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The maximal ideal space of the bounded analytic functions on a Riemann surface

Dedicated to Professor Yukio Kusunoki on his 60th birthday

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Introduction.

We denote by $H^{\infty}(R)$ the algebra of all bounded analytic functions on a Riemann surface R, by $\mathcal{M}(R)$ the maximal ideal space of the algebra $H^{\infty}(R)$ and by τ the canonical continuous mapping from R into $\mathcal{M}(R)$. In this note we shall answer negatively the following question (cf. [**3**], [**4**]): If $H^{\infty}(R)$ separates the points of R, does it follow that

(0.1) the mapping τ is a homeomorphism of R onto an open subset of $\mathcal{M}(R)$?

We shall show, in addition, that property (0.1) has several equivalent conditions; one of them asserts existence of a family of certain meromorphic functions on R (Theorem).

Property (0.1) is satisfied if R is an arbitrary domain on the complex plane or on any closed Riemann surface whenever $H^{\infty}(R)$ contains a nonconstant function. It is also satisfied for any Riemann surface of Parreau-Widom type (Stanton [**7**]). As indicated in Gamelin [**4**], property (0.1) has some applications ([**2**], [**5**]). For instance, one can show uniqueness (and existence by Theorem) of the Ahlfors function on R when (0.1) is valid.

Before stating the results, we fix notations. Equipped with the sup-norm $\|f\|=\sup_{a\in R}|f(a)|$, $H^{\infty}(R)$ is a Banach algebra. Let $H^{\infty}(R)^*$ be the dual space of the Banach space $H^{\infty}(R)$. One may identify the maximal ideal space $\mathcal{M}(R)$ as the set of all $\phi\in H^{\infty}(R)^*$ satisfying $\phi(fg)=\phi(f)\phi(g)$ $(f, g\in H^{\infty}(R))$ and $\|\phi\|=\phi(1)=1$. For each point $a\in R$, the point evaluation $f\to f(a)$ defines an element ϕ_a in $\mathcal{M}(R)$. A canonical map $\tau: R\to\mathcal{M}(R)$ is now defined by $\tau(a)=\phi_a$. Inheriting the weak* topology from $H^{\infty}(R)^*$, the set $\mathcal{M}(R)$ is a compact Hausdorff space and the map τ is continuous. Two points a, b of R are said to be *separated* by $H^{\infty}(R)$ if there is a function f in $H^{\infty}(R)$ with $f(a)\neq f(b)$, and *weakly separated* by $H^{\infty}(R)$ if there is a pair of functions f, g in $H^{\infty}(R)$ with $(f/g)(a)\neq(f/g)(b)$. The points of R are said to be (*weakly*) *separated* by $H^{\infty}(R)$

if $H^{\infty}(R)$ (weakly) separates any distinct two points of R.

Suppose that $H^{\infty}(R)$ contains nonconstants. It follows from Royden [**6**: Proposition 2] that there exists a Riemann surface R' and an analytic map σ of R into R' such that $H^{\infty}(R')$ weakly separates the points of R' and $H^{\infty}(R)=H^{\infty}(R')\circ\sigma$. This implies that $H^{\infty}(R)$ is isomorphic to $H^{\infty}(R')$ as Banach algebras. Thus, we may restrict our attention to the case in which $H^{\infty}(R)$ weakly separates the points of R. Finally, we shall denote by $M^{\infty}(R)$ the family of the meromorphic functions on R that are bounded off some compact subset K of R, where the set K may depend on the function.

Now we state our results.

THEOREM. *Suppose that $H^{\infty}(R)$ weakly separates the points of a Riemann surface R. For a given point a of R, one of the following properties implies all the others:*

(a) *There is a neighborhood U of the point a such that the restriction map $\tau|U: U\to\mathcal{M}(R)$ is open.*

(b) *There is a neighborhood U of the point a such that the map $\tau|U$ is a homeomorphism of U onto an open subset of $\mathcal{M}(R)$.*

(c) *There is a meromorphic function $g\in M^{\infty}(R)$ with a pole at the point a.*

(d) *There is a meromorphic function $g\in M^{\infty}(R)$ such that g is analytic on $R\setminus\{a\}$ and has a simple pole at the point a.*

(e) *There is a bounded linear operator T on $H^{\infty}(R)$ such that*

(e. 1) $T(fg)=gTf+f(a)Tg, \quad f, g\in H^{\infty}(R)$; *and*

(e. 2) $(Tf)(a)\neq 0$ *for some function* $f\in H^{\infty}(R)$.

