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The Euler characteristics and Weyl's curvature invariants of submanifolds in spheres

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§1. Introduction.

Let S^n be a unit n-sphere in R^{n+1}. Let M^p be a compact orientable p-dimensional Riemannian manifold which is imbedded in S^n. Let $\chi(M^p)$ be the Euler characteristics of M^p and $\tau(M^p)$ be the total curvature of M^p. One of Teufel's main results in [8] can be stated as follows.

$$\chi(M^p)=\tau(M^p)+\frac{1}{C_{n-1,n+1}}\int_{G_{n-1,n+1}}\chi(M^p\cap R^{n-1})\Omega_{n-1,n+1} \qquad \text{for } 3\leqq p, \tag{1.1}$$

where $G_{n-1,n+1}$ is the oriented Grassmann manifold of all oriented $(n-1)$-dimensional linear subspaces of R^{n+1}, $C_{n-1,n+1}$ its volume and $\Omega_{n-1,n+1}$ its standard volume element. Denote by $V(M^p)$ the volume of M^p. We can show (Theorem 4 in §4)

$$\chi(M^2)=\tau(M^2)+\frac{1}{2\pi}V(M^2). \tag{1.2}$$

In 1939, Weyl [10] found the formula for the volume of a tube of radius r about M^p. The coefficients in the power series expansion of the volume are expressed by the curvature invariants $k_e(M^p)$ (e even, $0\leqq e\leqq p$) (see (2.1)), which depend on the intrinsic geometry of M^p. Notice that $k_0(M^p)=V(M^p)$. Let $\tau_e(M^p)$ ($1\leqq e\leqq p$) be the e-th total mean curvature of M^p (see (2.2)). Then we have $\tau(M^p)=\tau_p(M^p)$, $\tau_e(M^p)=0$ for e odd, and for e even

$$\tau_e(M^p)=\frac{(p-2)!!}{(2\pi)^{p/2}(n-p+e-2)!!\binom{p}{e}}k_e(M^p), \tag{1.3}$$

where we mean that $m!!=m(m-2)\cdots 4\cdot 2$ or $m!!=m(m-2)\cdots 3\cdot 1$ according as m is even or odd. S. S. Chern [2] gives the kinematic formula and the linear kinematic formula in R^n. Following Chern, we introduce curvature invariants

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$\mu_e(M^p)$ (e even, $0 \leqq e \leqq p$), which are closely related to Weyl's invariants. In fact, we have

$$(1.4)\qquad \mu_e(M^p) = \frac{1}{(e-1)!!\binom{p}{e}} k_e(M^p) \qquad \text{for } e \text{ even}, \quad 0 \leqq e \leqq p.$$

We will show the following linear kinematic formula in S^n (Theorem 1).

$$(1.5)\qquad \int_{G_{q+1,n+1}} \mu_e(M^p \cap R^{q+1}) \Omega_{q+1,n+1} = \frac{(p+q-n-e)!!\, p!}{(2\pi)^{(n-q)/2}(p+q-n)!\,(p-e)!!} C_{q+1,n+1}\mu_e(M^p),$$

where $C_{q+1,n+1}$ is the volume of the oriented Grassmann manifold $G_{q+1,n+1}$.

In this paper, using (1.1), (1.2) and (1.5), we will prove

THEOREM. *Let M be a compact oriented 2p-dimensional manifold which is imbedded in a unit sphere S^n. Then, we have*

$$(1.6)\qquad \chi(M) = \frac{1}{(2\pi)^p} \sum_{k=0}^{p} (2k-1)!!\, k_{2p-2k}(M).$$

REMARKS. (i) Since we have (1.3) and (1.4), $k_{2p-2k}(M)$ in the above formula are replaced by $\tau_{2p-2k}(M)$ or $\mu_{2p-2k}(M)$. For example, we have

$$(1.7)\qquad \chi(M) = \sum_{k=0}^{p-1} \frac{(n-2k-2)!!\,(2p)!}{(2k)!!\,(n-2)!!\,(2p-2k)!} \tau_{2p-2k}(M) + \frac{(2p-1)!!}{(2\pi)^p} V(M).$$

