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On topologies of triangulated infinite-dimensional manifolds

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0. Introduction.

Consider $\boldsymbol{R}^n$ as the subset $\boldsymbol{R}^n \times \{(0, 0, \cdots)\}$ of the countable infinite product $\boldsymbol{R}^\omega$ of the real line $\boldsymbol{R}$. The set $\bigcup_{n\in\boldsymbol{N}}\boldsymbol{R}^n$ admits two different natural topologies. One is the weak topology with respect to the tower $\{\boldsymbol{R}^n\}_{n\in\boldsymbol{N}}$ and the space with this topology is called the *direct limit* of lines and denoted by $\operatorname{dir}\lim\boldsymbol{R}^n$ or simply by $\boldsymbol{R}^\infty$. Another is the relative topology inherited from the product topology of $\boldsymbol{R}^\omega$ and the space with this topology is denoted by σ, that is, σ is a subspace of the linear topological space $\boldsymbol{s}$ $(=\boldsymbol{R}^\omega)$ of all real sequences. (It is well-known that the pair $(\boldsymbol{s}, \sigma)$ is homeomorphic $(\approx)$ to the pair (l_2, l_2^f), where l_2^f is the linear span of the canonical orthonormal basis of Hilbert space l_2.) A separable topological manifold modeled on these spaces is called an $\boldsymbol{R}^\infty$*-manifold* or a σ*-manifold*, respectively. These are considered as two different topologizations on the same underlying set. The former is the direct limit of a tower of finite-dimensional (f. d.) compact metrizable spaces (compacta), that is, its topology is the weak topology with respect to the tower ([8, Prop. III. 2]). The latter is metrizable and coarser than the former. Both of these manifolds are triangulated, that is, each $\boldsymbol{R}^\infty$-manifold is homeomorphic to a simplicial complex with the weak (Whitehead) topology (cf. [**18**, Introduction]) and each σ-manifold is homeomorphic to a simplicial complex with the metric topology ([**11**, Theorem 15]). Let K be a simplicial complex and $|K|=\bigcup K$ the realization of K. By $|K|_{\mathrm{w}}$ and $|K|_{\mathrm{m}}$, we denote the spaces $|K|$ with the weak topology and the metric topology, respectively. We conjecture that *$|K|_{\mathrm{w}}$ is an $\boldsymbol{R}^\infty$-manifold if and only if $|K|_{\mathrm{m}}$ is a σ-manifold.* In this paper, we prove a half of this conjecture, that is,

THEOREM. *For a simplicial complex K, $|K|_{\mathrm{m}}$ is a σ-manifold if $|K|_{\mathrm{w}}$ is an $\boldsymbol{R}^\infty$-manifold.*

A map $f: X\to Y$ is a *fine homotopy equivalence* provided for each open cover $\mathcal{U}$ of Y there exists a map $g: Y\to X$ such that fg is $\mathcal{U}$-homotopic to id_Y and

gf is $f^{-1}(\mathcal{U})$-homotopic to id_X, where g is called a *$\mathcal{U}$-homotopy inverse* of f. It is not difficult to see that the natural bijection from $\boldsymbol{R}^\infty$ to σ is a fine homotopy equivalence (cf. [**19**]). For any simplicial complex K, the identity of $|K|$ is a fine homotopy equivalence from $|K|_{\mathrm{w}}$ to $|K|_{\mathrm{m}}$. This follows from Dowker's Theorem [**6**, (15.2)] combined with [**10**, Lemma V.7] and [**6**, (6.1)]. Recently, the author gave another proof in [**19**]. Then it is natural to conjecture more generally as follows,

CONJECTURE. *Let $h: \operatorname{dir}\lim X_n \to Y$ be a bijective fine homotopy equivalence from the direct limit of f.d. compacta to a metrizable space. Then* $\operatorname{dir}\lim X_n$ *is an $\boldsymbol{R}^\infty$-manifold if and only if Y is a σ-manifold.*

In fact, we can prove the "only if" part if we assume that each compact set A in Y is a *strong Z-set*, that is, for each open cover $\mathcal{U}$ of Y the identity of Y is $\mathcal{U}$-near to a map $f: Y \to Y$ such that $A \cap \operatorname{cl} f(Y) = \emptyset$. If $\operatorname{dir}\lim X_n$ is an $\boldsymbol{R}^\infty$-manifold then we have a map $f: Y \to Y \setminus A$ which is $\mathcal{U}$-near to id_Y. However *this conjecture is false.* In Section 4, we construct a bijective fine homotopy equivalence from $\boldsymbol{R}^\infty$ to an AR which is not a σ-manifold, and in Section 5, one to σ from the direct limit of a tower of f.d. compact AR's which is not an $\boldsymbol{R}^\infty$-manifold.

We give some applications in Section 3. Especially we have a generalization of [**11**, Theorem 16] and [**21**, Theorem 3]. In Section 6, using the example in Section 5, we answer negatively Problem 6-4 in [**17**]. We should remark that all results in this paper have the Q^∞- and Σ-versions which are mentioned in Section 7.

1. Strong universality of towers.

Let $X_1 \subset X_2 \subset \cdots$ be a tower of closed sets in a metric space $X=(X, d)$. We say that $\{X_n\}_{n\in\boldsymbol{N}}$ is *strongly universal for f.d. compacta* [**5**] if, for each f.d. compacta $A \supset B$, for each map $f: A \to X$ such that $f|B$ is an embedding of B into some X_m, and for each $\varepsilon > 0$, there exists an embedding $h: A \to X_n$ for some $n \geqq m$ such that $h|B = f|B$ and $d(h, f) = \sup\{d(h(x), f(x)) \mid x \in A\} < \varepsilon$. A tower $\{X_n\}_{n\in\boldsymbol{N}}$ has the *mapping absorption property for f.d. compacta,* provided for each map $f: A \to X$ of an f.d. compactum, for each $m \in \boldsymbol{N}$ and for each $\varepsilon > 0$, there exists a map $g: A \to X_n$ for some $n \geqq m$ such that $g|f^{-1}(X_m) = f|f^{-1}(X_m)$ and $d(g, f) < \varepsilon$ (cf. [**3**, Def. 4.5]). A tower $\{X_n\}_{n\in\boldsymbol{N}}$ is said to be *finitely expansive* [**3**, Def. 4.7] if for each $m \in \boldsymbol{N}$ there exists an embedding $h: X_m \times I \to X_n$ for some $n \geqq m$ such that $h(x, 0) = x$ for all $x \in X_m$.

