J. Math. Soc. Japan
Vol. 39, No. 1, 1987

A sharp sufficient geometric condition for the existence of global real analytic solutions on a bounded domain

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(Received Nov. 21, 1984)
(Revised July 26, 1985)

1. Let $P(D)$ denote a linear partial differential operator with constant coefficients. Let $\Omega\subset \boldsymbol{R}^n$ be open and let $\mathcal{A}(\Omega)$ denote the space of real analytic functions on Ω. In this paper we give a result which refines the following one on the existence of global real analytic solutions abstracted from the work of Kawai [7]:

THEOREM 1. *Let $P(D)$ be locally hyperbolic and let K_ξ denote the local propagation cone of $P(D)$ corresponding to the direction ξ, which we assume can be chosen depending in an upper semi-continuous way on $\xi\in \boldsymbol{S}^{n-1}$. Assume that Ω is bounded and that $\partial\Omega\times \boldsymbol{S}^{n-1}$ can be covered by two closed subsets $X^{\pm}$ such that*

(1) $\quad (x, \xi)\in X^{\pm}$ *implies either* $P_m(\xi)\neq 0$ *or* $(\{x\}\pm K_\xi)\cap\Omega=\emptyset$

(with the double signs in the same order). Then we have $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$.

Our refined theorem is as follows:

THEOREM 2. *Let $P(D)$ be locally hyperbolic and $\Omega\subset \boldsymbol{R}^n$ a bounded open set. Assume that*

(2) $\quad (x, \xi)\in\partial\Omega\times \boldsymbol{S}^{n-1}$ *implies either* $P_m(\xi)\neq 0$ *or* $(\{x\}+K_\xi)\cap\Omega=\emptyset$
for some choice of local propagation cone K_ξ at ξ (depending on x).

Then we have $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$.

Generically there are only two choices of local propagation cones at every ξ, namely $\pm K_\xi$. In that case the condition (2) is simply written as follows:

(2)′ $\quad (x, \xi)\in\partial\Omega\times \boldsymbol{S}^{n-1}$ *implies either of*
$P_m(\xi)\neq 0$, $(\{x\}+K_\xi)\cap\Omega=\emptyset$, $(\{x\}-K_\xi)\cap\Omega=\emptyset$.

Theorem 2 improves Theorem 1 in the following points: We do not need the global upper semi-continuity of $\xi \mapsto K_\xi$, and we do not assume the possibility of classifying points $(x, \xi) \in \partial\Omega \times S^{n-1}$ according to the property (2)' to two "closed" subsets $X^{\pm}$ (with non-void intersection). In fact as for the latter we have an explicit example by Zampieri [11] (see the last paragraph of this paper) showing that (2)'$\Rightarrow$(1) does not follow in general.

Concerning the problem of global existence of real analytic solutions there already exists a necessary and sufficient condition by Hörmander [5] when Ω is convex (but not necessarily bounded). However, his condition is not so intuitive, and Kawai's method based on the micro-local analysis has still value though it is restricted to a locally hyperbolic operator $P(D)$ and to a bounded Ω. Recently Zampieri [10] gave a paraphrase of Hörmander's condition also in terms of K_ξ as above. More precisely, he gave it in terms of local propagation cones of every local irreducible factor of the principal part $P_m(\xi)$, and thus succeeded in giving a very sharp sufficient condition in general which in particular was shown to be also necessary when the multiplicity of the zeros of every local irreducible factor is at most two. Note however that his result applies only to a convex Ω because it relies on Hörmander's work.

We have already given our main Theorem 2 in Kaneko [6]. There we have also treated the case of unbounded Ω, thus unifying the result of Andersson [2] and of Kawai. Since the theory of Fourier hyperfunctions in a little generalized sense is deeply employed there in order to treat systematically hyperfunctions with unbounded analytic singular supports, we believe that it is useful to give here a direct proof of the above main theorem for bounded Ω without employing Fourier hyperfunctions.

For further references on this problem see e. g. the introduction of Hörmander [5] or Cattabriga [3].

2. Let $\mathcal{B}$ denote the sheaf of hyperfunctions on $\boldsymbol{R}^n$. The following lemma is the basis of all our calculations when considering a bounded Ω.

LEMMA 3. *Let f, g be two hyperfunctions on $\boldsymbol{R}^n$ one of which has compact (analytic) singular support. Then the convolution $f*g$ is well defined as a section of the quotient sheaf $\mathcal{B}/\mathcal{A}$ on $\boldsymbol{R}^n$. We can always choose its global hyperfunction representative on $\boldsymbol{R}^n$. Denoting any such representative by the same symbol $f*g$, we have*

$$\text{(3)}\qquad \mathrm{S.S.}\, f*g \subset \{(x+y, \xi)\ ;\ (x, \xi)\in \mathrm{S.S.}\, f,\ (y, \xi)\in \mathrm{S.S.}\, g\}.$$

Here S.S. *denotes the singular spectrum (i.e. the equivalent notion of analytic wave front set for distributions) of a hyperfunction.*

This lemma immediately follows from the general theory on the manipulation of microfunctions given in Chapter I of S-K-K [9]. For the convenience of the general reader we give here an elementary proof based only on the knowledge of usual convolution for hyperfunctions.