(ii) When p is odd, it follows that $\tau_p(M^p) = \tau_{p-2}(M^p \cap R^{n-1}) = \cdots = \tau_1(M^p \cap R^{n-p+2}) = 0$. Hence applying Teufel's formula (1.1), we get

$$\chi(M) = \frac{1}{C_{n-p+2,n+1}} \int_{G_{n-p+2,n+1}} \chi(M^p \cap R^{n-p+2}) \Omega_{n-p+2,n+1}.$$

In general, $M^p \cap R^{n-p+2}$ is a 1-dimensional compact manifold without boundary and it holds $\chi(M^p \cap R^{n-p+2}) = 0$. Thus we obtain $\chi(M^p) = 0$. But it is well known that the Euler characteristic of a compact odd dimensional manifold without boundary is zero.

§2. The kinematic formula in S^n.

Let M be a p-dimensional submanifold in S^n. We take a local field of orthonormal frames $e_0, \cdots, e_n$ in R^{n+1} such that for $x \in M$, $e_0(x) = x$ and $e_1(x), \cdots, e_p(x)$ are tangent to M. We shall make use of the following convention on the range of indices unless otherwise stated;

$$0 \leqq A, B, C \leqq n, \quad 1 \leqq a, b, c \leqq n, \quad 1 \leqq i, j, k \leqq p, \quad p+1 \leqq \alpha, \beta, \gamma \leqq n.$$

The structure equations of R^{n+1} are given by

$$de_A = \sum_B \omega_{AB} e_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$
$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}.$$

By putting $\omega_a = \omega_{0a}$, the standard Riemannian metric on S^n is given by $ds^2 = \sum_a \omega_a^2$. From the above, we obtain

$$d\omega_a = \sum_b \omega_b \wedge \omega_{ba},$$
$$d\omega_{ab} = \sum_c \omega_{ac} \wedge \omega_{cb} - \omega_a \wedge \omega_b.$$

We restrict these forms to M and denote them by the same symbols. On M, it holds $\omega_\alpha = 0$. This implies that $0 = d\omega_\alpha = -\sum_i \omega_i \wedge \omega_{i\alpha}$. Thus we obtain

$$\omega_{i\alpha} = \sum_j h_{\alpha ij} \omega_j, \qquad h_{\alpha ij} = h_{\alpha ji}.$$

The structure equations of M are given as follows

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji},$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l.$$

Put $H_{ijkl} = \sum_\alpha (h_{\alpha ik} h_{\alpha jl} - h_{\alpha il} h_{\alpha jk})$. Then we get

$$H_{ijkl} = K_{ijkl} - (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

which we call the modified curvature tensor of M. For e even and $0 \leqq e \leqq p$, Weyl's curvature invariants $k_e(M)$ are defined as follows.

$$(2.1) \qquad k_e(M) = \frac{1}{2^e (e/2)!} \int_M \left(\sum \delta\begin{pmatrix} i_1, \cdots, i_e \\ j_1, \cdots, j_e \end{pmatrix} H_{i_1 i_2 j_1 j_2} \cdots H_{i_{e-1} i_e j_{e-1} j_e} \right) dM,$$

where dM is the volume element of M and $\begin{pmatrix} i_1, \cdots, i_e \\ j_1, \cdots, j_e \end{pmatrix}$ is equal to $+1$ (-1) according as $(i_1, \cdots, i_e)$ is an even (odd) permutation of $(j_1, \cdots, j_e)$ respectively, and is otherwise zero. We may define Chern's curvature invariants $\mu_e(M)$ by (1.4).

Let $y = \sum_\alpha y_\alpha e_\alpha$ be a vector normal to M. Then we define the e-th mean curvature $K_e(y)$ of M by

$$\det(\delta_{ij} + t h_{ij}(y)) = \sum_e \binom{n}{e} K_e(y) t^e,$$

where $h_{ij}(y) = \sum_\alpha y_\alpha h_{\alpha ij}$. We call the integral

(2.2) $$\tau_e(M)=\frac{1}{O_n}\int_N K_e(y)dN$$

the e-th total mean curvature of M, where N is the unit normal bundle over M, dN is the canonical volume element of N and O_n is the volume of S^{n-1}, that is, $O_n=2\pi^{n/2}/\Gamma(n/2)$. Using Weyl's result (see pp. 467–471 in [10] or Lemma 7.3 in [4]), we get (1.3).