(f) *There is a homeomorphism Φ of the open unit disc D onto an open subset of $\mathcal{M}(R)$ satisfying*

(f. 1) $\Phi(0)=\phi_{a}$; *and*

(f. 2) $\hat{f}\circ\Phi$ *is analytic on D for every $f\in H^{\infty}(R)$, where $\hat{f}(\phi)=\phi(f)$ is the Gelfand transform of f.*

In section 3 we shall construct two examples. Both answer negatively our question; the first one is easier and the second says more. In fact, we shall construct a Riemann surface such that $\tau(E)$ is never open in $\mathcal{M}(R)$ for any non empty subset E of R (in particular, for $E=R$). This also shows that $M^{\infty}(R)\setminus H^{\infty}(R)$ may be empty even if $H^{\infty}(R)$ separates the points of R.

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1. A Lemma.

We shall need the following lemma (cf. [**5**: Chap. II, section 3]).

LEMMA. *Suppose that* $a\in R$ *is the pole of a function in* $M^{\infty}(R)$, *and that* $h\in M^{\infty}(R)$ *has minimal total number of poles, among all functions in* $M^{\infty}(R)$ *with pole at* a. *Then* $fh\in H^{\infty}(R)$ *for all* $f\in H^{\infty}(R)$ *satisfying* $f(a)=0$. *Moreover, there is* $f\in H^{\infty}(R)$ *such that* $f(a)=0$ *and* $(fh)(a)\neq 0$.

PROOF. Let $\{a=a_1, a_2, \cdots, a_m\}$ be the pole set of the function h. Note that $H^{\infty}(R)$ identifies the points $a_1, a_2, \cdots, a_m$. Let $g\in M^{\infty}(R)$ have minimal total number of poles, among all functions in $M^{\infty}(R)$ with pole set included in $\{a_1, a_2, \cdots, a_m\}$. Say the pole set of g is $\{b_1, b_2, \cdots, b_k\}$. Let $f\in H^{\infty}(R)$ have minimal total number of zeros at $\{b_1, b_2, \cdots, b_k\}$, among all functions in $H^{\infty}(R)$ vanishing at $\{b_1, b_2, \cdots, b_k\}$. Then, by the minimality of g and f, we see that $fg\in H^{\infty}(R)$ and $(fg)(b_j)\neq 0$ for all j. Hence, $(fg)(a)\neq 0$ and g must have a pole at a. By the minimality of h and g, g has the same order of pole at each a_j as h. Thus, $fh\in H^{\infty}(R)$ and $(fh)(a)\neq 0$. For any $F\in H^{\infty}(R)$ satisfying $F(a)=0$, we have $Fh\in H^{\infty}(R)$ by the minimality of f. This proves the lemma.

2. Proof of Theorem.

(a)$\Rightarrow$(b): It follows from [**6**: Proposition 1] that there are functions f, g in $H^{\infty}(R)$ such that f/g has a simple zero at the point a. We may assume that (U, z) is the local coordinate satisfying $f/g=z$. Replacing U by a smaller one, if necessary, we may assume that the point a is the only zero of f on U. Now it is easy to see that f and g separate the points of U. Thus, $\tau|U$ is one-to-one, and hence, homeomorphic.