LEMMA 1. *A tower $\{X_n\}_{n\in\boldsymbol{N}}$ of f.d. compacta in a metric space X is strongly universal for f.d. compacta if and only if $\{X_n\}_{n\in\boldsymbol{N}}$ is finitely expansive and has* *the mapping absorption property for f.d. compacta.*

PROOF. For each f.d. compacta $A\supset B$, we have a map $k: A\to I^n$ for some $n\in \boldsymbol{N}$ such that $k(B)=0$ and $k|A\setminus B$ is injective. Then the "if" part is easily seen from the definitions. For any map $f: A\to X$ of an f.d. compactum, we can apply the strong universality of $\{X_n\}_{n\in \boldsymbol{N}}$ to the map $f': A\cup_{f|f^{-1}(X_m)}X_m\to X$ of the adjunction space induced by f and the inclusion $X_m\subset X$, and easily construct an approximation $g: A\to X_n$ of f with $g|f^{-1}(X_m)=f|f^{-1}(X_m)$. A strongly universal tower is clearly finite expansive. Thus the proof is completed. □

In [**14**], Mogilski gave a characterization of σ-manifolds. The next lemma is a version using a strongly universal tower.

LEMMA 2. *An ANR X is a σ-manifold if and only if X has a strongly universal tower $\{X_n\}_{n\in \boldsymbol{N}}$ for f.d. compacta such that $X=\bigcup_{n\in \boldsymbol{N}}X_n$ and each X_n is an f.d. compact strong Z-set.*

PROOF. Since each compact set in an ANR which is a countable union of strong Z-sets is a strong Z-set [**4**, Lemma 7.2], the "if" part is an immediate consequence of [**5**, Prop. 2.2 & Prop. 2.3] and Mogilski's characterization [**14**]. We will see the "only if" part. By [**2**, Theorem 9], $X\approx |K|\times\sigma$ where K is a countable locally finite simplicial complex (hence $|K|_{\mathrm{m}}=|K|_{\mathrm{w}}$). Let $\{K_n\}_{n\in \boldsymbol{N}}$ be a tower of finite subcomplexes of K with $K=\bigcup_{n\in \boldsymbol{N}}K_n$. Then we can prove similarly as [**19**, Lemma 3] that the tower $\{|K_n|\}_{n\in \boldsymbol{N}}$ has the mapping absorption property for f.d. compacta. Similarly as [**19**, Lemma 4], it is proved that the tower $\{[-n, n]^n\}_{n\in \boldsymbol{N}}$ in σ has the mapping absorption property for f.d. compacta. Let $X_n=|K_n|\times[-n, n]^n$, $n\in \boldsymbol{N}$. Since each compact set in a σ-manifold is a strong Z-set (cf. [**10**]), each X_n is an f.d. compact strong Z-set in X. The tower $\{X_n\}_{n\in \boldsymbol{N}}$ has the mapping absorption property for f.d. compacta and is obviously finitely expansive, hence it is strongly universal for f.d. compacta by Lemma 1. □

LEMMA 3. *Let $\{X_n\}_{n\in \boldsymbol{N}}$ be a tower of f.d. compacta in a σ-manifold N with $N=\bigcup_{n\in \boldsymbol{N}}X_n$ which is strongly universal for f.d. compacta. Then for each tower $\{Y_i\}_{i\in \boldsymbol{N}}$ of f.d. compacta in N and for each open cover $\mathcal{U}$ of N, there exists a homeomorphism $f: N\to N$ of N onto itself such that f is $\mathcal{U}$-near to* id *and each $f(Y_i)$ is contained in some X_n.*

PROOF. Let d be a metric for N such that

$$\{\{y\in N|\, d(x, y)<1\}\mid x\in N\} < \mathcal{U}.$$

Using the strong universality of $\{X_n\}_{n\in \boldsymbol{N}}$ and the Homeomorphism Extension Theorem (= the Unknotting Theorem for Z-sets) (see [**2**, Theorem 25]), we can easily obtain a homeomorphism $h_1: N \to N$ such that $h_1(Y_1) \subset X_{n_1}$ for some $n_1 > 0$ and $d(h_1, \mathrm{id}) < 2^{-1}$. Define a metric d_1 for N as follows:

$$d_1(x, y) = \max\{d(x, y), d(h_1^{-1}(x), h_1^{-1}(y))\}.$$

Similarly as above, we have a homeomorphism $h_2: N \to N$ such that $h_2|X_{n_1} = \mathrm{id}$, $h_2(h_1(Y_2)) \subset X_{n_2}$ for some $n_2 > n_1$ and $d_1(h_2, \mathrm{id})$ $(= d_1(h_2^{-1}, \mathrm{id})) < 2^{-2}$. Thus, inductively, we obtain homeomorphisms $h_i: N \to N$, $i = 1, 2, \cdots$ and integers $0 < n_1 < n_2 < \cdots$ such that

(1) $h_i | X_{n_{i-1}} = \mathrm{id}$,

(2) $h_i(h_{i-1} \cdots h_1(Y_i)) \subset X_{n_i}$,

(3) $d(h_i, \mathrm{id}) = d(h_i h_{i-1} \cdots h_1, h_{i-1} \cdots h_1) < 2^{-i}$, and

(4) $d(h_1^{-1} \cdots h_{i-1}^{-1} h_i^{-1}, h_1^{-1} \cdots h_{i-1}^{-1}) < 2^{-i}$.

