J. Math. Soc. Japan
Vol. 39, No. 2, 1987

Construction of Hopf G-spaces

Dedicated to the memory of late Professor Shichirô Oka

By Ken-ichi MARUYAMA

(Received Jan. 10, 1985)
(Revised Sept. 14, 1985)

§1. Introduction.

Let G be a topological group. The notion of a Hopf G-space is first noted by G. E. Bredon [**3**]. He defined a Hopf G-space to be a space which has a G-equivariant multiplication. Some people did their works in this area, K. Iriye [**6**] on Hopf Z_2-spheres, G. Triantafillou [**11**] on rational cases, etc.

In this paper we shall construct some examples of equivariant Hopf spaces by a method analogous to Zabrodsky's. Actually, A. Zabrodsky exploited his mixing homotopy method ([**13**], [**14**] and [**15**]) to obtain many non-classical Hopf spaces (including the Hilton-Roitberg's example, etc.). We shall discuss an equivariant version of his method under some conditions. For this, we shall use the equivariant localization of J. P. May, et al. [**9**].

Our main results are the following two theorems. Throughout the paper, we assume that G is a compact Lie group.

THEOREM 1.1. *Let S^n be the n-sphere with n odd >1, on which G acts desuspendably, i.e., the action is the suspension of a G-action on S^{n-1}. Let E be a compact Lie group on which G acts by automorphisms. Moreover assume that E acts on S^n transitively and the induced fibration $\pi: E\to S^n$ is a G-fibration, i.e. π is a G-map and has a G-homotopy covering property. Let $h_\lambda: S^n\to S^n$ be the G-map which is of degree $\lambda\in Z$, that is, h_λ is λ times the identity map of S^n in $[S^n, S^n]_G$, and T be a collection of prime numbers. Then the pull back W_{h_λ} in the following diagram*

$$\begin{array}{ccc} W_{h_\lambda} & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ S^n & \xrightarrow{h_\lambda} & S^n \end{array}$$

has a Hopf G-structure, if the following three conditions are satisfied.

a) *E^H and $(S^n)^H$ are connected, and $(S^n)^H$ is also a sphere, for each closed subgroup H of G.*

b) *The equivariant localization S_T^n is a G-homotopy commutative Hopf G-space and the set $[S_T^n \times S_T^n, S_T^n]_G$ becomes a group.*

c) *$p|\lambda$ implies $p \in T$.*

REMARK. We shall prove Theorem 1.1 under more general situation by introducing the G-equivariant operation. But all examples of section 4 will satisfy the conditions written above.

THEOREM 1.2. *Let S^m be an odd dimensional sphere and G acts on S^m such that $(S^m)^H$ is an odd dimensional sphere for each closed subgroup H. Let l be a collection of prime numbers such that $2m < p(t_0+1)-3$ for each $p \in l$, where $t_0 = \min\{\dim(S^m)^H ; H < G\}$. Then S_l^m (equivariant localization at l) is a G-homotopy commutative Hopf G-space.*

This paper will be organized as follows. In section 2 we prove an equivariant version of Zabrodsky's theorem. In section 3, Theorem 1.2 above will be shown by means of previous sections. In the final section, we present examples of Theorem 1.1.

We shall use the following notation throughout the paper. If X is a G-CW complex and T is a collection of primes, we shall write X_T for an equivariant localization of X at T in the sense of [**9**]. We write $[\ ,\ ]_G$ for a G-homotopy classes of G-maps. One should refer Bredon [**3**], Matumoto [**7**], Warner [**12**], for general references on G-CW theory.

The author would like to thank Professors A. Kono and M. Mimura.

§ 2. Equivariant Zabrodsky's theorem.

In this section we shall prove an equivariant version of Zabrodsky's theorem ([**15**]) under some conditions. We would rather use the proof by M. Arkowitz [**2**] (Zabrodsky did not use any words "localization").

DEFINITION 2.1. Let $X = S^1 \wedge Y$ be the reduced suspension of a space Y. A G-action on X is said to be desuspendable if there is a G-action on Y such that $g(t, y) = (t, gy)$ for $(t, y) \in X$, $g \in G$.

Then the G-homotopy set $[X, Z]_G$ becomes a group in the usual way.

DEFINITION 2.2. Suppose G acts desuspendably on S^n, $n \geqq 1$. A G-map $h_\lambda : S^n \to S^n$ is said to be of degree $\lambda \in Z$ if h_λ is λ times the identity map in $[S^n, S^n]_G$.