(b)$\Rightarrow$(c): Our proof is similar to the proof of Rossi's "Local peak set theorem" ([**1**: III. 8. 1]). We can choose a function f in $H^{\infty}(R)$ so that a local coordinate (U, z) satisfies $f(z)=z^k$, $|f|=1$ on ∂U and τ is homeomorphic on a neighborhood of $\bar{U}$. Put $f_1=f$. Now we choose functions $f_2, f_3, \cdots, f_n\in H^{\infty}(R)$ such that $f_j(a)=0$ and

$$\{\phi\in\mathcal{M}(R) : |\phi(f_j)|\leqq 1,\ 2\leqq j\leqq n\}\subset\tau(U).$$

Choose a small $\varepsilon>0$ so that $|f_j(p)|<1$, $j=2, \cdots, n$ when $p\in U$ and $|z(p)|<\varepsilon$. Write $w=(w_1, \cdots, w_n)$. Put $V=\{w\in \boldsymbol{C}^n : |w_j|<1,\ j=1, \cdots, n\}$ and

$W=\{w\in C^n : |w_1|>\varepsilon\}\cup \bar{V}^c$. Then,

$$\sigma(f_1, \cdots, f_n)\subset V\cup W,$$

where the left hand side denotes the joint spectrum $\{(\phi(f_1), \cdots, \phi(f_n)) : \phi\in \mathcal{M}(R)\}$. Now choose functions $f_{n+1}, \cdots, f_N\in H^\infty(R)$ and a polynomial polyhedron P such that $\sigma(f_1, \cdots, f_N)\subset P$ and $\pi(P)\subset V\cup W$, where $\pi(w_1, \cdots, w_N)=(w_1, \cdots, w_n)$. Put

$$V' = P\cap\pi^{-1}(V) \qquad \text{and} \qquad W' = P\cap\pi^{-1}(W).$$

We regard the holomorphic function $1/w_1$ on $V'\cap W'$ as Cousin data for the cover $\{V', W'\}$ of P. By Oka's theorem, there exist holomorphic functions h_1 on V' and h_2 on W' such that $h_2-h_1=1/w_1$ on $V'\cap W'$. Now the desired function g is defined by

$$g(p) = \begin{cases} h_2(f_1(p), \cdots, f_N(p)), & (f_1(p), \cdots, f_N(p))\in W', \\ 1/f(p)+h_1(f_1(p), \cdots, f_N(p)), & (f_1(p), \cdots, f_N(p))\in V'. \end{cases}$$

(b)⇒(d): We prove this implication by a slight modification of the above proof. Note that $\sqrt[k]{f}=\sqrt[k]{w_1}$ has a single-valued analytic branch on $\sigma(f_1, \cdots, f_n)\cap V$ and that $w_1^{-1}(\zeta)$ intersects the set $\sigma(f_1, \cdots, f_n)\cap V\cap W$ at exactly n distinct points for every $\varepsilon<|\zeta|<1$. Hence, we can find an open subset V_1 of V so that $\sigma(f_1, \cdots, f_n)\subset V_1\cup W$ and $\sqrt[k]{w_1}$ has a single-valued holomorphic branch on $V_1\cap W$. Now we may use V_1 and $\sqrt[k]{w_1}$ instead of V and w_1 to obtain a meromorphic function g with a simple pole at the point a.

(c)⇒(e): Take a function $h\in M^\infty(R)$ whose total number of poles is minimal, among all functions with pole at a. We define an operator T on $H^\infty(R)$ by $Tf=h(f-f(a))$. By the lemma, T satisfies (e. 1) and (e. 2).

(e)⇒(f): This is the Bishop-Banaschewski theorem ([**8**: Theorem 15.12]).

(f)⇒(b): The map $f\to\hat{f}\circ\Phi$ gives a representation of the algebra $H^\infty(R)$ on the open unit disc D in the sense of [**6**: p. 116]. Since $H^\infty(R)^\wedge\circ\Phi$ separates the points of D, there is a one-to-one analytic map of D into the representation space $X=\mathrm{Rep}\,H^\infty(R)$ [**6**: Proposition 1]. We may identify D and R as open sets of X. For $b\in R$ with $\phi_b\in\Phi(D)$, points b and $\Phi^{-1}(\phi_b)$ in X are not separated by $H^\infty(X)$. Since τ is continuous, there are many such b in a neighborhood of a. By [**6**: Lemma 1], $0\in D$ must be the same point as a in the space X. This means that D is identified with a neigborhood of a, and shows (b).