For each $x \in N$, $\{h_i \cdots h_2 h_1(x)\}_{i \in \boldsymbol{N}}$ and $\{h_1^{-1} h_2^{-1} \cdots h_i^{-1}(x)\}_{i \in \boldsymbol{N}}$ converge to points $f(x)$ and $g(x)$ in N respectively in view of (1). According to (3) and (4), $\{h_i \cdots h_2 h_1\}_{i \in \boldsymbol{N}}$ and $\{h_1^{-1} h_2^{-1} \cdots h_i^{-1}\}_{i \in \boldsymbol{N}}$ are uniformly Cauchy, so they are uniformly convergent to functions (hence maps) $f: N \to N$ and $g: N \to N$, respectively. It is easy to see that $fg = \mathrm{id}$ and $gf = \mathrm{id}$. Then f is a homeomorphism of N onto itself with $f^{-1} = g$. From (1) and (2), $f(Y_i) \subset X_{n_i}$ for each $i \in \boldsymbol{N}$. Observe

$$d(f, \mathrm{id}) \leqq \sum_{i \in \boldsymbol{N}} d(h_i, \mathrm{id}) < \sum_{i \in \boldsymbol{N}} 2^{-i} = 1,$$

so f is $\mathcal{U}$-near to id. □

The following is an easy version of the author's characterization of $\boldsymbol{R}^\infty$-manifolds [**16**].

LEMMA 4. *Let $\{X_n\}_{n \in \boldsymbol{N}}$ be a tower of f.d. compacta. Then* $\operatorname{dir\,lim} X_n$ *is an $\boldsymbol{R}^\infty$-manifold if and only if* $\operatorname{dir\,lim} X_n$ *is an ANE for f.d. compacta and $\{X_n\}_{n \in \boldsymbol{N}}$ is finitely expansive.*

The following can be proved similarly as [**19**, Theorem 1].

LEMMA 5. *Let $\{X_n\}_{n \in \boldsymbol{N}}$ be a tower of f.d. compacta in an ANR X which has the mapping absorption property for f.d. compacta. Then the identity of X induces a fine homotopy equivalence h:* $\operatorname{dir\,lim} X_n \to X$.

2. Proof of Theorem.

First, we will prove the "only if" part of Conjecture under the assumption that each compact set in Y is a strong Z-set, that is,

PROPOSITION 1. *Let $h: M \to Y$ be a bijective fine homotopy equivalence from an $\boldsymbol{R}^\infty$-manifold M to a metric space $Y = (Y, d)$. If each compact set in Y is a strong Z-set then Y is a σ-manifold.*

PROOF. Since M is an ANE for metrizable spaces, Y is an ANR [**13**, Ch. IV, Theorem 6.3]. Write $M=\text{dir}\lim X_n$ where $\{X_n\}_{n\in N}$ is a tower of f.d. compacta and put $Y_n=h(X_n)$, $n\in N$. Then the tower $\{Y_n\}_{n\in N}$ of f.d. compact strong Z-sets in Y is finitely expansive by Lemma 4. We will see that $\{Y_n\}_{n\in N}$ has the mapping absorption property for f.d. compacta. Let $f: A\rightarrow Y$ be a map of an f.d. compactum, let $m\in N$ and $\varepsilon>0$. For an open cover $\mathcal{U}$ of Y with $\text{mesh}\,\mathcal{U}=\sup\{\text{diam}\,U \mid U\in\mathcal{U}\}<\varepsilon/2$, we have a $\mathcal{U}$-homotopy inverse g of h. Note that $h^{-1}|Y_m: Y_m\rightarrow M$ is a map which is $h^{-1}(\mathcal{U})$-homotopic to $g|Y_m$. By the Homotopy Extension Theorem (see [**13**, Ch. IV, Theorem 2.2] and its proof), $h^{-1}|Y_m$ extends to a map $g': Y\rightarrow M$ which is $h^{-1}(\mathcal{U})$-homotopic to g. From compactness, $g'(f(A))\subset X_n$ for some $n\geqq m$. Then the map $hg'f: A\rightarrow Y_n$ satisfies the required conditions, $hg'f|f^{-1}(X_m)=f|f^{-1}(X_m)$ and $d(hg'f, f)<\varepsilon$. The proposition follows from Lemmas 1 and 2. □

Let K be a simplicial complex and L a subcomplex of K. The *simplicial neighborhood* of L in K is the subcomplex

$$N(L, K)=\{S\in K \mid \exists S'\in K \text{ s.t. } S<S' \ \&\ S'\cap|L|\neq\emptyset\}$$

and the *simplicial complement* of L in K is the subcomplex

$$C(L, K)=\{S\in K \mid S\cap|L|=\emptyset\}.$$

Note $|N(L, K)|$ is a (topological) neighborhood of $|L|$ in both $|K|_{\mathrm{w}}$ and $|K|_{\mathrm{m}}$. By Sd^2K, we denote the second barycentric subdivision of K. Note that $|\mathrm{Sd}^2K|_{\mathrm{m}}=|K|_{\mathrm{m}}$ though $|K'|_{\mathrm{m}}\neq|K|_{\mathrm{m}}$ for some subdivision K' of K. Then $|N(\mathrm{Sd}^2L, \mathrm{Sd}^2K)|$ is also a neighborhood of $|L|$ in both $|K|_{\mathrm{w}}$ and $|K|_{\mathrm{m}}$. In general, a neighborhood of $|L|$ in $|K|_{\mathrm{w}}$ is not a neighborhood in $|K|_{\mathrm{m}}$. However we have

LEMMA 6. *Let K be a simplicial complex and L a subcomplex of K. Then for each neighborhood U of $|L|$ in $|K|_{\mathrm{w}}$ there exists a homeomorphism h: $|K|_{\mathrm{w}}\rightarrow|K|_{\mathrm{w}}$ such that $h|(|L|\cup|C(L, K)|)=\mathrm{id}$, $h(S)=S$ for each $S\in K$ and $h(U)\supset|N(\mathrm{Sd}^2L, \mathrm{Sd}^2K)|$ (hence $h(U)$ is a neighborhood of $|L|$ in $|K|_{\mathrm{m}}$).*

The desired homeomorphism can be constructed by the skeletonwise induction. This is not so difficult. Perhaps the result may be known. Then we omit the proof.

Now we will prove Theorem.

