$ should be isomorphic to one of four or five dimensional Lie algebras in that list. Let us identify it. First, we note that $\mathfrak{n}_3$ is an ideal of $\mathfrak{s}$. Then the quotient $\mathfrak{s}/\mathfrak{n}_3$ is isomorphic to $\mathfrak{b}_5$ in the Poguntke's list, where $\mathfrak{b}_5$ is the five dimensional Lie algebra with the basis e_k $(0\leqq k\leqq 4)$ such that

$$[e_2, e_3]=e_4,\quad [e_1, e_2]=-e_2,\quad [e_1, e_3]=e_3,$$
$$[e_0, e_2]=e_2,\quad [e_0, e_4]=e_4,$$

other brackets being zero or deduced by skew-symmetry. The isomorphism of $\mathfrak{s}/\mathfrak{n}_3$ onto $\mathfrak{b}_5$ is given by

$$[1, 0, 0; 0, 0, 0]+\mathfrak{n}_3 \longleftrightarrow e_0, \qquad [-1/2, 1/2, 0; 0, 0, 0]+\mathfrak{n}_3 \longleftrightarrow e_1,$$
$$[0, 0, 1; 0, 0, 0]+\mathfrak{n}_3 \longleftrightarrow e_2, \qquad [0, 0, 0; 0, 1, 0]+\mathfrak{n}_3 \longleftrightarrow e_3,$$
$$[0, 0, 0; 0, 0, 1]+\mathfrak{n}_3 \longleftrightarrow e_4.$$

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