The implications (d)⇒(c) and (b)⇒(a) are trivial. This completes the proof.

3. Examples.

By Schwarz's lemma the canonical map $\tau : R\to\mathcal{M}(R)$ is continuous even if $\mathcal{M}(R)$ is equipped with the norm topology induced from $H^\infty(R)^*$.

EXAMPLE 1. We shall construct a Riemann surface with the following properties:

(3.1) $H^\infty(R)$ separates the points of R; and

(3.2) τ is not an open map (even in the norm topology for $\mathcal{M}(R)$).

Let D be the open unit disc. Choose a sequence of closed disjoint slits I_k in D such that I_k has no accumulation points in the interior of D. Put

$$D^* = D \setminus \bigcup_{k=1}^{\infty} I_k .$$

For the moment, we think of D_0, D_1, $\cdots$ as a sequence of copies of D^*. Roughly speaking, the required Riemann surface R will be constructed by joining every D_k ($k \geqq 1$) to D_0 along two sides of the cut I_k crosswise (Fig. 1), though actually

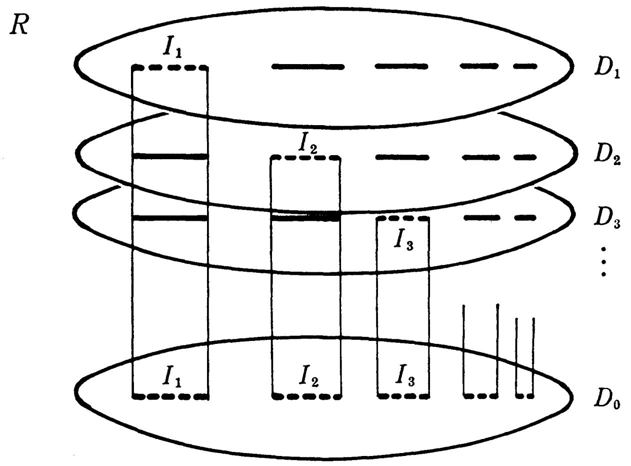

Figure 1.

we will do this in a little more complex way. We consider finite disjoint subintervals $J_{k1}, \cdots, J_{kn_k}$ of I_k for each k, and what we really do is to join D_k with D_0 along two sides of the cuts J_{kj} ($1 \leqq j \leqq n_k$) crosswise. So, the precise definition of D_k is as follows:

$$D_0 = D \setminus \Big(\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{n_k} J_{kj} \Big)$$

$$D_k = D \setminus \Big(\Big(\bigcup_{j=1}^{n_k} J_{kj} \Big) \cup \Big(\bigcup_{n \neq k} I_n \Big) \Big), \qquad k \geqq 1.$$

We shall show that the surface R has the desired property if the integers n_k are chosen to be sufficiently large. Let $D^* = \bigcup_k K_k$, where $\{K_k\}$ is an increasing sequence of compact sets. Denote by D_k^0 the corresponding copy of D_k in D_0. The subset $D_k \cup D_k^0$ of R can be considered as a two-sheet cover of the domain $D_k^* = D \setminus \bigcup_{n\neq k} I_n$. Now let $f \in H^\infty(R)$, $|f| \leqq 1$. Every non-branching point $z \in D_k^*$ has two pre-images z_k^+ and z_k^- in $D_k \cup D_k^0$. Set $g_k(z) = [f(z_k^+) - f(z_k^-)]^2$. As is well-known, $g_k(z)$ extends to an analytic function on D_k^*, and it is zero at the branch points, which are the end points of the slits J_{kj}. Since $|g_k| \leqq 4$, we can choose an integer n_k, independent of f, so that