PROOF OF THEOREM. We will apply Lemmas 1 and 2. First note $|K|_{\mathrm{w}}=\text{dir}\lim|L_n|$ where $\{L_n\}_{n\in N}$ is a tower of finite subcomplexes of K. From the proof of Proposition 1, the tower $\{|L_n|\}_{n\in N}$ of f.d. compacta in $|K|_{\mathrm{m}}$ is strongly universal for f.d. compacta. It remains to see that each $|L_n|$ is a strong Z-set in $|K|_{\mathrm{m}}$. We show that for each finite subcomplex L of K, $|L|$ is a strong

Z-set in $|K|_m$. Let $\mathcal{U}$ be an open cover of $|K|_m$ and $\mathcal{V}$ a star-refinement of $\mathcal{U}$. By [**10**, Lemma V. 7], we can assume $K<\mathcal{V}$. Since $h=\mathrm{id}: |K|_w \to |K|_m$ is a fine homotopy equivalence and $|L|_w=|L|_m$, h admits a $\mathcal{V}$-homotopy inverse $g: |K|_m \to |K|_w$ with $g||L|=\mathrm{id}$ (see the proof of Proposition 1). Since any compact set in an $\boldsymbol{R}^\infty$-manifold is contained in a collared closed submanifold [**17**, Cor. 1-5 & Theorem 3-1], we have a map $k: |K|_w \to |K|_w$ $\mathcal{V}$-near to id such that $U \cap k(|K|)=\emptyset$ for some neighborhood U of $|L|$ in $|K|_w$. By Lemma 6, we have a homeomorphism $f: |K|_w \to |K|_w$ $\mathcal{V}$-near to id such that $f(|L|)=|L|$ and $f(U)$ is a neighborhood of $|L|$ in $|K|_m$. Then $hf(U)$ is a neighborhood of $|L|$ in $|K|_m$ such that $hf(U) \cap hfkg(|K|)=\emptyset$. Thus id is $\mathcal{U}$-near to the map $hfkg: |K|_m \to |K|_m$ with $|L| \cap \mathrm{cl}\, hfkg(|K|)=\emptyset$. The proof is completed. □

3. Corollaries.

By Δ^∞, we denote the countably infinite full complex (∞-simplex), that is, the countably infinite simplicial complex such that each finite subset of vertices spans a simplex of Δ^∞. Then $|\Delta^\infty|_w \approx \boldsymbol{R}^\infty$ (see [**9**] or [**16**]), hence $|\Delta^\infty|_m \approx \sigma$. A *combinatorial* ∞*-manifold* is a countable simplicial complex such that the star of each vertex is combinatorially equivalent to Δ^∞, that is, they admit simplicially isomorphic subdivisions ([**18**]). For each combinatorial ∞-manifold K, $|K|_w$ is an $\boldsymbol{R}^\infty$-manifold. Then we have

COROLLARY 1. *For each combinatorial* ∞*-manifold* K, $|K|_m$ *is a* σ*-manifold.*

We should remark that for any subdivision K' of K the topology of $|K'|_m$ is not necessarily the same as $|K|_m$. We have the following Combinatorial Triangulation Theorem for σ-manifolds by [**2**, Theorem 9] and [**18**, Theorem 3.6].

COROLLARY 2. *Each* σ*-manifold is homeomorphic to a combinatorial* ∞*-manifold with the metric topology.*

Since the identity of $|K|$ is a fine homotopy equivalence from $|K|_w$ to $|K|_m$, it follows

COROLLARY 3. (a) *Each* $\boldsymbol{R}^\infty$*-manifold* M *has a continuous metric* d *such that the metric space* (M, d) *is a* σ*-manifold and the identity of* M *is a fine homotopy equivalence from* M *to* (M, d). (b) *Each* σ*-manifold* N *can be obtained from an* $\boldsymbol{R}^\infty$*-manifold* M *by changing topology so that the identity of* N *is a fine homotopy equivalence from* M *to* N.

A *metric direct limit* (or *system*) is a direct limit (or system) in the category of metric spaces and isometries. By Lemma 4 and Corollary 3 (a), we can generalize [**11**, Theorem 16] and [**21**, Theorem 3] as follows,

COROLLARY 4. *If* $X_1 \subset X_2 \subset \cdots$ *is a finitely expansive tower of f.d. compact ANR's, then each* X_n *can be metrized by a metric* d_n *so that* $(X_1, d_1) \subset (X_2, d_2) \subset \cdots$ *is a metric direct system whose limit is a* σ*-manifold and the identity induces a fine homotopy equivalence from* $\operatorname{dir\,lim} X_n$ *(which is an* $\boldsymbol{R}^\infty$*-manifold) to the metric direct limit.*

We remark that Corollary 3 (b) can be proved in the following strong form by using Lemmas 2, 3, 4 and 5.

PROPOSITION 2. *Let* $f: \operatorname{dir\,lim} Y_n \to N$ *be a continuous bijection from the direct limit of a tower* $\{Y_n\}_{n \in N}$ *of f.d. compacta to a* σ*-manifold* N*. Then there exists an* $\boldsymbol{R}^\infty$*-manifold* M *and* f *is factored by a continuous bijection* $g: \operatorname{dir\,lim} Y_n \to M$ *and a bijective fine homotopy equivalence* $h: M \to N$*.*

