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Movability and homotopy, homology pro-groups of Whitney continua

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0. Introduction.

By a *continuum* we mean a nonempty compact connected metric space. Let X be a continuum with metric d. By the *hyperspace* of X we mean $C(X)=\{A \mid A$ is a (nonempty) subcontinuum of $X\}$ with the Hausdorff metric H_d. Let $F_1(X)=\{\{x\} \mid x \in X\}$. A *Whitney map* for $C(X)$ is a continuous function $\omega: C(X) \to [0, \omega(X)]$ such that

(0.1) if $A \subset B$ and $A \neq B$, then $\omega(A)<\omega(B)$, and
(0.2) $\omega(\{x\})=0$ for each $\{x\} \in F_1(X)$.

In [**33**] and [**34**], H. Whitney showed that for any continuum X there exists a Whitney map ω for $C(X)$. In 1942, Kelley's important paper [**13**] appeared. J. L. Kelley was the first person to introduce Whitney map into the study of $C(X)$. After Kelley's work, several papers on Whitney maps have been written and Whitney maps have become standard tool and has since been used in almost all papers about hyperspaces (e.g., see the references).

Let $\mathfrak{P}$ be a topological property. The property $\mathfrak{P}$ is called a *Whitney property* provided whenever X has the property $\mathfrak{P}$, so does $\omega^{-1}(t)$ for any Whitney map ω for $C(X)$ and $0 \leqq t < \omega(X)$. It is known that many properties are Whitney properties (e.g., see [**5**], [**8**], [**9**], [**13**], [**14**], [**15**], [**16**], [**18**], [**20**], [**23**], [**24**], [**27**], [**28**], [**29**], [**30**] and [**31**], etc.). Also, it is known that many properties are not Whitney properties (e.g., see [**3**], [**4**], [**10**], [**11**], [**18**], [**24**] and [**26**], etc.).

In [**8**], we proved that the property of being pointed 1-movable is a Whitney property. Also, in [**9**] we proved that the property of being movable is a Whitney property for $\theta(2)$-curves. Naturally, the following problem is raised: Is the property of being movable a Whitney property? In section 1, we give a negative answer to the problem. In fact, there exist a movable curve X and a Whitney map ω for $C(X)$ such that for some $0<t<\omega(X)$, $\omega^{-1}(t)$ is not 2-movable. In [**27**], J. T. Rogers, Jr. proved that if X is any continuum and ω is any Whitney map for $C(X)$, then there is an injection $\gamma^*: \check{H}^1(\omega^{-1}(t))$

$\to \check{H}^1(X)$ for $0 \leqq t \leqq \omega(X)$, where $\check{H}^1(X)$ denotes the first Čech cohomology group of X. Also, if $\check{H}^1(X)$ is finitely generated, then there is a positive number t_0 such that $\check{H}^1(\omega^{-1}(t)) \cong \check{H}^1(X)$ for $0 \leqq t \leqq t_0$ (see [8, (1.14)]). In section 2, we prove the similar theorems concerning the first homotopy and homology pro-groups.

We refer readers to see [**1**] and [**21**] for shape theory, and we refer readers to see [**24**] for hyperspace theory.

1. The property of being movable is not a Whitney property.

A compactum X lying in the Hilbert cube $Q=[0, 1]^\infty$ is said to be *movable* ([**1**] or [**21**]) provided that for every neighborhood V of X in Q there is a neighborhood U of X in Q such that for any neighborhood W of X in Q there is a homotopy $\varphi_W : U \times [0, 1] \to V$ satisfying the following condition.

(i) $\varphi_W(x, 0)=x$, $\varphi_W(x, 1) \in W$ for every point $x \in U$.

A compactum X lying in Q is said to be *n-movable* ($n \geqq 1$) ([**1**] or [**21**]) provided that for every neighborhood V of X in Q there is a neighborhood U of X in Q such that for any neighborhood W of X in Q, any compactum A with $\dim A \leqq n$ and any map $f : A \to U$, there is a homotopy $\varphi_W : A \times [0, 1] \to V$ satisfying the following condition.

(ii) $\varphi_W(a, 0)=f(a)$, $\varphi_W(a, 1) \in W$ for every point $a \in A$.

Similarly, "pointed movable" and "pointed n-movable" are defined (see [**1**] or [**21**]). It is well-known that those properties are topological properties ([**1**] or [**21**]). Clearly, "(pointed) movable" implies "(pointed) n-movable" for each $n=1, 2, \cdots$.

The main theorem in this section is the following

(1.1) THEOREM. *The property of being movable is not a Whitney property. More precisely, there exist a movable curve X and a Whitney map ω for $C(X)$ such that for some $0<t<\omega(X)$, $\omega^{-1}(t)$ is not 2-movable. Hence, the property of being (pointed) 2-movable is not a Whitney property.*

To prove (1.1), we need the following

(1.2) (L. E. Ward, Jr. [**32**]). *Let P be a compact metric partially ordered space such that $\min P$ and $\max P$ are disjoint closed subsets and let Q be a closed subset of P such that $\min Q \subset \min P$ and $\max Q \subset \max P$. Then a Whitney map for Q can be extended to a Whitney map for P.*

The next result is obvious. We omit the proof.