Let M^p and M^q be compact submanifolds of dimensions p, q in S^n respectively. Let G be the group of motions in S^n, that is, $G=SO(n+1)$. By gM^q, we mean the submanifold obtained from M^q by a transformation $g\in G$. Then M^p and gM^q generally intersect in a submanifold of dimension $p+q-n$. The kinematic density in S^n is given (see, for example, [6]) by

$$dg=\bigwedge_{i,j<k}\omega_{0i}\wedge\omega_{jk}=\bigwedge_{i,j<k}\omega_i\wedge\omega_{jk}\,.$$

Since the following kinematic formula is shown by the same argument in §3-§6 in [2], we describe the result only.

(2.3) $$\int_G\mu_e(M^p\cap gM^q)dg=\sum_{l=0}^{e}D_{el}\mu_l(M^p)\mu_{e-l}(M^q)\,,$$

where D_{el} are constants depending on n, p, q, e, l, and e is an even integer satisfying $0\leqq e\leqq p+q-n$. In this paper, unfortunately we can not decide constant D_{el}. But in a particular case, we will determine them and get the linear kinematic formula. This will be achieved in § 3.

§ 3. The linear kinematic formula in S^n.

In this section, we assume that $M^q=S^n\cap R^{q+1}$, where R^{q+1} is a $(q+1)$-plane through the origin in R^{n+1}. The kinematic density dg has an expression

(3.1) $$dg=dg_1\wedge dg_2\wedge\Omega_{q+1,\,n+1}\,,$$

where dg_1, dg_2 and $\Omega_{q+1,\,n+1}$ are the volume elements of $SO(q+1)$, $SO(n-q)$ and $G_{q+1,\,n+1}$ respectively. From the definition, it follows that $\mu_e(S^q)=0$ for $2\leqq e\leqq p+q-n$. Since the volume of $SO(n+1)$ is equal to $O_{n+1}O_n\cdots O_2$, using (3.1), we get from (2.3)

(3.2) $$\int_{G_{q+1,\,n+1}}\mu_e(M^p\cap R^{q+1})\Omega_{q+1,\,n+1}=E_e\mu_e(M^p)\,,$$

where $E_e=D_{ee}/(O_{q+1}\cdots O_2O_{n-q}\cdots O_2)$.

In this section, we take an arbitrary a with $0\leqq a\leqq 1$ and put $R^{p+2}=\{\boldsymbol{x}=(x_0, x_1, \cdots, x_n)\,;\,x_{p+2}=\cdots=x_n=0\}$, $R^{p+1}=\{\boldsymbol{x}\in R^{p+2}\,;\,x_{p+1}=a\}$ and $S^p_a=S^{n+1}\cap R^{p+1}$. Let $\boldsymbol{f}_0, \cdots, \boldsymbol{f}_n$ be the standard base of R^{n+1}. Take local frame fields $e_0, \cdots, e_n$ in S_a^p such that for $x \in S_a^p$, $e_0(x)=x$, $e_{p+1}(x)=(1/\sqrt{1-a^2})(f_{p+1}-ae_0)$, $e_{p+2}(x)=f_{p+2}, \cdots, e_n(x)=f_n$ and $e_1(x), \cdots, e_p(x)$ are tangent to S_a^p. We will use the same notations as in § 2. Then the coefficients of the second fundamental form of $M=S_a^p$ in S^n are given by

$$h_{p+1\,ij}=\frac{a}{\sqrt{1-a^2}}\delta_{ij}, \qquad h_{\alpha ij}=0 \quad \text{for } p+2 \leqq \alpha \leqq n\,.$$

Hence the modified curvature tensor of M has the expression

$$H_{ijkl}=\frac{a^2}{1-a^2}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\,.$$