4. Counter-example for the "only if" part of Conjecture.

Here we will construct a bijective fine homotopy equivalence from $\boldsymbol{R}^\infty$ to an AR which is not a σ-manifold. Let

$$X = (0, 1] \times \{0\} \cup \{2^{-n} \mid n \in \boldsymbol{N}\} \times [0, 1],$$
$$Y = (0, 1] \times [-1, 0] \cup \{(0, -1)\},$$
$$Z = \{(s, t) \mid 0 \leqq -t \leqq s \leqq 1\}$$

and let $A = \{(0, -1)\}$. Since any locally compact separable metric space is the direct limit of a tower of compacta which covers the space, the product of an f.d. locally compact separable ANR (resp. AR) and $\boldsymbol{R}^\infty$ is an $\boldsymbol{R}^\infty$-manifold (resp. homeomorphic to $\boldsymbol{R}^\infty$). Hence $X \times \boldsymbol{R}^\infty \approx \boldsymbol{R}^\infty$. Observe

$$Y \times \boldsymbol{R}^\infty / A \times \boldsymbol{R}^\infty \approx \operatorname{cone}((0, 1] \times \{0\} \cup \{1\} \times [-1, 0]) \times \boldsymbol{R}^\infty$$
$$\approx \operatorname{cone} \boldsymbol{R}^\infty \approx \boldsymbol{R}^\infty$$

and moreover

$$(X \times \boldsymbol{R}^\infty) \cap (Y \times \boldsymbol{R}^\infty / A \times \boldsymbol{R}^\infty) = (0, 1] \times \boldsymbol{R}^\infty \approx \boldsymbol{R}^\infty.$$

Then by [**16**, Theorem 7-1] we have

$$M = (X \cup Y) \times \boldsymbol{R}^\infty / A \times \boldsymbol{R}^\infty = (X \times \boldsymbol{R}^\infty) \cup (Y \times \boldsymbol{R}^\infty / A \times \boldsymbol{R}^\infty) \approx \boldsymbol{R}^\infty.$$

Let $B = \{(0, 0)\}$ and $N = ((X \cup Z) \times \sigma)_B$, i.e.,

$$N = B \cup ((X \cup Z \setminus B) \times \sigma)$$

equipped with the finest topology in which the projection $\pi: (X \cup Z) \times \sigma \to N$ is continuous. Then N is a σ-f.d. compact AR which is not a σ-manifold because the compact set B is not a strong Z-set (see [1] or [20]). Let $h: M \to N$ be a continuous bijection defined by $h(*)=(0, 0)$ (where $*=A\times \boldsymbol{R}^\infty/A\times \boldsymbol{R}^\infty$) and for $(s, t, v)\in(X\cup Y\setminus A)\times \boldsymbol{R}^\infty$,

$$h(s, t, v)=\begin{cases}(s, t, i(v)) & \text{if } (s, t)\in X,\\ (s, st, i(v)) & \text{if } (s, t)\in Y\setminus A.\end{cases}$$

where $i: \boldsymbol{R}^\infty \to \sigma$ is the natural bijection. We will show that h is a fine homotopy equivalence. Let $\mathcal{U}$ be an open cover of N. Choose $U_0\in\mathcal{U}$ so that $B\subset U_0$. For each $m\in\boldsymbol{N}$, let

$$\begin{aligned}W_m &= \{(s, t)\in X\cup Z \mid s, t<2^{-m}\}\\ &= \{2^{-n} \mid n>m\}\times[0, 2^{-m})\cup\{(s, t)\in Z \mid s<2^{-m}\}.\end{aligned}$$

Then $(\mathrm{cl}W_{m-1}\setminus B)\times\sigma\subset U_0$ for some $m\in\boldsymbol{N}$ because $\{B\cup(W_m\setminus B)\times\sigma \mid m\in\boldsymbol{N}\}$ is a neighborhood base of B in N. Observe

$$\begin{aligned}\mathrm{cl}W_{m-1}\setminus W_m &= \{2^{-n} \mid n>m\}\times[2^{-m}, 2^{-m+1}]\cup\{2^{-m}\}\times[0, 2^{-m+1}]\\ &\qquad \cup\{(s, t)\in Z \mid 2^{-m}\leqq s\leqq 2^{-m+1}\}, \quad \text{and}\\ \mathrm{cl}W_m &= \{2^{-n} \mid n>m\}\times[0, 2^{-m}]\cup\{(s, t)\in Z \mid s\leqq 2^{-m}\}.\end{aligned}$$

Since $(X\cup Z)\setminus W_m$ is compact, we have an open cover $\mathcal{V}$ of σ such that

$$\{\{(s, t)\}\times V \mid (s, t)\in(X\cup Z)\setminus W_m,\ V\in\mathcal{V}\} < \mathcal{U}.$$

Let $j: \sigma\to\boldsymbol{R}^\infty$ be a $\mathcal{V}$-homotopy inverse of i. And then define a map $g: N\to M$ by $g(0, 0)=*$ and for $(s, t, v)\in(X\cup Z\setminus B)\times\sigma$,

$$g(s, t, v)=\begin{cases}(s, t, j(v)) & \text{if } (s, t)\in X\setminus W_{m-1},\\ (s, s^{-1}t, j(v)) & \text{if } (s, t)\in Z\setminus W_{m-1},\\ (s, t, 2^{m-1}t\cdot j(v)) & \text{if } s=2^{-m},\ 0\leqq t\leqq 2^{-m+1},\\ (s, t, (2^m t-1)\cdot j(v)) & \text{if } s=2^{-n}\ (n>m),\ 2^{-m}\leqq t\leqq 2^{-m+1},\\ (s, s^{-1}t, (2^m s-1)\cdot j(v)) & \text{if } (s, t)\in Z,\ 2^{-m}\leqq s\leqq 2^{-m+1},\\ (s, 2t-2^{-m}, 0) & \text{if } s=2^{-n}\ (n>m),\ 2^{-m-1}\leqq t\leqq 2^{-m},\\ (s, 2^{m+1}(1-2^m s)t+2^m s-1, 0) & \text{if } s=2^{-n}\ (n>m),\ 0\leqq t\leqq 2^{-m-1},\\ (s, 2^m t+2^m s-1, 0) & \text{if } (s, t)\in Z,\ 0<s\leqq 2^{-m}.\end{cases}$$

It is easy to verify that g is a $\mathcal{U}$-homotopy inverse of h.

5. Counter-example for the "if" part of Conjecture.

Here we will construct a tower of f.d. compact AR's in σ such that the identity of σ induces a fine homotopy equivalence from the direct limit of the tower to σ but the limit is not an $\boldsymbol{R}^{\infty}$-manifold.