(1.3) *Let* $\{X_i\}_{i=1,2,\cdots,n}$ *be a finite family of compact* ARs *such that* $\bigcap_{i\in E} X_i$ *is empty or an* AR *for each subset* E *of* $\{1, 2, \cdots, n\}$. *Assume that* $A, A_1, A_2, \cdots, A_n$ *are compacta such that* $A=A_1\cup A_2\cup \cdots \cup A_n$. *If* $f, g: A\to\bigcup_{i=1}^n X_i$ *are any maps such that* $f(A_i), g(A_i)\subset X_i$ *for each* $i=1, 2, \cdots, n$, *then there is a homotopy* $F: A\times[0, 1]\to\bigcup_{i=1}^n X_i$ *such that* $F(a, 0)=f(a)$, $F(a, 1)=g(a)$ *for each* $a\in A$ *and* $F(A_i\times[0, 1])\subset X_i$ *for each* $i=1, 2, \cdots, n$.

(1.4) (M. Lynch [**19**]). *Let* X *be any continuum and let* $A\in C(X)$. *Then for any Whitney map* ω *for* $C(X)$ *and any* $t\in[0, \omega(X)]$, *the set*

$$C(A, \omega, t) = \{B\in\omega^{-1}(t) \mid B\supset A\}$$

is an AR, *where* $\omega(A)\leqq t$.

Proof of Theorem (1.1). Consider the following sets in the Euclidean 3-dimensional space E^3:

$X_0 = [0, 1]\times[0, \infty)\times[0, 1]$.

$X_n = [1/3, 2/3]\times[n-(1/3), n+(1/3)]\times[0, 1]$ $(n\geqq 1)$.

$N = X_0 - \bigcup_{n=1}^{\infty} X_n$.

$M = \partial N$, where ∂N denotes the manifold's boundary of N.

Then M is a non compact 2-dimensional manifold (see Fig. 1). Let $Y=M\cup\{\infty\}$ denote the one point compactification of M, which is well-known Borsuk's continuum [**2**] (see Fig. 2). In [**2**], K. Borsuk proved that Y is not movable, more precisely Y is not 2-movable. Let $M_n=M\cap([0, 1]\times[n, n+1]\times[0, 1])$ $(n=1, 2, \cdots)$ and let K_n be a simplicial complex which is a triangulation of M_n, i.e., $|K_n|=M_n$. We may assume that $K=\bigcup_{n=1}^{\infty} K_n$ is a simplicial complex which is a triangulation of M, i.e., $|K|=M$.

Now, we consider the set X which is a disjoint union of $|K^1|$ and an arc A, i.e., $X=|K^1|\cup A$, where K^1 denotes the 1-skeleton of K. We define a metric d on the space X satisfying the following condition:

(*) For any $\varepsilon>0$, there is a subcompactum C of $|K^1|$ such that if $\langle V, W\rangle$ is an edge of K^1 with $\langle V, W\rangle\cap C=\emptyset$, then $U_\varepsilon(A)\supset\langle V, W\rangle$ and $U_\varepsilon(\langle V, W\rangle)\supset A$, where $U_\varepsilon(B)=\{x\in X \mid d(x, B)<\varepsilon\}$ for a subset B of X.

Then X is a compact connected 1-dimensional metric space which contains $|K^1|$ and A. First, we shall show that X is pointed movable. In fact, consider the decomposition space X/A which is obtained by identifying A to a point. Let $p: X\to X/A$ be the quotient map. Since A is cell-like, p induces a shape equivalence. Since X/A is a locally connected curve, X/A is a pointed movable

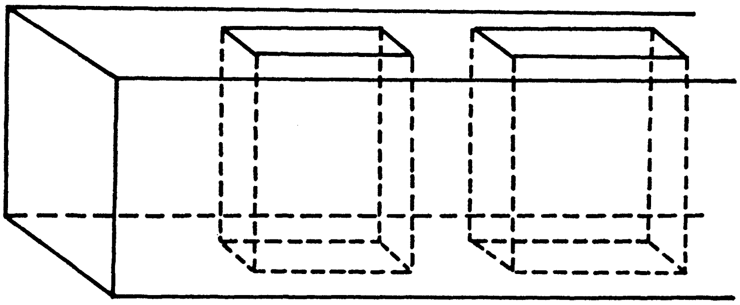

Figure 1.