PROOF. Assume that $\operatorname{sing\,supp} g \subset \{|x| \leqq r\}$ and choose $R \gg r$. Let f_R be a modification of f with support in $|x| \leqq R$. Then the convolution $f_R * g$ is well-defined as a hyperfunction. In view of a corresponding formula for hyperfunctions, S.S. $f_R * g$ for $|x| < R-r$ is estimated by the right-hand side of (3). Moreover, for $R' > R$, $f_{R'} * g$ defines the same section of $\mathcal{B}/\mathcal{A}$ as $f_R * g$ on $|x| < R-r$, because the difference $(f_{R'} - f_R) * g$ becomes real analytic there. Since R is arbitrary, an element of $(\mathcal{B}/\mathcal{A})(\boldsymbol{R}^n)$ is thus determined. In view of Malgrange's vanishing theorem $H^1(\boldsymbol{R}^n, \mathcal{A}) = 0$, the canonical mapping $\mathcal{B}(\boldsymbol{R}^n) \to (\mathcal{B}/\mathcal{A})(\boldsymbol{R}^n)$ is surjective, hence we can always choose a global hyperfunction representative for $f * g$ which is determined modulo $\mathcal{A}(\boldsymbol{R}^n)$. This justifies the above abuse of notation to consider $f * g$ also as denoting a hyperfunction representative on $\boldsymbol{R}^n$. q. e. d.

Next we remember the definition of locally hyperbolic operators.

DEFINITION 4 (cf. Andersson [1]). $P(D)$ is called *locally hyperbolic* if for every $\xi^0 \in \boldsymbol{S}^{n-1}$ there exist a vector $v = v(\xi^0) \in \boldsymbol{R}^n \setminus \{0\}$, a neighborhood Δ of ξ^0 in $\boldsymbol{S}^{n-1}$ and $\varepsilon_0 > 0$ such that

$$P_m(\xi + itv) \neq 0 \quad \text{if} \quad \xi \in \Delta \quad \text{and} \quad 0 < |t| < \varepsilon_0 .$$

Here P_m denotes the principal part of P.

Let $(P_m)_\xi(\eta)$ denote the localization of P_m at ξ. If $P(D)$ is locally hyperbolic, then the localization at ξ becomes hyperbolic to the direction $v(\xi)$ appearing in Definition 4, hence $v(\xi)$ can in fact move inside the normal cone $\Gamma((P_m)_\xi(\eta), v(\xi))$ of $(P_m)_\xi(\eta)$ containing $v(\xi)$ (which may contain a characteristic direction of the original operator). Its dual cone is denoted by K_ξ and is called the *local propagation cone* of $P(D)$ at ξ corresponding to $v(\xi)$. For any choice of K_{ξ^0} we can always imbed it locally to an upper semi-continuous correspondence $\xi \mapsto K_\xi$. Here the meaning of upper semi-continuity is the following:

(4) For any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi - \xi^0| \leqq \delta$ implies that K_ξ is contained in the ε-neighborhood of K_{ξ^0}.

(However we may obtain the opposite cone $-K_\xi$ when we let ξ make a round trip on $\boldsymbol{S}^{n-1}$.)

THEOREM 5. *Let $P(D)$ be locally hyperbolic. Then for any $\xi^0 \in \boldsymbol{S}^{n-1}$ and for any upper semi-continuous choice of the local propagation cone $\xi \mapsto K_\xi$ on a neighborhood Δ of ξ^0 we can construct a "good" micro-local fundamental solution*

$E^{\Delta}(x)$ of $P(D)$ on $\boldsymbol{R}^n \times \Delta$. That is, we can find a hyperfunction $E^{\Delta}(x)$ on $\boldsymbol{R}^n$ such that

i) *$P(D)E^{\Delta}(x)-\delta(x)$ is micro-analytic on $\boldsymbol{R}^n \times \Delta$.*

ii) $\mathrm{S.S.}\, E^{\Delta}(x) \subset (\{0\} \times \bar{\Delta}) \cup (\boldsymbol{R}^n \times \partial\Delta) \cup \bigcup_{\xi \in N(P_m) \cap \Delta} (K_\xi \times \{\xi\})$.