Modifying the example of J. P. Henderson and J. J. Walsh in [**12**, § 3], we will construct a σ-f. d. compact AR X containing no embedded 2-cell but $X\times I\approx\sigma$ and moreover admitting a point $x_0\in X$, towers $\{X_n\}_{n\in\boldsymbol{N}}$ and $\{Y_n\}_{n\in\boldsymbol{N}}$ of f. d. compact AR's and ANR's, respectively, such that $X=\bigcup_{n\in\boldsymbol{N}}X_n$, $X\setminus\{x_0\}=\bigcup_{n\in\boldsymbol{N}}Y_n$ and $Y_n\subset X_n$ for each $n\in\boldsymbol{N}$. First we recall that each n-manifold M with $\partial M=\emptyset$ and $n\geqq 3$ admits an upper semi-continuous (u. s. c.) CE-decomposition $\mathcal{G}$ such that

(i) if M is non-compact then any sequence of elements $G_1, G_2, \cdots\in\mathcal{G}$ approaching infinity (i. e., having at most finitely many members contained in any compact subset of M) satisfies $\lim_{i\to\infty}\operatorname{diam}G_i=0$,

(ii) the decomposition space $M/\mathcal{G}$ contains no embedded 2-cell, and

(iii) the map $q\times\mathrm{id}: M\times\boldsymbol{R}\to(M/\mathcal{G})\times\boldsymbol{R}$ is a near homeomorphism, where $q: M\to M/\mathcal{G}$ is the quotient map (cf. [**12**, § 3]).

For each $n\in\boldsymbol{N}$, let

$$C_n=\partial[-1, 1]^{n+3} \quad\text{and}\quad D_n=(-1, 1)^{n+3}\setminus[-2^{-1}, 2^{-1}]^{n+3}.$$

Then each $C_n\setminus C_{n-1}$ and $D_n\setminus D_{n-1}$ (where $C_0=D_0=\emptyset$) admit u. s. c. CE-decompositions $\mathcal{E}_n$ and $\mathcal{F}_n$, respectively, which satisfy the conditions (i), (ii) and (iii). In general, for $t\in\boldsymbol{R}$ and $E\subset\boldsymbol{R}^n$ we denote $tE=\{tx \mid x\in E\}$ and for a collection $\mathcal{E}$ of subsets of $\boldsymbol{R}^n$, $t\mathcal{E}=\{tE \mid E\in\mathcal{E}\}$. From the condition (i),

$$\mathcal{G}_{n,i}=\bigcup_{m=1}^{n}\Big(\bigcup_{j=0}^{i}2^{-j}\mathcal{E}_m\cup\bigcup_{j=0}^{i-1}2^{-j}\mathcal{F}_m\Big)$$

is an u. s. c. CE-decomposition of $[-1, 1]^{n+3}\setminus(-2^{-i}, 2^{-i})^{n+3}$, and then

$$\mathcal{G}_n=\bigcup_{i\in\boldsymbol{N}}\mathcal{G}_{n,i}\cup\{\{0\}\}$$

is also an u. s. c. CE-decomposition of $[-1, 1]^{n+3}$. Since each C_{n-1} and D_{n-1} are bicollared in C_n and D_n, and since each $2^{-j}C_n$ is collared in $2^{-j}C_n\cup 2^{-j}D_n$ and $2^{-j}C_n\cup 2^{-j+1}D_n$, we can use the pseudo-isotopies implicitly in the condition (iii) to see that $\mathcal{G}_n$ and $\mathcal{G}_{n,i}$ satisfy the condition (iii) (cf., the proof in [**12**, § 3]). Of course the condition (ii) is satisfied. Note that $[-1, 1]^{\omega}_f=\sigma\cap[-1, 1]^{\omega}\approx\sigma$. It is easy to see that $\mathcal{G}=\bigcup_{n\in\boldsymbol{N}}\mathcal{G}_n$ is an u. s. c. CE-decomposition of $[-1, 1]^{\omega}_f$. Since each $[-1, 1]^{n+3}$ has a bicollar in $[-1, 1]^{n+4}$, it can be seen that $\mathcal{G}$ satisfies the conditions (ii) and (iii) (see the proof of [**12**, § 3]). Let $X=[-1, 1]^{\omega}_f/\mathcal{G}$, $x_0=q(0)$, where $q: [-1, 1]^{\omega}_f\to X$ is the quotient map, and for each $n\in\boldsymbol{N}$,

$$X_n=[-1, 1]^{n+3}/\mathcal{G}_n=q([-1, 1]^{n+3}) \quad\text{and}$$

$$Y_n=([-1, 1]^{n+3}\setminus(-2^{-n}, 2^{-n})^{n+3})/\mathcal{G}_{n,n}=q([-1, 1]^{n+3}\setminus(-2^{-n}, 2^{-n})^{n+3}).$$

Then each X_n is an f. d. compact AR and each Y_n is an f. d. compact ANR with $Y_n \subset X_n$. And X is a σ-f.d. compact AR with $X = \bigcup_{n\in\boldsymbol{N}} X_n$ and $X \setminus \{x_0\} = \bigcup_{n\in\boldsymbol{N}} Y_n$. Moreover X contains no embedded 2-cell, but $X \times \boldsymbol{R} \approx \sigma$ equivalently $X \times I \approx \sigma$ by [**12**, Theorem 3].

Now let $N = X \times I \setminus \{x_0\} \times (0, 1]$. Then $N \approx X \times I$ (e.g., see [**15**, Cor. 2-7]), hence $N \approx \sigma$. For each $n \in \boldsymbol{N}$, let