Thus we have

$$\mu_e(S_a^p)=O_{p+1}a^e(1-a^2)^{(p-e)/2}\,. \tag{3.3}$$

Let R^{q+1} be moving $(q+1)$-dimensional linear subspaces in R^{n+1}. For each R^{q+1}, R^{p+1} and R^{q+1} intersect in a $(p+q-n+1)$-dimensional linear subspace which we denote by R^{k+1}, where $k=p+q-n$. Put $S^k=R^{k+1} \cap S^n=S_a^p \cap R^{q+1}$. Let O' and O'' be the centers of S_a^p and S^k respectively. Let t be the distance of O'' from O'. Then the distance of O'' from the origin O is equal to $\sqrt{a^2+t^2}$, and the radius of the sphere S^k is $\sqrt{1-a^2-t^2}$. Thus we have

$$\mu_e(S_a^p \cap R^{q+1})=\mu_e(S^k)=O_{k+1}(a^2+t^2)^{e/2}(b^2-t^2)^{(k-e)/2},$$

where $k=p+q-n$, $t=$ the distance of O'' from O', and $b=\sqrt{1-a^2}$.

Let $\pi: G_{q+1,n+1} \to G_{k+2,p+2}$ be a projection defined by $\pi(R^{q+1})=R^{k+2}$, where we put $R^{k+2}=R^{q+1} \cap R^{p+2}$. Then the above integral invariants $\mu_e(S_a^p \cap R^{q+1})$ are constant on each fibre $\pi^{-1}(R^{k+2})$ of the projection π. Hence we obtain

$$\begin{aligned}\int_{G_{q+1,n+1}} & \mu_e(S_a^p \cap R^{q+1})\Omega_{q+1,n+1} \\ &= \frac{O_{n+1}O_n \cdots O_{p+3}}{O_{q+1}O_q \cdots O_{k+3}}\int_{G_{k+2,p+2}} \mu_e(S_a^p \cap R^{k+2})\Omega_{k+2,p+2}\,.\end{aligned} \tag{3.4}$$

Now we may consider that S_a^p is a sphere in R^{p+2} and that R^{k+1} are moving linear subspaces in R^{p+2}. Let t and x be the vectors from O' to O'' and O to O'' respectively. Put $R^{k+1}=R^{p+1} \cap R^{k+2}$. We choose two local orthonormal frame fields $\{e_0, \cdots, e_{p+1}\}$ and $\{f_0, \cdots, f_{p+1}\}$ such that $\{e_0, e_1, \cdots, e_{k+1}\}$ is a base of R^{k+2} and

$$\begin{cases} e_0=x/\|x\|, \quad e_1=f_1, \quad \cdots, \quad e_{k+1}=f_{k+1}, \\ t=tf_0, \quad f_{p+1}=(0, \cdots, 0, 1), \\ e_0=(tf_0+af_{p+1})/\sqrt{a^2+t^2}, \quad e_{p+1}=(af_0-tf_{p+1})/\sqrt{a^2+t^2}. \end{cases} \tag{3.5}$$

In this section, from now on we use the following notations of indices;

$$1 \leqq i \leqq k+1, \quad 0 \leqq a, b \leqq k+1, \quad k+2 \leqq u, v \leqq p, \quad q+1 \leqq \alpha, \beta \leqq p+1.$$

The volume element of $G_{k+2,\,p+2}$ is given by

$$\begin{aligned} \Omega_{k+2,\,p+2} &= \bigwedge_{\substack{u=k+2,\cdots,p \\ a,b=0,\cdots,k+1}} (e_u,\, de_a) \wedge (e_{p+1},\, de_b) \\ &= \pm \wedge (e_u,\, de_i) \wedge (e_{p+1},\, de_i) \wedge (e_v,\, de_0) \wedge (e_{p+1},\, de_0). \end{aligned}$$

Using (3.5), we can express $\Omega_{k+2,\,p+2}$ as

$$\Omega_{k+2,\,p+2} = \pm a^{k+2}(a^2+t^2)^{-(p+2)/2} t^{p-k-1} dt \wedge \Omega_{k+1,\,p} \wedge dS^p,$$

where $\Omega_{k+1,\,p} = \wedge (f_u,\, df_i)$ is the volume element of $G_{k+1,\,p}$ and $dS^p = \wedge (f_i,\, df_0) \wedge (f_u,\, df_0)$ is the volume element of a unit sphere in R^{p+1}. Thus we get