Here $N(P_m)$ denotes the set of zeros of P_m and K_ξ is the assigned choice of local propagation cone.

This theorem is essentially proved in Kawai [7] (where he constructs a pair of global "good" fundamental solutions assuming the global upper semi-continuity of $\xi \mapsto K_\xi$). Another proof based on the inverse Fourier transformation may be found in Kaneko [6].

The correspondence $\xi \mapsto K_\xi$ is in general not lower semicontinuous. That is, K_ξ may expand suddenly as ξ varies. This is the cause of the difficulty for deriving (2)′⇒(1). Therefore we now introduce on the cotangential sphere $\boldsymbol{S}^{n-1}$ the following stratification:

$$\boldsymbol{S}^{n-1} = \Xi_0 \sqcup \Xi_1 \sqcup \cdots \sqcup \Xi_s ,$$

where we put

$$\Xi_k = \{\xi \in \boldsymbol{S}^{n-1} \text{ ; the localization of } P_m \text{ at } \xi \text{ has order } m_k\} ,$$

with some finite sequence of integers

$$0 = m_0 < m_1 < \cdots < m_s \leqq m = \deg P .$$

Thus Ξ_0 is the set of non-characteristic directions, and (if $m_1=1$) Ξ_1 is the set of simply-characteristic directions. We have obviously the usual property of stratification

$$\Xi_{k+1} \subset \overline{\Xi_k} , \qquad \Xi_k = \overline{\Xi_k} \setminus \overline{\Xi_{k+1}} , \qquad k = 0, \cdots, s$$

with the convention $\Xi_{s+1} = \emptyset$.

LEMMA 6. *Any choice of upper semi-continuous correspondence of local propagation cones $\xi \mapsto K_\xi$ is in fact continuous on each connected component of every stratum Ξ_k. Here the word "continuity" implies both the upper semi-continuity* (4) *and the lower semi-continuity in the following sense:*

(5) *For any point $x^0 \in K_{\xi^0}$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\xi - \xi^0| \leqq \delta$ implies that K_ξ contains some point in the ε-neighborhood of x^0.*

(Here the assertion of the lemma is that this condition (5) holds if ξ is restricted to the connected component Ξ_k^0 of the stratum Ξ_k containing ξ^0. Note that we do not assume that K_ξ varies homeomorphically with $\xi \in \Xi_k^0$.)

PROOF. Let

$$P_m(\xi+t\eta) = t^{m_k}(P_m)_\xi(\eta)+o(t^{m_k})$$

be the Taylor expansion defining the localization of P_m at $\xi\in\Xi_k^0$. For ξ sufficiently near ξ^0, we have $(P_m)_\xi(v)\neq 0$ with some fixed vector v. Employ a linear coordinate transformation bringing v to $(1, 0, \cdots, 0)$. Then we have

$$(P_m)_\xi(\eta) = a(\xi)\eta_1^{m_k}+\cdots,$$

with $a(\xi)\neq 0$ for ξ close to ξ^0. By the assumption $(P_m)_\xi(\eta)$ becomes a hyperbolic polynomial when $\xi\in\Xi_k^0$. Let $\tau(\eta';\xi)$ be the maximal root of the equation $(P_m)_\xi(\eta_1, \eta')=0$ for η_1. We have by definition

$$K_\xi = \Gamma((P_m)_\xi(\eta), v)^\circ,$$

where

$$\begin{aligned}\Gamma((P_m)_\xi(\eta), v) &= \text{the connected component of } \{\eta\in\boldsymbol{R}^n ; (P_m)_\xi(\eta)\neq 0\} \\ &\quad\ \text{containing } v \\ &= \{\eta\in\boldsymbol{R}^n ; \eta_1>\tau(\eta';\xi)\}.\end{aligned}$$

By the continuity of the roots of an algebraic equation, $\tau(\eta';\xi)$ varies continuously with ξ. Hence K_ξ varies lower semi-continuously in the above sense (5) as long as $\xi\in\Xi_k^0$. (In fact if for some $\varepsilon>0$ there exists a sequence $\xi^l\in\Xi_k^0$ tending to ξ^0 such that $K_{\xi^l}\cap\{|x-x^0|\leqq\varepsilon\}=\emptyset$, then we could choose a sequence of linear functions $\langle\cdot, \eta^l\rangle$, $|\eta^l|=1$, separating them, i. e.

$$\langle x, \eta^l\rangle > 0 \quad \text{for } x\in K_{\xi^l}, \qquad \langle x, \eta^l\rangle < 0 \quad \text{for } |x-x^0|\leqq\varepsilon,$$

hence especially $\eta^l\in\Gamma((P_m)_{\xi^l}, v)$ in view of the convexity of the latter cone. Replacing by a subsequence if necessary, we can assume that $\eta^l\to\eta^0$, $|\eta^0|=1$. Then we would have $\eta^0\in\overline{\Gamma((P_m)_{\xi^0}, v)}$ by the above continuity of $\partial\Gamma((P_m)_{\xi^l}, v)$, and on the other hand,

$$\langle x, \eta^0\rangle \leqq 0 \qquad \text{for } |x-x^0|\leqq\varepsilon,$$

hence $\langle x, \eta^0\rangle<0$ for some $x\in K_{\xi^0}$. This is a contradiction. q. e. d.