$$|g_k| \leqq (\varepsilon_k)^2 \qquad \text{on } K_k, \tag{3.3}$$

where $\varepsilon_k \downarrow 0$ is a sequence of positive numbers. Hence,

$$\|\phi_{z_k^+} - \phi_{z^-}\| = \sup\{|f(z_k^+) - f(z^-)| \;:\; f \in H^\infty(R),\ |f| \leqq 1\} < \varepsilon_k,$$

where $z^- = z_k^-$ is the point on the sheet D_0 for $z \in D^*$. This shows that $\phi_{z_k^+} \to \phi_{z^-}$ in the norm topology as $k \to \infty$, and hence, $\phi_{z_k^+} \to \phi_{z^-}$ in the weak* topology. Now it remains to show that $H^\infty(R)$ separates the points of R. To this end, we consider a Riemann surface R_k obtained by joining two copies of $D \setminus \bigcup_{j=1}^{n_k} J_{kj}$ along all two sides of J_{kj}. Denote by π, π_k the ramified covering maps from R, R_k onto the open unit disc D, respectively. Note that there is a natural inclusion map ψ_k from the subset $D_0 \cup D_k$ of R into R_k such that $\pi_k \circ \psi_k = \pi$. Of course, ψ_k maps D_0 to one sheet of R_k and D_k to the other sheet. Now, we extend ψ_k to the whole of R by mapping all the other D_j's in R to the sheet containing $\psi_k(D_0)$ so that $\pi_k \circ \psi_k = \pi$. Inspecting analytic coordinates at each point, we see that ψ_k is an analytic map of R into R_k. Here, it is crucial that no D_j other than D_0 and D_k contain the segment I_k, where the conformal structure in R_k is changed by ramifications. Since R_k is a finite bordered Riemann surface, $H^\infty(R_k)$ separates the points of R_k. Since $H^\infty(R_k) \circ \psi_k \subset H^\infty(R)$ for all k, we see that any two distinct points of R are separated by $H^\infty(R)$.

EXAMPLE 2. In the above example the sheets D_k $(k \geqq 1)$ are openly imbedded in the maximal ideal space $\mathcal{M}(R)$. With a little more effort, we next construct a Riemann surface R satisfying (3.1),

(3.4) $\tau(E)$ is never open in $\mathcal{M}(R)$ (with respect to the norm topology) for any non empty subset E of R; and

(3.5) $M^\infty(R) = H^\infty(R)$.

We take I_k, J_{kj}, D^* and K_k as in Example 1. To define D_k, we first order them as follows. At stage 0, only D_0 exists. At stage 1, D_0 has one child, which we name D_1. At stage 2, each of D_0, D_1 has one child, which we name D_2 and D_3, respectively, and we continue this process for all D_k. According to this ordering, we see that each D_l, $2^{k-1} \leqq l < 2^k$, has only one parent $D_{p(l)}$, so that $0 \leqq p(l) < 2^{k-1}$. Now we define

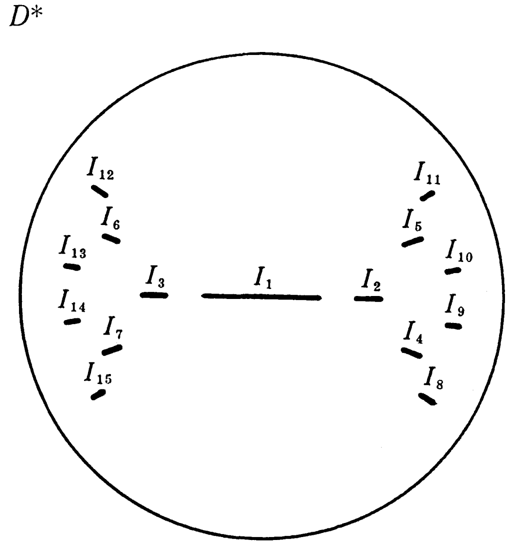

Figure 2.