$$\text{(3.6)} \quad \int_{G_{k+2,\,p+2}} \mu_e(S^k) \Omega_{k+2,\,p+2} = \frac{O_p \cdots O_2 O_{p+1} O_{k+1}}{O_{k+1} \cdots O_2 O_{p-k-1} \cdots O_2} \times a^{k+2} \int_0^b (a^2+t^2)^{(e-p-2)/2} (b^2-t^2)^{(k-e)/2} t^{p-k-1} dt,$$

where $b = \sqrt{1-a^2}$. Now put $t = abs^{1/2}/\sqrt{1-b^2 s}$, $s \geqq 0$, we have

$$\begin{aligned} \text{(3.7)} \quad \int_0^b (a^2+t^2)^{(e-p-2)/2} (b^2-t^2)^{(k-e)/2} t^{p-k-1} dt &= \frac{a^e b^{p-e}}{2} \int_0^1 s^{(p-k-2)/2} (1-s)^{(k-e)/2} ds \\ &= \frac{a^e b^{p-e}}{2} B\Big(\frac{p-k}{2},\, \frac{k-e+2}{2}\Big). \end{aligned}$$

Taking account of $B(u, v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$, we get from (3.2), (3.4), (3.6) and (3.7)

$$E_e = \frac{(p+q-n-e)!!\, p!}{(2\pi)^{(n-q)/2} (p-e)!!\, (p+q-n)!} C_{q+1,\,n+1}.$$

Thus we have proved

THEOREM 1. *Let M^p be a compact oriented submanifold in S^n. For an even integer e with $0 \leqq e \leqq p$, let $\mu_e(M^p)$ be Chern's curvature invariants given by* (1.4). *Then we have the formula* (1.5).

Since we have (1.3) and (1.4), we can write (1.5) as follows.

COROLLARY 2. *Under the same assumption as Theorem* 1, *we have*

$$\text{(3.8)} \quad \int_{G_{q+1,\,n+1}} k_e(M^p \cap R^{q+1}) \Omega_{q+1,\,n+1} = \frac{(p-e-1)!!}{(2\pi)^{(n-q)/2} (p+q-n-e-1)!!} C_{q+1,\,n+1} k_e(M^p),$$

$$\text{(3.9)} \quad \int_{G_{q+1,\,n+1}} \tau_e(M^p \cap R^{q+1}) \Omega_{q+1,\,n+1} = \frac{(p+q-n-e)!!\, (n-p+e-2)!!\, p!}{(p-e)!!\, (2n-p-q+e-2)!!\, (p+q-n)!} C_{q+1,\,n+1} \tau_e(M^p).$$

§4. The Euler characteristic of a compact 2-dimensional submanifold in S^n.

In this section, we treat a 2-dimensional submanifold M in S^n. For $L=R^{n-1}\in G_{n-1,n+1}$, let R^2 be the orthogonal complement of $L=R^{n-1}$. Following Teufel [7], we define a level function $h_L: S^n-L\to S^1=S^n\cap R^2$ by $h_L(x)=(x\wedge L)\cap S^1\cap S_+^n$, $x\in S^n-L$, where $x\wedge L$ is the n-dimensional linear subspace spanned by x and L, and S_+^n is the hemisphere of S^n which contains the point x. We denote by h_L^M the restriction of h_L to M. Then it is defined on $M-L$. Let $\beta_k(h_L^M)$ be the number of critical points of index k of h_L^M. Put

$$\tilde{\beta}(h_L^M)=\sum_{k=0}^{2}(-1)^k\beta_k(h_L^M).$$

In general, M and L are transversal in R^{n+1}. In this case $M\cap L$ is a finite set of points in R^{n+1}. We set $M\cap L=\{P_1, \cdots, P_l\}$. Denote by $\#(M\cap L)$ the number of points in $M\cap L$, that is, $\#(M\cap L)=l$. We will prove