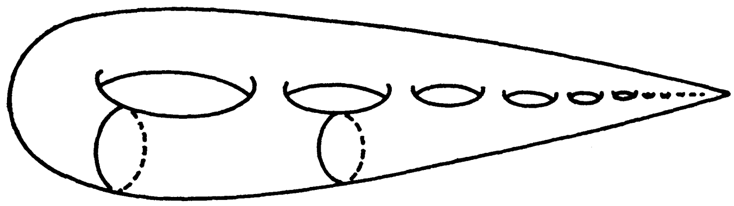

Figure 2.

(see [17]). Since the property of being pointed movable is shape invariant, X is also pointed movable.

Next, we shall construct a Whitney map ω for $C(X)$ as follows. Consider the following sets:

$\mathfrak{A}_1=\{|L|\in C(X) \mid L$ is a subcomplex of K^1 such that $|L|$ is contained in some simplex of $K\}$.

$\mathfrak{A}_2=\{|L|\in C(X) \mid L$ is a subcomplex of K^1 such that $|L|$ is not contained in any simplex of K and $|L|\subset M_n\cup M_{n+1}$ for some $n=1, 2, \cdots\}$.

Then $\mathfrak{A}_1\cup\mathfrak{A}_2\cup F_1(X)\cup\{A\}$ is closed in $C(X)$. Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are discrete subsets of $C(X)$, we can define a map $\omega'\colon \mathfrak{A}_1\cup\mathfrak{A}_2\cup F_1(X)\cup\{A\}\to[0, \infty)$ satisfying the following conditions:

(1) $\omega'(A)=1$,

(2) $\omega'(|L|)<1$ for $|L|\in\mathfrak{A}_1$,

(3) $\omega'(|L|)>1$ for $|L|\in\mathfrak{A}_2$,

(4) $\omega'(\{x\})=0$ for $\{x\}\in F_1(X)$,

(5) if $Z_1, Z_2\in\mathfrak{A}_1\cup\mathfrak{A}_2\cup F_1(X)\cup\{A\}$ and $Z_1\subsetneqq Z_2$, then $\omega'(Z_1)<\omega'(Z_2)$, and

(6) for any $\varepsilon>0$, there is a natural number n such that if $|L|\in\mathfrak{A}_1\cup\mathfrak{A}_2$, $|L|$ is non-degenerate and $|L|\subset M_n\cup M_{n+1}\cup\cdots$, then $|\omega'(|L|)-1|<\varepsilon$.

Then (6) implies that ω' is continuous. By (1.2), there exists a Whitney map ω for $C(X)$ which is an extension of ω'.

Now, we shall show that $\omega^{-1}(1)$ is homotopy equivalent to Y, i.e., $\omega^{-1}(1)\simeq Y$. By (*), A is terminal in X, i.e., if $C\in C(X)$, $C\cap A\neq\emptyset$, then $C\subset A$ or $C\supset A$. Hence we see that

(7) $\omega^{-1}(1)=\bigcup_{V\in K^0}C(V, \omega, 1)\cup\{A\}$, where K^0 denotes the 0-skeleton of K and $C(V, \omega, 1)=\{Z\in\omega^{-1}(1) \mid Z\ni V\}$.

By the same way as in the proof of [**10**, (3.1)], for any subset $\{V_0, V_1, \cdots, V_m\}$ of K^0, $\langle V_0, V_1, \cdots, V_m\rangle\in K$ if and only if $\bigcap_{i=0}^m C(V_i, \omega, 1)$ is nonempty. Also, if $\langle V_0, V_1, \cdots, V_m\rangle\in K$, then $\bigcap_{i=1}^m C(V_i, \omega, 1)$ is an AR. Hence we can construct maps $f:\bigcup_{V\in K^0}C(V, \omega, 1)\to M$ and $g: M\to\bigcup_{V\in K^0}C(V, \omega, 1)$ such that

(8) $f(C(V, \omega, 1))\subset \mathrm{St}(V, \mathrm{Sd}\,K)$ and $g(\mathrm{St}(V, \mathrm{Sd}\,K))\subset C(V, \omega, 1)$ for every $V\in K^0$, where $\mathrm{Sd}\,K$ denotes the barycentric subdivision of K.

Note that for each natural number n, there is a positive number $\delta>0$ such that if $Z\in\omega^{-1}(1)$, $Z\neq A$ and $H_d(A, Z)<\delta$, then $Z\subset M_n\cup M_{n+1}\cup\cdots$. Also, we show that if $Z_n\in\omega^{-1}(1)$ and $Z_n\subset M_n\cup M_{n+1}\cup\cdots$, then $\lim Z_n=A$. Suppose, on the contrary, that there is a sequence $Z_1, Z_2, \cdots$, of points of $\omega^{-1}(1)$ such that $Z_n\subset M_n\cup M_{n+1}\cup\cdots$, and $H_d(A, Z_n)\geqq\varepsilon>0$ for some $\varepsilon>0$. We may assume that $\lim Z_n=Z\subset A$. Then $\omega(Z)=\lim\omega(Z_n)=1$. Since $Z\subset A$ and $\omega(A)=1$, $A=Z$. This is a contradiction. Hence we can obtain (continuous) maps $f^*:\bigcup_{V\in K^0}C(V, \omega, 1)\cup\{A\}\to M\cup\{\infty\}$, and $g^*: M\cup\{\infty\}\to\bigcup_{V\in K^0}C(V, \omega, 1)\cup\{A\}$, which are defined by $f^*\mid\bigcup_{V\in K^0}C(V, \omega, 1)=f$, $f^*(A)=\infty$, $g^*\mid M=g$ and $g^*(\infty)=A$. By using (1.3), $g^*f^*\simeq 1$ and $f^*g^*\simeq 1$. Hence $\omega^{-1}(1)\simeq Y$. Since Y is not 2-movable, $\omega^{-1}(1)$ is also not 2-movable. This completes the proof.