DEFINITION 2.3. Let $f: X \to Y$ be a G-map. A G-equivariant operation of X on Y is a G-map $\mu : X \times Y \to Y$ such that

$$\mu i_1 \underset{G}{\simeq} f : X \to Y, \qquad \mu i_2 \underset{G}{\simeq} \mathrm{Id} : Y \to Y,$$

where i_1, i_2 are inclusions.

To state our result, we prepare the following data.

(2.4) (a) an n-sphere S^n, n odd >1, on which G acts desuspendably.
(b) a Hopf G-space E with multiplication μ_E.
(c) a G-fibration $\pi : E \to S^n$.
(d) a G-equivariant operation $\mu : E \times S^n \to S^n$ compatible with the multiplication μ_E via π, that is,

$$\pi\mu_E \underset{G}{\simeq} \mu(1\times\pi) : E\times E \to S^n.$$

(e) a G-map $h_\lambda : S^n \to S^n$ of degree $\lambda \in Z$.
(f) a set T of primes.
(g) the pull back $W = W_{h_\lambda}$ given by the pull back diagram:

$$\begin{array}{ccc} W_{h_\lambda} & \longrightarrow & E \\ \downarrow{\scriptstyle r} & & \downarrow \\ S^n & \xrightarrow{h_\lambda} & S^n. \end{array}$$

Then our theorem is stated as follows:

THEOREM 2.5. *Assume* (2.4) *and that*
(a) E^H *and* (S^n) *are connected and* $(S^n)^H$ *is also a sphere, for each closed subgroup* H.
(b) *The equivariant localization* S^n_T *is an abelian Hopf G-space and* $[S^n_T \times S^n_T, S^n_T]_G$ *becomes a group.*
(c) $p|\lambda$ *implies* $p \in T$.
Then the pull back W *has a Hopf G-structure.*

REMARK 2.6. In case $G=e$, i.e., the nonequivariant case, a compact Lie group E acting transitively on S^n satisfies conditions (b)-(d) of (2.4). Theorem 1.1 is the equivariant version in this special situation.

REMARK 2.7. If $G=e$ and T is a set of odd primes, then the above theorem is (the nonequivariant) Zabrodsky theorem [**15**].

We need some definitions before the proof.

DEFINITION 2.8. Given spaces and maps

$$X \xrightarrow{f} A \xleftarrow{g} Y,$$

define

$$W(f, g) = \{(x, w, y) \mid x \in X, w \in A^I, y \in Y,\\ f(x) = w(0), g(y) = w(1)\},$$

the weak pull back of f and g. So we obtain the following homotopy commutative diagram;

$$\begin{tikzcd} W(f, g) \arrow[r, "s"] \arrow[d, "r"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

where r and s are the canonical maps. We note that $W(f, g)$ becomes canonically a G-space when f and g are G-maps. In this case we should note that $W(f, g)^H = W(f^H, g^H)$ is a weak pull back of f^H and g^H, for each closed subgroup H.

Here we should note that $W(f, g)$ can be replaced by a G-CW complex up to weak G-equivalence (see [**12**], [8]).

DEFINITION 2.9. Let f, g be G-maps. The following G-homotopy commutative square

$$\begin{tikzcd} W \arrow[r, "b"] \arrow[d, "a"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & A \end{tikzcd}$$

is called a G-weak pull back diagram if there exists a G-homotopy equivalence $\delta : W \to W(f, g)$ such that $r\delta \underset{G}{\simeq} a$, $s\delta \underset{G}{\simeq} b$.

Also here we should note that we obtain a nonequivariant sense weak pull back diagram for every W^H.

PROOF OF THEOREM 2.5. Let T' be the complementary set of primes for T, i.e. $T \cap T' = \emptyset$, $T \cup T' =$ all primes. The following square is a G-weak pull back diagram, where j and j' are the G-localization maps.

$$\begin{tikzcd} W_{T \cup T'} \arrow[rr] \arrow[dd] & & W_T \arrow[d, "s_T"] \\ & & E_T \arrow[d, "j"] \\ W_{T'} \arrow[r, "s_{T'}"] & E_{T'} \arrow[r, "j'"] & E \end{tikzcd}$$

Therefore there is a G-map $d: W_{T\cup T'} \to W(js_T, j's_{T'})$, which is a G-homotopy equivalence.