LEMMA 3. *Assume that M and L are transversal in R^{n+1}, and that h_L^M has no degenerate critical point. Then it holds*

$$\chi(M)=\tilde{\beta}(h_L^M)+\#(M\cap L). \tag{4.1}$$

PROOF. We will show that there is a neighborhood V of $M\cap L$ and a vector field X on M such that $X=\operatorname{grad} h_L^M$ on $M-V$, points of $M\cap L$ are zeros of X and there is no other zero of X in V. Take a point P_1 of $M\cap L=\{P_1, \cdots, P_l\}$. We may assume that $R^2=\{(0, x_1, x_2, 0, \cdots, 0), x_1, x_2\in R\}$ and $P_1=(1, 0, \cdots, 0)$. Since for $x=(x_0, x_1, \cdots, x_n)\in S^n$, $h_L(x)=(0, x_1, x_2, 0, \cdots, 0)\in S^1$, to consider h_L as a function, we may assume that it is represented as $h_L(x)=\tan^{-1}(x_2/x_1)$. As M and L are transversal, we can take a local coordinate neighborhood U of P_1 in M

$$U=\{(f_0(u_1, u_2), f_1(u_1, u_2), \cdots, f_n(u_1, u_2))=f(u_1, u_2),\ u_1^2+u_2^2<\varepsilon\}$$

such that $f(0,0)=p_1=(1, 0, \cdots, 0)$, $f_1(u_1, u_2)=u_1$, $f_2(u_1, u_2)=u_2$ and $U\cap\{P_2, \cdots, P_l\}=\emptyset$. Moreover we may assume

$$\frac{\partial f}{\partial u_1}(0, 0)=(0, 1, 0, \cdots, 0), \qquad \frac{\partial f}{\partial u_2}(0, 0)=(0, 0, 1, 0, \cdots, 0).$$

Then we can put $f_\alpha(u_1, u_2)=A_{11}^\alpha u_1^2+2A_{12}^\alpha u_1u_2+A_{22}^\alpha u_2^2+O(u^3)$ for $3\leqq\alpha\leqq n$, where we set $u=\sqrt{u_1^2+u_2^2}$. We consider that $(x_1, \cdots, x_n)$ is a local coordinate of a point $x=(x_0, x_1, \cdots, x_n)$ in a neighborhood of P_1 in S^n. With respect to $(x_1, \cdots, x_n)$, the standard Riemannian metric g_{ab} of S^n is expressed by

$$g_{ab}=\delta_{ab}+\frac{x_a x_b}{x_0^2}, \qquad x_0^2=1-\sum_{a=1}^{n} x_a^2.$$

Hence the Riemannian metric g'_{ij} of M is expressed near P_1 by

$$g'_{ij}=\delta_{ij}+O(u^2).$$

Hence we have, near P_1,

$$g'^{ij}=\delta_{ij}+O(u^2).$$

Put $h=h_L^M=\tan^{-1}(u_2/u_1)$. With respect to the local coordinate (u_1, u_2), grad h is written locally as

$$\text{(4.2)} \qquad \begin{aligned} \operatorname{grad} h &= \sum_{i,j} g'^{ij}\frac{\partial h}{\partial u_i}\frac{\partial}{\partial u_j} \\ &= \Big(\frac{-u_2+O(u^3)}{u_1^2+u_2^2}\Big)\frac{\partial}{\partial u_1}+\Big(\frac{u_1+O(u^3)}{u_1^2+u_2^2}\Big)\frac{\partial}{\partial u_2}. \end{aligned}$$

For a sufficiently small positive ε, let a and b be constants with $0<a<b<\varepsilon$. Then we can construct a C^∞ function λ on R^2 which satisfies

$$\lambda(u_1, u_2)=\begin{cases} u_1^2+u_2^2 & \text{for } u_1^2+u_2^2<a, \\ 1 & \text{for } b<u_1^2+u_2^2<\varepsilon. \end{cases}$$

Now we set

$$X_1=\begin{cases} \lambda(u_1, u_2)\operatorname{grad} h_L^M & \text{on } U, \\ \operatorname{grad} h_L^M & \text{on } M-U. \end{cases}$$

Then X_1 is a vector field on $M-\{P_2, P_3, \cdots, P_l\}$ and the zeros of X_1 are those of grad h_L^M and P_1. Since we have (4.2), by an easy calculation, we can show that P_1 is a zero of index 1 of X_1 (see, for example, pp. 132-136 in [**3**]).