(1.5) REMARK. By the similar arguments in the proof of (1.1), we can conclude that if P is a non compact locally finite polyhedron and $P^*=P\cup\{\infty\}$ is the one point compactification of P, then there are a movable curve X and a Whitney map ω for $C(X)$ such that for some $0<t<\omega(X)$, $\omega^{-1}(t)\simeq P^*$.

2. The first homotopy and homology pro-groups of Whitney continua.

Let (X, x_0) be a pointed continuum. By pro-$\pi_n(X, x_0)$ we mean the n-th homotopy pro-group of (X, x_0) and by pro-$H_n(X)$ the n-th homology pro-group of X with coefficients in integers Z (e.g., see [**21**]). Let ω be any Whitney map for $C(X)$. In [**8**] and [**9**], we defined a shape deformation retraction $\underline{r}_{st}:\omega^{-1}([s, t])\to\omega^{-1}(t)$ $(s\leqq t)$ (see the proof of [**8**, (1.3)]). Now, consider the following subset $C[x_0]$ in $C(X)$:

$$C[x_0]=\{A\in C(X) \mid A\ni x_0\}.$$

Note that $C[x_0]\cap\omega^{-1}(t)=C(x_0, \omega, t)$ is an AR (see (1.4)). Consider the decomposition space $C^*(X)=C(X)/C[x_0]$ which is obtained by identifying $C[x_0]$ to a point $*$ and let $h: C(X)\to C^*(X)$ be the projection. By the construction of $\underline{r}_{st}$ (see the proof of [8, (1.3)]), $\underline{r}_{st}$ induces a shape deformation retraction $\underline{r}^*_{st}: (\omega^{-1}([s, t])^*, *)\to(\omega^{-1}(t)^*, *)$, where $\omega^{-1}([s, t])^*=\omega^{-1}([s, t])/(C[x_0]\cap\omega^{-1}([s, t])$. Clearly, the restriction $\underline{f}^*_{st}=\underline{r}^*_{st} \mid \omega^{-1}(s)^*: (\omega^{-1}(s)^*, *)\to(\omega^{-1}(t)^*, *)$ is a pointed shape morphism. Then $\underline{f}^*_{tu}\underline{f}^*_{st}=\underline{f}^*_{su}$ for $0\leqq s\leqq t\leqq u\leqq\omega(X)$. Since $C(x_0, \omega, t)$ is an AR, $h_t=h \mid \omega^{-1}(t): \omega^{-1}(t)\to\omega^{-1}(t)^*$ induces a pointed shape equivalence, i.e., $\mathrm{Sh}(\omega^{-1}(t), A)=\mathrm{Sh}(\omega^{-1}(t)^*,*)$, where $A\in C(x_0, \omega, t)$. Set $\underline{f}_{st}=\underline{r}_{st} \mid \omega^{-1}(s)$. Then we have a commutative diagram in (unpointed) shape category as follows:

$$\begin{array}{ccc}
\omega^{-1}(s) & \xrightarrow{\underline{f}_{st}} & \omega^{-1}(t) \\
{\scriptstyle h_s}\downarrow & & \downarrow{\scriptstyle h_t} \\
(\omega^{-1}(s)^*, *) & \xrightarrow[\underline{f}^*_{st}]{} & (\omega^{-1}(t)^*, *)
\end{array}$$

Hence, in order to study the shape properties of $\omega^{-1}(t)$, we will study the pointed space $(\omega^{-1}(t)^*, *)$.

First, we shall show the following

(2.1) THEOREM. *Let (X, x_0) be a pointed continuum and let ω be any Whitney map for $C(X)$. Then* $\text{pro-}\pi_1(\underline{f}^*_{st}): \text{pro-}\pi_1(\omega^{-1}(s)^*, *)\to\text{pro-}\pi_1(\omega^{-1}(t)^*, *)$ *is an epimorphism of pro-groups for each* $0\leqq s\leqq t\leqq\omega(X)$.