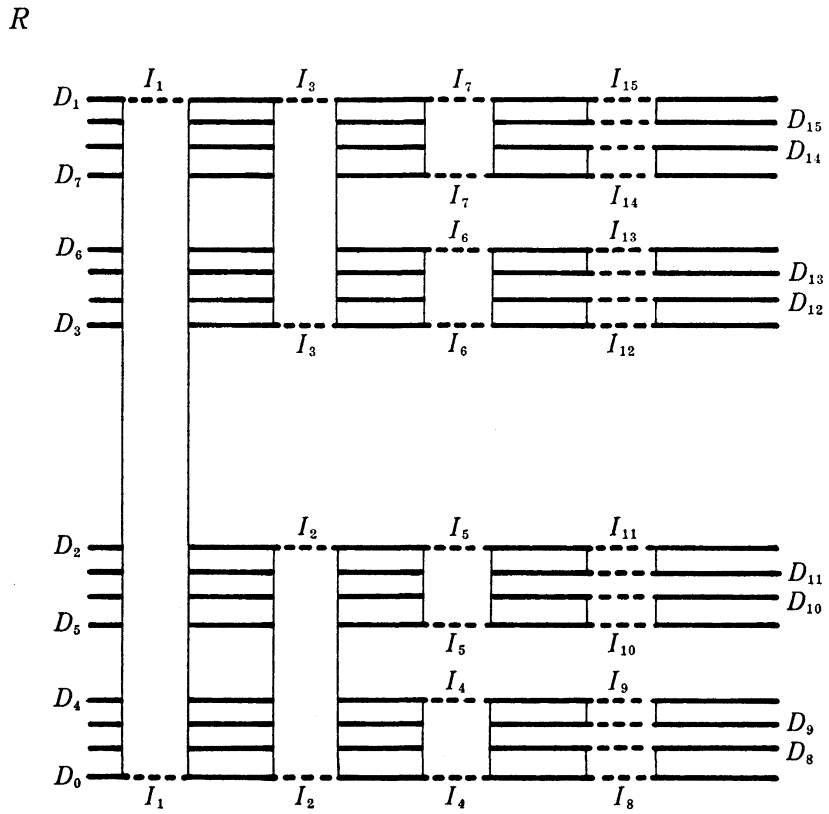

Figure 3.

$$D_l = D \setminus \left(\bigcup_{i=1}^{\infty} I_i^{(l)} \right), \qquad l=0, 1, 2, \cdots,$$

where $I_i^{(l)} = \bigcup_{j=1}^{n_i} J_{ij}$ for $i=l$ or $p(i)=l$, and $I_i^{(l)} = I_i$, otherwise. The desired Riemann surface is now obtained by joining D_l with $D_{p(l)}$ along the two sides of J_{lj} $(1 \leqq j \leqq n_l)$ crosswise for each $l=1, 2, \cdots$. In other words, at stage k, we join $D_0, \cdots, D_{2^{k-1}-1}$ with $D_{2^{k-1}}, \cdots, D_{2^k-1}$ so that (3.1) holds for each joined pair (Fig. 2, Fig. 3). Define a finite bordered Riemann surface R_l and covering maps π, π_l as in Example 1. Furthermore, we define a one-to-one map ψ_l from the subset $D_l \cup D_{p(l)}$ of R onto the corresponding subdomain of R_l so that $\pi_k \circ \psi_k = \pi$. We extend ψ_l analytically to the whole of R by mapping all the D_j's that are D_l's descendants to the same sheet as the one containing $\psi_l(D_l)$ and all the remaining D_j's to the other sheet of R_l. As before, we see that $H^\infty(R)$ separates the points of R, and (3.5) follows from the theorem. In order to have (3.4), we choose ε_k satisfying $\sum_{k=1}^{\infty} \varepsilon_k < \infty$ in (3.3). Put $E = \pi^{-1}(\zeta)$ for $\zeta \in D^*$. Then, the set $\tau(E)$ is totally bounded in the norm metric $\|\phi_1 - \phi_2\|$. Therefore, the norm closure $\overline{\tau(E)}$ is a totally disconnected compact set, and hence, it is homeomorphic to Cantor's ternary set. Consequently, $\overline{\tau(E)} \setminus \tau(E)$ is dense in $\overline{\tau(E)}$, and (3.4) holds.

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