In the above method, we can construct a vector field X on M whose zeros are those of grad h_L^M and $\{P_1, \cdots, P_l\}$. Moreover, each P_k $(1\leqq k\leqq l)$ is a zero of index 1 of X. Thus the desired result follows from the Poincaré-Hopf index theorem. Q. E. D.

Teufel proved in [**7**]

$$\tau(M)=\frac{1}{C_{n-1,n+1}}\int_{G_{n-1,n+1}}\tilde{\beta}(h_{R^{n-1}}^M)\Omega_{n-1,n+1}.$$

Combining this with (4.1), we obtain

THEOREM 4. *Let M be a compact oriented 2-dimensional submanifold in S^n. Denote by $\#(M\cap R^{n-1})$ the number of points in $M\cap R^{n-1}$. Then we have*

$$\chi(M)=\tau(M)+\frac{1}{C_{n-1,n+1}}\int_{G_{n-1,n+1}}\#(M\cap R^{n-1})\Omega_{n-1,n+1}$$
$$=\tau(M)+\frac{1}{2\pi}V(M).$$

To get the second equation in the above theorem, we need the following integral formula (see the formula (14.70) in [6]).

LEMMA 5 (L. A. Santalo). *Let M^p be a compact p-dimensional Riemannian submanifold in S^n. Let O_k be the volume of S^{k-1}. Then we have*

$$\int_{G_{n-p+1,n+1}}\#(M^p\cap R^{n-p+1})\Omega_{n-p+1,n+1}=\frac{O_{n+1}O_n\cdots O_{p+2}}{O_{n-p+1}O_{n-p}\cdots O_2}V(M).$$

§ 5. Proof of the main theorem.

Our preparations are complete. We will prove the theorem stated in § 1. Let M be a compact oriented $2p$-dimensional submanifold in S^n. Applying Teufel's formula (1.1) to a submanifold $M\cap R^{n-1}$ in S^{n-2}, we get

$$\chi(M\cap R^{n-1})=\tau_{2p-2}(M\cap R^{n-1})+\frac{1}{C_{n-3,n-1}}\int\chi((M\cap R^{n-1})\cap R^{n-3})\Omega_{n-3,n-1}.$$

In this section, we omit domains of integration. From this and (1.1), it follows that

$$\chi(M)=\tau_{2p}(M)+\frac{1}{C_{n-1,n+1}}\int\tau_{2p-2}(M\cap R^{n-1})\Omega_{n-1,n+1}$$
$$+\frac{1}{C_{n-3,n+1}}\int\chi(M\cap R^{n-3})\Omega_{n-3,n+1}.$$

By a continuation of the above argument, we finally get

$$\chi(M)=\tau_{2p}(M)+\sum_{k=1}^{p-2}\frac{1}{C_{n+1-2k,n+3-2k}}\int\tau_{2p-2k}(M\cap R^{n+1-2k})\Omega_{n+1-2k,n+3-2k}$$
$$+\frac{1}{C_{n+3-2p,n+1}}\int\chi(M\cap R^{n+3-2p})\Omega_{n+3-2p,n+1}. \tag{5.1}$$

Since $M\cap R^{n+3-2p}$ are generally 2-dimensional submanifolds in S^{n+2-2p}. Hence, by Theorem 4, we have

$$\chi(M\cap R^{n+3-2p})=\tau_2(M\cap R^{n+3-2p})$$
$$+\frac{1}{C_{n+1-2p,n+3-2p}}\int\#((M\cap R^{n+3-2p})\cap R^{n+1-2p})\Omega_{n+1-2p,n+3-2p}. \tag{5.2}$$

Thus from (5.1) and (5.2) we get

$$\chi(M)=\tau_{2p}(M)+\sum_{k=1}^{p-1}\frac{1}{C_{n+1-2k,\,n+3-2k}}\int\tau_{2p-2k}(M\cap R^{n+1-2k})\Omega_{n+1-2k,\,n+3-2k}+\frac{1}{C_{n+1-2p,\,n+1}}\int\#(M\cap R^{n+1-2p})\Omega_{n+1-2p,\,n+1}. \tag{5.3}$$

Using (3.8) and Lemma 5, from (5.3) we obtain (1.7), from which the desired formula (1.6) follows.

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