PROOF. The proof is essentially due to Rogers [27]. First, we consider the case $s=0$. Consider the following subset Y of $X\times\omega^{-1}(t)$:

$$Y = \{(x, A) \mid x\in X,\ A\in\omega^{-1}(t) \text{ and } A\ni x\}.$$

Also, consider the decomposition space

$$Y^* = Y/\{x_0\}\times C(x_0, \omega, t),$$

which is obtained by identifying $\{x_0\}\times C(x_0, \omega, t)$ to a point $*$. Let $A\in C(x_0, \omega, t)$. Let $k: (Y, (x_0, A))\to(Y^*, *)$ be the quotient map and let $p: (Y, (x_0, A))\to(X, x_0)$ and $q: (Y, (x_0, A))\to(\omega^{-1}(t), A)$ be the projections. Clearly, there is a map $g: (Y^*, *)\to(\omega^{-1}(t)^*, *)$ such that $h_t\cdot q=g\cdot k$, where $h_t: \omega^{-1}(t)\to\omega^{-1}(t)^*$ is the projection. By using the unpointed shape morphism $\underline{f}_{0t}: X\to\omega^{-1}(t)$, we can easily obtain a shape morphism $\tilde{\underline{f}}_{0t}: (X, x_0)\to(Y, \{x_0\}\times C(x_0, \omega, t))$ such that $p\cdot\tilde{\underline{f}}_{0t}=\underline{1}_X$ and $q\cdot\tilde{\underline{f}}_{0t}=\underline{f}_{0t}$. Note that $h_t\cdot q\cdot\tilde{\underline{f}}_{0t}=\underline{f}^*_{0t}$. Then $k\cdot\tilde{\underline{f}}_{0t}: (X, x_0)\to(Y^*, *)$ is a pointed shape morphism. Note that $p'\cdot k\cdot\tilde{\underline{f}}_{0t}=p\cdot\tilde{\underline{f}}_{0t}=\underline{1}_{(X, x_0)}$, where $p': (Y^*, *)\to(X, x_0)$ is a map such that $p'\cdot k=p$. Then we have the following diagram in pointed shape category:

$$
\begin{tikzcd}
 & & & (Y^*, *) \arrow[dd, shift right, "p'"'] \arrow[lllu, phantom] \\
(\omega^{-1}(t)^*, *) & (\omega^{-1}(t), A) \arrow[l, "h_t"'] & (Y, (x_0, A)) \arrow[l, "q"'] \arrow[ur, "k"] \arrow[dr, "p"'] & \\
 & & & (X, x_0) \arrow[uu, shift right, "k\cdot\underline{\tilde{f}}_{0t}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\omega^{-1}(t)^*, *) & & & (Y^*, *) \arrow[lll, "g"']
\end{tikzcd}
$$

By (1.4), p, k and h_t are cell-like maps, and q is a monotone map, i.e., $q^{-1}(Z)\cong Z\in \mathrm{AC}^0$ for each $Z\in\omega^{-1}(1)$. Hence pro-$\pi_1(p)$, pro-$\pi_1(k)$ and pro-$\pi_1(h_t)$ are isomorphisms of pro-groups (e.g., see [**21**, p. 283]) and pro-$\pi_1(q)$ is an epimorphism of pro-groups (e.g., see [**6**, Theorem (8.5)]). Thus pro-$\pi_1(g)$ is an epimorphism of pro-groups. Since pro-$\pi_1(p')$ is an isomorphism of pro-groups, we conclude that pro-$\pi_1(k\cdot\underline{\tilde{f}}_{0t})$ is an isomorphism of pro-groups. Since pro-$\pi_1(\underline{f}^*_{0t})$=pro-$\pi_1(g)\cdot$pro-$\pi_1(k\cdot\underline{\tilde{f}}_{0t})$, pro-$\pi_1(\underline{f}^*_{0t})$ is an epimorphism of pro-groups. Next, we consider the case $s>0$. Note that $\underline{f}^*_{st}\underline{f}^*_{0s}=\underline{f}^*_{0t}$. Hence we can easily see that pro-$\pi_1(\underline{f}^*_{st})$ is also an epimorphism of pro-groups. This completes the proof.

A compactum X is said to be *n-shape connected* ($n\geqq 0$) (e.g., see [**21**]) provided that pro-$\pi_i(X, x_0)=0$ for each $x_0\in X$ and $0\leqq i\leqq n$.