$s_{T'}$ is a G-homotopy equivalence as we consider homotopy exact sequences of G-fibrations $W_{T'} \to S^n_{T'}$ and $E_{T'} \to S^n_{T'}$. Then $W_{T'}$ has a Hopf G-structure μ'_0 such that $s_{T'}$ is a Hopf G-map (that is, $(\mu_E)_{T'} \circ (s_{T'} \times s_{T'}) \underset{G}{\simeq} s_{T'} \circ \mu'_0$). If we can define a Hopf G-structure μ_0 on W_T such that $(\mu_E)_T \circ (s_T \times s_T) \underset{G}{\simeq} s_T \circ \mu_0$, then it is easy to obtain a Hopf G-structure μ_W on W, since js_T becomes a Hopf G-maps and $j's_{T'}$ is already so.

Now we shall construct a Hopf G-structure μ_0 on W_T. As the following square is a G-weak pull back diagram, it is enough to construct it on the weak pull back $W(h_T, \pi_T)$

$$\begin{tikzcd}
(W_h)_T \arrow[r, "s_T"] \arrow[d, "r'_T"'] & E_T \arrow[d, "\pi_T"] \\
S^n_T \arrow[r, "h_T"] & S^n_T .
\end{tikzcd}$$

We label G-homotopies as follows. Let μ_s be the Hopf G-structure on S^n_T.

$$F: E_T \times S^n_T \longrightarrow (S^n_T)^I; \qquad F(\ )(0) = \mu(\ ), \quad F(\ )(1) = wq + \mu_s(\pi_T \times 1).$$

Here $\mu \simeq wq + \mu_s(\pi_T \times 1)$ for some $w \in [E_T \wedge S^n_T, S^n_T]$, which is obtained from the G-Puppe sequence argument, and $q: E_T \times S^n_T \to E_T \wedge S^n_T$ is the projection.

$$H: E_T \times E_T \longrightarrow (S^n_T)^I; \qquad H(\ )(0) = \mu(1 \times \pi_T)(\ ), \quad H(\ )(1) = \pi_T(\mu_E)_T(\ ),$$

the homotopy as in (2.4) (d).

As we may take $h_T: S^n_T \to S^n_T$ as a map of λ times the identity map of S^n_T by the multiplication μ_S, h_T can be considered as a Hopf G-map. Therefore, there is a homotopy as follows.

$$\begin{aligned} J: S^n_T \times S^n_T \times S^n_T \longrightarrow (S^n_T)^I; \qquad & J(\ )(0) = h_T(\mu_s(1 \times \mu_s)), \\ & J(\ )(1) = \mu_s(1 \times \mu_s)(h_T \times h_T \times h_T). \end{aligned}$$

Finally, we prepare the following homotopy.

$$K: E_T \wedge S^n_T \longrightarrow (S^n_T)^I; \qquad K(\ )(0) = h_T w, \quad K(\ )(1) = w(1 \wedge h_T).$$

We remark that F and J are taken as relative to $E_T \vee S^n_T$ and $S^n_T \vee S^n_T \vee S^n_T$ respectively because of the following fact.

LEMMA 2.10. ((G-) James theorem, see [16]). *Suppose $\Sigma A \subset \Sigma B$ is G-retract and (X, μ) a Hopf G-space. If two G-homotopic maps g_0, g_1 from B to X satisfy that $g_0|A = g_1|A$ then g_0 and g_1 are G-homotopic* rel A.

PROOF. This is exactly the same as the nonequivariant case. So we omit the proof.

We define a path from $h_T(\mu_s(wq(y, x'), \mu_s(x, x')))$ to $\pi_T(\mu_E)_T(y, y')$ by the above homotopies as

$$H \circ F^{-1} \circ \mu_s(w(y, \phi'), \mu_s(\phi, \phi')) \circ K \circ J$$

for (x, ϕ, y), $(x', \phi', y') \in (W_h)_T$, with $h_T(x)=\phi(0)$, $\pi_T(y)=\phi(1)$, $h_T(x')=\phi'(0)$ and $\pi_t(y')=\phi'(1)$, $x, x' \in S_T^n$, $y, y' \in E_T$. F^{-1} denotes the inverse path of F.

We denote this path as $M=M((x, \phi, y), (x', \phi', y'))$. Then the desired Hopf G-structure μ_0 on W_T can be obtained by

$$\mu_0((x, \phi, y), (x', \phi', y')) = (\mu_s(wq(y, x'), \mu_s(x, x')), M, (\mu_E)_T(y, y')).$$

§3. A sufficient condition for S^n to be a Hopf G-space.