$$Z_n = X_n \times \{0\} \cup Y_n \times I.$$

Then each Z_n is an f.d. compact AR and $N = \bigcup_{n\in\boldsymbol{N}} Z_n$. We will show that the tower $\{Z_n\}_{n\in\boldsymbol{N}}$ has the mapping absorption property for f.d. compacta. It suffices to see that for each compact set A in N, $m \in \boldsymbol{N}$ and each open cover $\mathcal{U}$ of N, there exists a map $r: A \to Z_n$ from A to some Z_n, $n \geqq m$, such that $r | A \cap Z_m = \mathrm{id}$ and r is st($\mathcal{U}$)-near to id. Since Z_m is an ANR, we have an open cover $\mathcal{V}$ of N with $\mathcal{V} < \mathcal{U}$ such that arbitrary two $\mathcal{V}$-near maps in Z_m are $\mathcal{U}$-homotopic in Z_m. We will construct a map $r': A \to Z_n$ for some $n \geqq m$ such that r' is $\mathcal{V}$-near to id. Then the inclusion $A \cap Z_m \subset Z_n$ is $\mathcal{U}$-homotopic to $r' | A \cap Z_m$, hence extends to a map $r: A \to Z_n$ $\mathcal{U}$-homotopic to r' (hence st($\mathcal{V}$)-near to id) by Homotopy Extension Theorem [**13**, Ch. IV, Theorem 2.2]. Let $\mathcal{W}$ be a star-refinement of $\mathcal{V}$. By [**7**], there is a $\mathcal{W}$-homotopy inverse

$$g : N \longrightarrow [-1, 1]^{\omega}_f \times I \setminus \{0\} \times (0, 1] = (q \times \mathrm{id})^{-1}(N)$$

of $q \times \mathrm{id} | (q \times \mathrm{id})^{-1}(N)$. Since $g(A)$ is compact, we can choose $\varepsilon > 0$ and $n' \geqq m$ so that for $(x, s) \in g(A)$ and $(y, t) \in (q \times \mathrm{id})^{-1}(N)$ if $x_1 = y_1, \cdots, x_{n'} = y_{n'}$ and $|s - t| \leqq \varepsilon$ then $(q(x), s)$, $(q(y), t) \in W$ for some $W \in \mathcal{W}$. From compactness of $g(A) \cap [-1, 1]^{\omega}_f \times [\varepsilon, 1]$, there exists an $n \geqq n'$ $(\geqq m)$ such that

$$p_{n+3}^{-1}([-2^{-n}, 2^{-n}]^{n+3}) \times I \cap (g(A) \cap [-1, 1]^{\omega}_f \times [\varepsilon, 1]) = \emptyset$$

where $p_{n+3}: [-1, 1]^{\omega}_f \to [-1, 1]^{n+3}$ is the projection onto the first $(n+3)$-coordinates. Let $k: I \to I$ be the piecewise linear map with $k(0) = k(\varepsilon) = 0$ and $k(1) = 1$. Then

$$(q \times \mathrm{id})(p_{n+3} \times k)(g(A) \cap [-1, 1]^{\omega}_f \times [\varepsilon, 1]) \subset Y_n \times I \quad \text{and}$$

$$(q \times \mathrm{id})(p_{n+3} \times k)(g(A) \cap [-1, 1]^{\omega}_f \times [0, \varepsilon]) \subset X_n \times \{0\}.$$

Thus we have a map

$$r' = (q \times \mathrm{id})(p_{n+3} \times k) g | A : A \longrightarrow Z_n.$$

Observe that if $(x, s) \in g(A)$ and $(y, t) = (p_{n+3}(x), k(s))$ then $x_1 = y_1, \cdots, x_{n'} = y_{n'}$ and $|s - t| \leqq \varepsilon$. Thus it is seen that r' is $\mathcal{W}$-near to $(q \times \mathrm{id}) g | A$, hence $\mathcal{V}$-near to id.

By Lemma 5, the identity of N induces a fine homotopy equivalence from $\operatorname{dir\,lim} Z_n$ to N. However, $\operatorname{dir\,lim} Z_n$ is not an $\boldsymbol{R}^\infty$-manifold. In fact, if so, there is an embedding $u: I^2 \to \mathrm{dir\,lim}\, Z_n$ with $u(0, 0)=(x_0, 0)$. From compactness, $u(I^2) \subset Z_n$ for some $n \in \boldsymbol{N}$. Since $u(0, 0) \notin Y_n \times I$, $u([0, \delta]^2) \subset X_n \times \{0\} \subset X \times \{0\}$ for some $\delta > 0$. This contradicts the fact that X contains no embedded 2-cell.

6. Answer for a problem concerning enlargement of an R^∞-manifold.

Let X be an ANE for compacta which is the direct limit of a tower of f.d. compacta. In [**17**, Problem 6-4], the author asked whether X is an $\boldsymbol{R}^\infty$-manifold or not if X contains an $\boldsymbol{R}^\infty$-manifold M with $X \setminus M$ a D-set. (For the definition of D-sets, refer to [**17**, §1].) Here using the example in Section 5, we answer negatively this problem.

Let $\{Z_n\}_{n \in \boldsymbol{N}}$ be the tower of f.d. compact AR's constructed in Section 5. Then $Z = \mathrm{dir\,lim}\, Z_n$ is an AE for compacta which is not an $\boldsymbol{R}^\infty$-manifold. We will prove that $Z \setminus \{(x_0, 0)\}$ is an $\boldsymbol{R}^\infty$-manifold and $\{(x_0, 0)\}$ is a D-set in Z. First observe

$$Z \setminus \{(x_0, 0)\} = \mathrm{dir\,lim}(Y_n \times I) = (\mathrm{dir\,lim}\, Y_n) \times I .$$

By the arguments in the proof of [**12**, §3], we can see that the tower $\{Y_n \times \boldsymbol{R}\}_{n \in \boldsymbol{N}}$ is finitely expansive, hence by Lemma 4 $(\mathrm{dir\,lim}\, Y_n) \times \boldsymbol{R} = \mathrm{dir\,lim}(Y_n \times \boldsymbol{R})$ is an $\boldsymbol{R}^\infty$-manifold. This implies $Z \setminus \{(x_0, 0)\} = (\mathrm{dir\,lim}\, Y_n) \times I$ is an $\boldsymbol{R}^\infty$-manifold by [**17**, Theorem 6-2]. Next we show that $\{(x_0, 0)\}$ is a D-set in Z. Let $C \supset C_0$ be compact sets in Z and $\mathcal{U}$ an open cover of Z. We must construct an embedding $f: C \to Z$ such that f is $\mathcal{U}$-near to id, $f|C_0 = \mathrm{id}$ and $f(C \setminus C_0) \subset Z \setminus \{(x_0, 0)\}$. We may assume $(x_0, 0) \in C \setminus C_0$, otherwise the inclusion $C \subset Z$ is the desired embedding. Since C_0 is a D-set (cf. [**17**, Cor. 1-5]), it suffices by [**17**, Theorem 4-1] to construct a map $f: C \to Z \setminus \{(x_0, 0)\}$ such that f is $\mathcal{U}$-near to id and $f|C_0 = \mathrm{id}$. From compactness, $C \subset Z_n$ and $C_0 \subset Y_n \times I$ for some $n \in \boldsymbol{N}$. Choose $m > n$ so that $q([-2^{-m}, 2^{-m}]^{n+4}) \times \{0\} \subset U$ for some $U \in \mathcal{U}$. Note that