(2.2) COROLLARY. *The property of being* 1-*shape connected is a Whitney property.*

(2.3) THEOREM. *Let (X, x_0) be a pointed continuum and let ω be any Whitney map for $C(X)$. If* pro-$\pi_1(X, x_0)$ *is stable (e.g., see* [**21**]*), then there is a positive number $0<t_0<\omega(X)$ such that* pro-$\pi_1(\underline{f}^*_{0t})$: pro-$\pi_1(X, x_0)\rightarrow$pro-$\pi_1(\omega^{-1}(t)^*, *)$ *is an isomorphism of pro-groups for $0\leqq t\leqq t_0$.*

PROOF. Let $t_1>t_2>\cdots$, be a decreasing sequence of positive numbers such $\lim t_i=0$. Note that $\bigcap_{i=1}^{\infty}(\omega^{-1}([0, t_i])^*, *)=(X, x_0)$. Then we have the following commutative diagram in pointed shape category:

$$
\begin{tikzcd}
(\omega^{-1}([0, t_1])^*, *) \arrow[d, shift left, "\underline{r}^*_{0t_1}"] & (\omega^{-1}([0, t_2])^*, *) \arrow[l, hook'] \arrow[d, shift left, "\underline{r}^*_{0t_2}"] & (\omega^{-1}([0, t_3])^*, *) \arrow[l, hook'] \arrow[d, shift left, "\underline{r}^*_{0t_3}"] & \arrow[l, hook'] \\
(\omega^{-1}(t_1)^*, *) \arrow[u, hook, shift left] & (\omega^{-1}(t_2)^*, *) \arrow[l, "\underline{f}^*_{t_2t_1}"] \arrow[u, hook, shift left] & (\omega^{-1}(t_3)^*, *) \arrow[l, "\underline{f}^*_{t_3t_2}"] \arrow[u, hook, shift left] & \arrow[l]
\end{tikzcd}
$$

Note that the inclusion $i:(\omega^{-1}(t_j)^*, *)\hookrightarrow(\omega^{-1}([0, t_j])^*, *)$ induces a pointed shape equivalence (see [**8**, (1.3)]). By the shape continuity, $\underline{f}=\{\text{pro-}\pi_1(\underline{f}^*_{0t_n})\}_{n=1,2,\dots}$: pro-$\pi_1(X, x_0)\rightarrow\underline{W}$ induces an isomorphism of pro-groups, where $\underline{W}$ is the following inverse system:

$$
\text{pro-}\pi_1(\omega^{-1}(t_1)^*, *) \xleftarrow[\text{pro-}\pi_1(\underline{f}^*_{t_2t_1})]{} \text{pro-}\pi_1(\omega^{-1}(t_2)^*, *) \xleftarrow[\text{pro-}\pi_1(\underline{f}^*_{t_3t_2})]{} \text{pro-}\pi_1(\omega^{-1}(t_3)^*, *) \longleftarrow
$$

Since pro-$\pi_1(X, x_0)$ is stable, for some group G, pro-$\pi_1(X, x_0)\cong G$. By (2.1), pro-$\pi_1(f^*_{t_{n+1}t_n})$ is an epimorphism of pro-groups for each $n=1, 2, \cdots$. By the following lemma (2.4), we conclude that there is a natural number n such that for each $m\geqq n$, pro-$\pi_1(\underline{f}^*_{0t_m})$ is an isomorphism of pro-groups. Let $t_0=t_n$ and let $0\leqq t\leqq t_0$. Since pro-$\pi_1(\underline{f}^*_{tt_n})\cdot$pro-$\pi_1(\underline{f}^*_{0t})=$pro-$\pi_1(\underline{f}^*_{0t_n})$ is an isomorphism of pro-groups, pro-$\pi_1(\underline{f}^*_{0t})$ is a monomorphism of pro-groups. Hence we can conclude that pro-$\pi_1(\underline{f}^*_{0t})$ is an epimorphism and a monomorphism of pro-groups. By [**21**, p. 114, Theorem 6], pro-$\pi_1(\underline{f}^*_{0t})$ is an isomorphism of pro-groups.

(2.4) LEMMA. *Let $\underline{H}(i)=\{H(i)_1 \xleftarrow{h(i)_{21}} H(i)_2 \xleftarrow{h(i)_{32}} \cdots\}$ be an inverse sequence of groups for each $i=1, 2, \cdots$, and let $\underline{p}(i+1, i)=\{p(i+1, i)_k : H(i+1)_k\to H(i)_k\}_{k=1,2,\dots}$: $\underline{H}(i+1)\to\underline{H}(i)$ be an epimorphism of pro-groups for each i. Let G be a group and $\underline{f}=\{\underline{f}_i : G\to\underline{H}(i)\}_{i=1,2,\dots}$: $G\to\underline{H}$ be a morphism of pro-groups, where $\underline{H}$ denotes the following inverse system:*

$$\begin{tikzcd}
\arrow[d] & & \arrow[d] & & \arrow[d] & & \\
H(1)_3 \arrow[d, "h(1)_{32}"'] & & H(2)_3 \arrow[ll, "p(2,1)_3"'] \arrow[d, "h(2)_{32}"] & & H(3)_3 \arrow[ll, "p(3,2)_3"'] \arrow[d, "h(3)_{32}"] & & \cdots \arrow[ll] \\
H(1)_2 \arrow[d, "h(1)_{21}"'] & & H(2)_2 \arrow[ll, "p(2,1)_2"'] \arrow[d, "h(2)_{21}"] & & H(3)_2 \arrow[ll, "p(3,2)_2"'] \arrow[d, "h(3)_{21}"] & & \cdots \arrow[ll] \\
H(1)_1 & & H(2)_1 \arrow[ll, "p(2,1)_1"] & & H(3)_1 \arrow[ll, "p(3,2)_1"] & & \cdots \arrow[ll]
\end{tikzcd}$$

If $\underline{f} : G\to\underline{H}$ is an isomorphism of pro-groups, then there is a natural number n such that $\underline{f}_m : G\to\underline{H}(m)$ is an isomorphism of pro-groups for $m\geqq n$.