In [1], J. F. Adams has shown that the odd dimensional sphere can be considered as a Hopf space mod p. Here we give some condition for the equivariant analogue of his result to hold with some specific action on the sphere. Let S^m be an odd dimensional sphere and G acts on S^m. Assume that $(S^m)^H$ is always an odd dimensional sphere and we denote that $t_0 = \min\{\dim(S^m)^H ; H<G\}$, where H is any closed subgroup of G.

THEOREM 1.2. *Let l be a collection of primes. Assume that for any $p \in l$, $2m<p(t_0+1)-3$. Then S_l^m is a G-commutative Hopf G-space.*

PROOF. Embed S_l^m equivariantly in $\Omega^2\Sigma^2 S_l^m$ by the natural G-map j. It is well known that for an odd dimensional sphere S^t, $\pi_i(\Omega^2\Sigma^2 S^t, S^t)_{(p)}=0$, for $i<p(t+1)-2$, p an odd prime (see [10], p. 516). Therefore, $\pi_i(\Omega^2\Sigma^2(S_l^m)^H, (S_l^m)^H)=0$ if $i<p(t_0+1)-2$, $p \in l$. Making use of Proposition (3.3) of [7], we obtain the following bijection if $2m<p(t_0+1)-3$;

$$[S_l^m \times S_l^m, S_l^m]_G \xrightarrow{j^*} [S_l^m \times S_l^m, \Omega^2\Sigma^2 S_l^m]_G.$$

Let $\mu: \Omega^2\Sigma^2 S_l^m \times \Omega^2\Sigma^2 S_l^m \to \Omega^2\Sigma^2 S_l^m$ be the loop multiplication. Then μ is clearly a G-homotopy commutative G-structure. Now we define

$$m = (j^*)^{-1}(\mu(j^* \times j^*)) : S_l^m \times S_l^m \longrightarrow S_l^m.$$

It is clear that m is a Hopf G-structure on S_l^m which is G-homotopy commutative.

§4. Examples.

In this section we will give examples for Theorem 2.5. We will introduce some actions on Lie groups which induce the actions on odd dimensional spheres mentioned in section 3.

First we introduce the following lemma which is observed by Bredon [**3**] when G is finite.

LEMMA 4.1. *Let B be a G-space such that B^H is connected and simple for each closed subgroup H. Then a G-map $\pi: E \to B$ has the G-covering homotopy property for every G-CW complex, if and only if $\pi^H: E^H \to B^H$ is a fibration in a nonequivariant sense for each closed subgroup H.*

PROOF. Let X be a G-CW complex. Let $f_t: X \to B$ be a G-homotopy and $g: X \to E$ be a G-map such that $\pi g = f_0$. We are able to use the induction on G-cells of X to construct a required homotopy. Actually, if $G/H \times D^n$ is a G-cell of X, our lifting problem can be considered as a nonequivariant problem through the bijection

$$[G/H \times D^n \times I,\ G/H \times S^n \times I\,;\, B,\ E]_G \simeq [D^n \times I,\ S^n \times I\,;\, B^H,\ E^H].$$

EXAMPLE 4.2. The case $G = Z_2$, $E = U(n)$ and the usual transitive operation $\mu: U(n) \times S^{2n-1} \to S^{2n-1}$. We represent Z_2 as the subgroup of $U(n)$ generated by

$$a_t = \left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{t}} & \multicolumn{3}{c}{\overbrace{\qquad\qquad}^{n-t}} \\ -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ \hline & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{array}\right), \qquad 1 \leqq t \leqq n$$

which defines an action $*$ of Z_2 on $U(n)$ by the conjugation:

$$a_t * A = a_t A a_t, \qquad A \in U(n).$$

We call this action to be of type t. With this action, $U(n)$ can be considered as a Hopf $G\,(=Z_2)$-space. The induced action on $U(n)/U(n-1) = S^{2n-1}$ can be seen as follows. Take $x \in S^{2n-1}$, $x = (x_1, \cdots, x_n)$, $\sum_i |x_i|^2 = 1$, $x_i \in C$. Then the induced action $*$ on S^{2n-1} becomes

$$a_t * x = (-x_1, \cdots, -x_t, x_{t+1}, \cdots, x_n).$$

Now we observe fixed point sets of this action as follows.