$$q([-1, 1]^{n+3} \times [0, 1] \setminus (-2^{-m}, 2^{-m})^{n+4})$$

is an AR which is a closed subset of $X_{n+1} = q([-1, 1]^{n+4})$ containing Y_n. Let

$$r : X_{n+1} \longrightarrow q([-1, 1]^{n+3} \times [0, 1] \setminus (-2^{-m}, 2^{-m})^{n+4})$$

be the retraction. Then the map $f = r \times \mathrm{id}|C : C \to Z \setminus \{(x_0, 0)\}$ is the desired embedding.

7. Q^∞-manifolds and Σ-manifolds.

Let $Q = [-1, 1]^\omega$ be the Hilbert cube. Similarly as $\bigcup_{n \in \boldsymbol{N}} \boldsymbol{R}^n$, the set $\bigcup_{n \in \boldsymbol{N}} Q^n$ $(\subset Q^\omega)$ admits two different natural topologies and then the spaces $Q^\infty = \mathrm{dir\,lim}\, Q^n$

and Σ are obtained. It is well-known that the pair (Q^{ω}, Σ) is homeomorphic to $(Q, B(Q))$, where $B(Q)=\{(x_i)_{i\in N}\in Q \mid x_i=\pm 1$ for some $i\}$ is the pseudo-boundary of Q and that Σ is homeomorphic to the linear span of the Hilbert cube $\prod_{i\in N}[-2^{-i}, 2^{-i}]$ in Hilbert space l_2. A separable topological manifold modeled on these spaces is called a *Q^∞-manifold* or a *Σ-manifold*, respectively. Similarly as manifolds modeled on $\boldsymbol{R}^\infty$ and σ, these manifolds are also considered as two topologizations on the same underlying sets. In Section 1, by replacing $\boldsymbol{R}^\infty$, σ, I and $[-n, n]^n$ by Q^∞, Σ, Q and Q^n, respectively, and by deleting the phrases "f. d." and "finitely", we can obtain the corresponding definitions, lemmas and proofs. Then we have the following version of Proposition 1.

PROPOSITION 1′. *Let $h: M\to Y$ be a bijective fine homotopy equivalence from a Q^∞-manifold M to a metric space Y. If each compact set in Y is a strong Z-set then Y is a Σ-manifold.*

Moreover the corresponding conjecture is also false. In fact, by suitable modifications of Sections 4 and 5, we have a bijective fine homotopy equivalence from Q^∞ to an AR which is not a Σ-manifold and one to Σ from the direct limit of a tower of compact AR's which is not a Q^∞-manifold. And then Problem 6-4 for Q^∞-manifolds in [**17**] is also negatively answered.

For a simplicial complex K, if $|K|_{\mathrm{w}}\times Q$ is a Q^∞-manifold then we can show that for each finite subcomplex L of K, $|L|\times Q$ is a strong Z-set in $|K|_{\mathrm{m}}\times Q$ by the same arguments in the proof of Theorem. Thus we can prove the following proposition.

PROPOSITION 3. *For a simplicial complex K, if $|K|_{\mathrm{w}}\times Q$ is a Q^∞-manifold then $|K|_{\mathrm{m}}\times Q$ is a Σ-manifold.*

Notice $Q^\infty\approx\boldsymbol{R}^\infty\times Q$ and $\Sigma\approx\sigma\times Q$. By the Triangulation Theorem (cf. [**16**] and [**2**]) and the above proposition, we have the version of Corollary 3.

COROLLARY 3′. (a) *Each Q^∞-manifold M has a continuous metric d such that the metric space (M, d) is a Σ-manifold and the identity of M is a fine homotopy equivalence from M to (M, d).* (b) *Each Σ-manifold N can be obtained from a Q^∞-manifold M by changing topology so that the identity of N is a fine homotopy equivalence from M to N.*

And then the version of Corollary 4 is also obtained.

COROLLARY 4′. *If $X_1\subset X_2\subset\cdots$ is an expansive tower of compact ANR's, then each X_n can be metrized by metric d_n so that $(X_1, d_1)\subset(X_2, d_2)\subset\cdots$ is a metric direct limit system whose limit is a Σ-manifold and the identity induces a fine homotopy equivalence from* $\operatorname{dir}\lim X_n$ *(which is a Q^∞-manifold) to the metric direct limit.*

Similarly we have also

PROPOSITION 2′. *Let* $f:\operatorname{dir}\lim Y_n \to N$ *be a continuous bijection from the direct limit of a tower* $\{Y_n\}_{n\in N}$ *of compacta to a* Σ*-manifold* N. *Then there exists a* Q^∞*-manifold* M *and* f *is factored by a continuous bijection* $g:\operatorname{dir}\lim Y_n \to M$ *and a bijective fine homotopy equivalence* $h: M \to N$.

Addendum. Recently, the converse of Theorem has been proved. In fact, the author [22] has proved that a simplicial complex K is a combinatorial ∞-manifold if $|K|_{\mathrm{m}}$ is a σ-manifold. Thereby *for any simplicial complex* K, *the following are equivalent:*

(i) K *is a combinatorial* ∞*-manifold;*

(ii) $|K|_{\mathrm{w}}$ *is an* $\boldsymbol{R}^\infty$*-manifold;*

(iii) $|K|_{\mathrm{m}}$ *is a* σ*-manifold.*

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