PROOF. Set $\underline{f}_i=\{f_{ij}\}_{j=1,2,\dots}$, where $f_{ij} : G\to H(i)_j$. Let $\underline{g} : \underline{H}\to G$ be a morphism of pro-groups such that $\underline{g}\cdot\underline{f}=1_G$ and $\underline{f}\cdot\underline{g}=1_{\underline{H}}$. We may assume that $\underline{g}=\{g : H(1)_1\to G\}$. We shall show that $\underline{f}_i : G\to\underline{H}(i)$ is an isomorphism of pro-groups for each $i\geqq 1$. In fact, $(\underline{g}\cdot\underline{p}(i, 1))\cdot\underline{f}_i=\underline{g}\cdot\underline{f}_1=1_G$, where $\underline{p}(i, 1)=\underline{p}(2, 1)\cdot\underline{p}(3, 2)\cdots\cdot\underline{p}(i, i-1)$. Since $\underline{f}\cdot\underline{g}=1_{\underline{H}}$, for each j, there are i_1 and j_1 such that $i_1\geqq i$, $j_1\geqq j$ and $f_{ij}\cdot g\cdot p(i, 1)_1\cdot h(i)_{j_1 1}\cdot p(i_1, i)_{j_1}=h(i)_{j_1 j}\cdot p(i_1, i)_{j_1}$. Since $\underline{p}(i_1, i)$ is an epimorphism of pro-groups, by [**21**, p. 109, Theorem 3], there is a $j_2\geqq j_1$ such that $\mathrm{Im}(h(i)_{j_2 j_1})\subset\mathrm{Im}(p(i_1, i)_{j_1})$. Hence $f_{ij}\cdot g\cdot p(i, 1)_1\cdot h(i)_{j_2 1}=h(i)_{j_2 j}$, which implies that $\underline{f}_i\cdot(\underline{g}\cdot\underline{p}(i, 1))=1_{\underline{H}(i)}$. Thus $\underline{f}_i : G\to\underline{H}(i)$ is an isomorphism of pro-groups.

Next, we shall study the first homology pro-groups of Whitney continua.

By [**21**, p. 136, Theorem 2], there is a natural epimorphism φ_1 : pro-$\pi_1(X, x_0)$ $\to$pro-$H_1(X)$ of pro-groups for any pointed continuum (X, x_0). Hence (2.1) and (2.3) imply the following

(2.5) COROLLARY. *Let X be a continuum and let ω be any Whitney map for $C(X)$. Then* pro-$H_1(\underline{f}_{st})$: pro-$H_1(\omega^{-1}(s))\to$pro-$H_1(\omega^{-1}(t))$ *is an epimorphism of pro-groups for each $0\leqq s\leqq t\leqq\omega(X)$. Moreover, if* pro-$H_1(X)$ *is stable, then there is a positive number $0<t_0<\omega(X)$ such that* pro-$H_1(\underline{f}_{0t})$: pro-$H_1(X)\to$pro-$H_1(\omega^{-1}(t))$ *is an isomorphism of pro-groups for $0\leqq t\leqq t_0$.*

(2.6) REMARK. In [26], A. Petrus showed that there exists a Whitney map ω for $C(D)$ such that D is a 2-cell and for some $t>0$ the 2-sphere S^2 is homotopy dominated by $\omega^{-1}(t)$. Hence (2.1), (2.2), (2.3) and (2.5) are not true for the second homotopy and homology pro-groups.

(2.7) REMARK. In the statements of (2.3) and (2.5), we cannot omit the condition that pro-$\pi_1(X, x_0)$ and pro-$H_1(X)$ are stable. Consider the following continuum in the plane E^2.

$$X=\bigcup_{n=1}^{\infty}\{(x, y) \mid (x-(1/n))^2+y^2=1/n^2\} \quad \text{(see Fig. 3).}$$

Then pro-$\pi_1(X, (0, 0))$ and pro-$H_1(X)$ are not stable. If ω is any Whitney map for $C(X)$, $\omega^{-1}(t)$ is an ANR for $0<t\leqq\omega(X)$ (see [9, (3.12)]). Hence pro-$\pi_1(\underline{f}^*_{0t})$ and pro-$H_1(\underline{f}_{0t})$ are not isomorphisms of pro-groups for each $0<t\leqq\omega(X)$.