$$U(n)^{Z_2} = \left\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in U(n)\ ;\ A \in U(t),\ B \in U(n-t)\right\},$$

$$(S^{2n-1})^{Z_2} = \{(0, \cdots, 0, x_{t+1}, \cdots, x_n) \in S^{2n-1}\}.$$

Therefore we see that $U(n)^{Z_2} \to (S^{2n-1})^{Z_2}$ is the fibration. This corresponds to the condition (c) of (2.4) by Lemma 4.1. For adopting (2.4), we have to check the conditions other than (c) above. But for the condition (d), the above operation μ can be considered as a $G(=Z_2)$-operation. For (a) and (e), we should add the assumption $n-t\geqq 1$. Finally, for the condition (b) of Theorem 2.5, we appeal to Theorem 1.2. Let λ be an integer such that if a prime p dividing λ satisfies $p>(4n+1)/(2n-2t)$. We should take T in (2.4) to be satisfied that $p>(4n+1)/(2n-2t)$ for each $p\in T$. Then we obtain a Hopf Z_2-space W_{h_λ} by the following pull back, where h_λ is the map of degree λ.

$$\begin{tikzcd} U(n-1) \arrow[r, equal] \arrow[d] & U(n-1) \arrow[d] \\ W_{h_\lambda} \arrow[r] \arrow[d] & U(n) \arrow[d] \\ S^{2n-1} \arrow[r, "h_\lambda"] & S^{2n-1}. \end{tikzcd}$$

For small values of t, we may choose the degree λ as follows,

$t=1$; $$\lambda=\begin{cases} \text{an odd integer} & \text{if } n\geqq 4 \\ \text{an odd integer with } (\lambda, 3)=1 & \text{if } n=2, 3 \end{cases}$$

$t=2$; $$\lambda=\begin{cases} \text{an odd integer} & \text{if } n\geqq 7 \\ \text{an odd integer with } (\lambda, 3)=1 & \text{if } n=4, 5, 6 \\ \text{an odd integer with } (\lambda, 3)=1 \text{ and } (\lambda, 5)=1 & \\ & \text{if } n=3. \end{cases}$$

For example, in case of $n=4$, $t=1$, $\lambda=3$, we obtain a Hopf Z_2-space by the map h of degree 3.

$$\begin{tikzcd} W_{h_3} \arrow[r] \arrow[d] & U(4) \arrow[d] \\ S^7 \arrow[r, "h_3"] & S^7. \end{tikzcd}$$

This corresponds to the example of Curtis-Mislin [**4**].

EXAMPLE 4.3. The case $G=Z_2$, $E=Sp(n)$, Z_2-action, of type t, $n-t\geqq 1$, as above. We then obtain Hopf Z_2-spaces W_{h_λ} in the same way as above if $p|\lambda$ implies $p>(8n+1)/4(n-t)$.

For example take $n=2$, $t=1$, then the pull back

$$\begin{tikzcd} W_{h_5} \arrow[r] \arrow[d] & Sp(2) \arrow[d] \\ S^7 \arrow[r, "h_5"] & S^7 \end{tikzcd}$$

is a Hopf Z_2-space. This corresponds to the Hilton-Roiterberg's example.

Next we consider a Z_p-action (p: an odd prime). Let $\rho_{k,l}$ be the following representation

$$\rho_{k,l}(\gamma)=\left(\begin{array}{ccccccc} e^{-\xi} & & & & & & \\ & \ddots & & & & 0 & \\ & & e^{-\xi}, & & & & \\ & & & e^{\xi} & & & \\ & & & & \ddots & & \\ & & & & & e^{\xi}, & \\ & 0 & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{array}\right)\in U(n),$$

where $\xi=2\pi i/p$ and γ is a generator of Z_p.

EXAMPLE 4.4. The case $G=Z_p$, $E=U(n)$ (as in 4.1) with Z_p-action $*$ on $U(n)$:

$$a*A=\rho_{k,l}(a)A\rho_{k,l}(-a), \qquad a\in Z_p,\ A\in U(n).$$

Let $n>k+l$ and λ be an integer satisfying the condition that if a prime p divides λ then $p>(4n+1)/2(n-k-l)$. Then the pull back W_{h_λ} by the map $h_\lambda: S^{2n-1}\to S^{2n-1}$ is a Hopf Z_p-space.