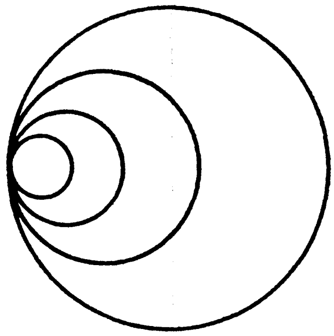

Figure 3.

Let G be a graph (=compact connected 1-dimensional polyhedron) with a triangulation T. For each edge $e=\langle V, W\rangle\in T$, let $\mathfrak{A}(e)=\{A \mid A$ is an arc in G from V to $W\}$ and let $|\mathfrak{A}(e)|$ denote the cardinal number of $\mathfrak{A}(e)$. Set $n(G)=\max\{|\mathfrak{A}(e)| \mid e$ is an edge of $T\}$. A curve X is said to be a *$\theta(m)$-curve* [9] provided that there exists an inverse sequence $\{G_i, p_{ii+1}\}$ of graphs such that $X\cong\varprojlim\{G_i, p_{ii+1}\}$ and $n(G_i)\leqq m$ for each i. Note that X is tree-like if and only if X is a $\theta(1)$-curve. In (2.7), the continuum X is a $\theta(2)$-curve.

Finally, we prove the following

(2.8) COROLLARY (cf. [9, (3.11)]). *Let X be a $\theta(2)$-curve and let ω be any Whitney map for $C(X)$. Let $0\leqq t\leqq\omega(X)$. Then the shape morphism $\underline{f}_{0t}: X\to\omega^{-1}(t)$*

is a weak domination in shape category (e.g., see [**21**, p. 186]). *Moreover, if X is an* FANR (e.g., see [**1**] or [**21**]), $\underline{f}_{0t}$ *is a shape domination. Also, there is a positive number $0<t_0<\omega(X)$ such that $\underline{f}_{0t}$ is a shape equivalence for $0\leqq t\leqq t_0$.*

PROOF. Since X is a $\theta(2)$-curve, $\mathrm{Fd}(\omega^{-1}(t))\leqq 1$ (see [**9**, (3.9)]). By (2.1), pro-$\pi_1(\underline{f}^*_{0t})$ is an epimorphism of pro-groups. By the proof of [**21**, p. 150, Theorem 5], $\underline{f}^*_{0t}$ is a weak domination, hence $\underline{f}_{0t}$ is also a weak domination. Assume that X is an FANR. By [**9**, (3.10)], $\omega^{-1}(t)$ is movable. Since $\mathrm{Fd}(\omega^{-1}(t))\leqq 1$, it is well-known that $\mathrm{Sh}(\omega^{-1}(t))=\mathrm{Sh}(\bigvee_{i=0}^{n} S^1)$, where $\bigvee_{i=0}^{n} S^1$ denotes the one point union of n circles for $n=0, 1, \cdots, \infty$. Since $\underline{f}_{0t}$ is a weak domination, we see that $n<\infty$. Hence $\underline{f}_{0t}$ is a shape domination. By (2.3), there is a positive number $0<t_0<\omega(X)$ such that pro-$\pi_1(\underline{f}^*_{0t})$ is an isomorphism of pro-groups for $0\leqq t\leqq t_0$. Note that $\mathrm{Fd}\, X\leqq 1$ and $\mathrm{Fd}(\omega^{-1}(t))\leqq 1$. By the Whitehead theorem in shape theory (e.g., see [**21**, p. 152, Theorem 8]), we conclude that $\underline{f}^*_{0t}$ is a pointed shape equivalence. Hence $\underline{f}_{0t}$ is a shape equivalence for $0\leqq t\leqq t_0$.

(2.9) REMARK. In the statement of (2.8), we cannot omit the condition that X is a $\theta(2)$-curve. Let G be the graph as follows (see Fig. 4).

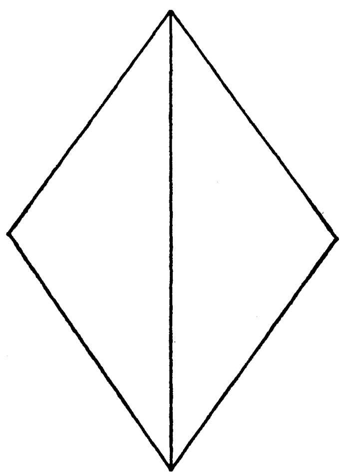

Figure 4.

Note that $n(G)=3$. In [**9**, (2.6)], we showed that for any Whitney map ω for $C(X)$, there is a positive number $0<t_0<\omega(X)$ such that $\omega^{-1}(t)\simeq S^2$ (=the 2-sphere) for $t_0\leqq t<\omega(X)$. Hence $\underline{f}_{0t}: X\to\omega^{-1}(t)$ is not a weak domination.

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