In this case the fixed point sets of a subgroup $K<Z_p$ are obtained as follows.

$$U(n)^K=\begin{cases}\begin{pmatrix}A & & 0\\ & B & \\ 0 & & C\end{pmatrix}; & A\in U(k),\ B\in U(l) \text{ and } C\in U(n-k-l), \quad \text{if } K\neq e,\\ U(n) & \text{if } K=e.\end{cases}$$

$$(S^{2n-1})^K=\begin{cases} S^{2(n-k-l)-1} & \text{if } K\neq e,\\ S^{2n-1} & \text{if } K=e.\end{cases}$$

Next we consider the case $G=S^1$ $(=e^{i\theta\pi},\ 0\leqq\theta\leqq 2\pi)$. Let $\rho_{k,0}$ be the representation as above, that is;

$$\rho_{k,l}(e^{i\theta\pi})=\left(\begin{array}{ccccccc} e^{i\theta\pi} & & & & & & \\ & \ddots & & & & 0 & \\ & & e^{i\theta\pi}, & & & & \\ & & & e^{-i\theta\pi} & & & \\ & & & & \ddots & & \\ & & & & & e^{-i\theta\pi}, & \\ & 0 & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{array}\right).$$

EXAMPLE 4.5. The case $G=S^1$ and $E=U(n)$, with action as in the example above. Let $n>l+k$ and λ be an integer satisfying the condition that if a prime p divides λ then $p>(4n+1)/2(n-k-l)$. Then the pull back W_{h_λ} by the map h_λ is a Hopf S^1-space.

In this case the fixed point sets are as follows.

$$U(n)^K = \begin{cases} \begin{pmatrix} A & & 0 \\ & B & \\ 0 & & C \end{pmatrix} & \text{if } K \neq e, Z_2, \\ \begin{pmatrix} A' & 0 \\ 0 & C \end{pmatrix} & \text{if } K = Z_2. \end{cases}$$

Here, $A \in U(k)$, $A' \in U(l+k)$, $B \in U(l)$ and $C \in U(n-l-k)$. Define an S^1-action on $Sp(n)$ by;

$$\begin{pmatrix} e^{i\theta\pi} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} e^{-i\theta\pi} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \qquad A \in Sp(n).$$

We also obtain the following example.

EXAMPLE 4.6. The case $G=S^1$, $E=Sp(n)$, with the above action. Let $n \geqq 2$ and λ be an integer satisfying the condition that if a prime p divides λ then $p > (8n+1)/4(n-1)$. Then the pull back W_{h_λ} is a Hopf S^1-space.

References

[1] J. F. Adams, The sphere considered as an H-space mod p, Quart. J. Math., 12 (1961), 52-60.

[2] M. Arkowitz, Localization and H-spaces, Lecture Notes Series, 44, Mathematik Institut, Aarhus Univ., Aarhus, 1976.

[3] G. E. Bredon, Equivariant cohomology theories, Lecture Notes in Math., 34, Springer, 1967.

[4] M. L. Curtis and G. Mislin, H-spaces which are bundles over S^7, J. Pure Appl. Alg., 1 (1971), 27-40.

[5] P. Hilton and J. Roitberg, On principal S^3-bundles over spheres, Ann. of Math., 90 (1969), 91-107.

[6] K. Iriye, Hopf τ-spaces and τ-homotopy groups, J. Math. Kyoto Univ., 22 (1983), 719-727,

[7] T. Matumoto, On G-CW complexes and a theorem of J. H. C. Whitehead, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 18 (1971-72), 363-374.

[8] T. Matumoto, N. Minami and M. Sugawara, On the set of free homotopy classes and Brown's construction, Hiroshima Math. J., 14 (1984), 359-369.

[9] J. P. May, J. McClure and G. Triantafillou, Equivariant localization, Bull. London Math. Soc., 14 (1982), 223-230.

[10] E. H. Spanier, Algebraic topology, McGraw-Hill, 1966.

[11] G. Triantafillou, Rationalization of Hopf G-spaces, Math. Z., 182 (1983), 485-500.

[12] S. Warner, Equivariant homotopy theory and Milnor's theorem, Trans. Amer. Math. Soc., 258 (1980), 351-368.

[13] A. Zabrodsky, Homotopy associativity and finite CW-complexes, Topology, 9 (1970) 121-128.

[14] A. Zabrodsky, On sphere extensions of classical Lie groups, Algebraic Topology, Proc. Symp. Pure Math., **22**, 1971, pp. 279–283.
[15] A. Zabrodsky, On the construction of new finite CW H-spaces, Invent. Math., **16** (1972), 200–216.
[16] A. Zabrodsky, Hopf spaces, North-Holland Math. Studies, **22**, 1976.

Ken-ichi MARUYAMA
Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka 812
Japan