PROOF OF THEOREM 2. Let $f\in\mathcal{A}(\Omega)$. Choose a hyperfunction extension with minimal support $\tilde{f}\in\mathcal{B}[\bar{\Omega}]$. Let $\tilde{f}=\sum_\Delta\tilde{f}^\Delta$ be a decomposition to a finite sum such that $\mathrm{S.S.}\tilde{f}\Subset\partial\Omega\times\Delta$, where Δ is a small neighborhood of some point of $\boldsymbol{S}^{n-1}$ on which a set of "good" micro-local fundamental solutions as in Theorem 5 is available. Employing it we shall find a solution $u^\Delta\in\mathcal{B}(\boldsymbol{R}^n)$ of $P(D)u^\Delta=\tilde{f}^\Delta$ such that $u^\Delta|_\Omega\in\mathcal{A}(\Omega)$. Then setting $u=\sum u^\Delta$ we will have $P(D)u=\tilde{f}+h$, where $h\in\mathcal{A}(\boldsymbol{R}^n)$. Since we can always choose a real analytic solution v of $P(D)v=h$ on a compact ball containing $\bar{\Omega}$ (see e. g. Komatsu [8]), we will thus obtain a true real analytic solution $(u-v)|_{\Omega}$.

Now we solve the equation $P(D)u=f_0=\tilde{f}^{\Delta}$ for a fixed $\Delta\subset S^{n-1}$, where the data $f_0\in\mathcal{B}(\boldsymbol{R}^n)$ satisfies S.S. $f_0\Subset\partial\Omega\times\Delta$. We shall solve it step by step on $\boldsymbol{R}^n\times\Xi_k$. In the sequel the convolution is understood always in the sense of Lemma 3. First put

$$u_0=E*f_0,$$

where E is any fundamental solution of P. Then we have

$$\text{S.S. } u_0\Subset(\boldsymbol{R}^n\setminus\Omega)\times(\overline{\Xi_0}\cap\Delta)\cup\bar{\Omega}\times(\overline{\Xi_1}\cap\Delta). \tag{6}$$

Namely, the singularity does not propagate into Ω except for the one with the directional components in $\overline{\Xi_1}$. (In this first step this is due to the so-called Sato fundamental theorem on the micro-analyticity of solutions at non-characteristic directions, and we may write $\partial\Omega\times\Delta$ instead of the first component on the right-hand side of (6).) Choose a decomposition

$$u_0=v_0+w_0,$$

such that S.S. v_0 resp. S.S. w_0 are contained in the first resp. second component in the right hand side of the estimate (6). (The possibility of such a decomposition comes from the flabbiness of the so called sheaf $\mathcal{C}$ of micro-functions (see S-K-K [**9**]). Recently de Roever [**4**] gave a direct elementary way of giving such a decomposition employing a new concrete formula of singular spectral decomposition.) Here abandon w_0 and put

$$f_1=f_0-P(D)v_0=P(D)w_0.$$

Then we have

$$\text{S.S. } f_1\Subset(\boldsymbol{R}^n\setminus\Omega)\times(\overline{\Xi_0}\cap\Delta)\cap\bar{\Omega}\times(\overline{\Xi_1}\cap\Delta)=\partial\Omega\times(\overline{\Xi_1}\cap\Delta).$$

Next we solve $P(D)u_1=f_1$. Let $\xi\mapsto K_\xi^j$, $j=1,\cdots,N_\Delta$ be all the possible upper semi-continuous correspondences of local propagation cones on Δ, and let $E^{\Delta,j}$ be the corresponding "good" micro-local fundamental solutions given by Theorem 5. Put

$$X_1^j=\{(x,\xi)\in\partial\Omega\times\Delta\ ;\ P_m(\xi)\neq 0\text{ or }(\{x\}+K_\xi^j)\cap\Omega=\emptyset\},\ j=1,\cdots,N_\Delta. \tag{7}$$

In view of Lemma 6 we see easily that this constitutes a relatively closed covering of $\partial\Omega\times(\Xi_1\cap\Delta)$. Thus $\{\overline{X^j}\}_{j=1}^{N_\Delta}$ constitutes a closed covering of $\partial\Omega\times(\overline{\Xi_1}\cap\Delta)$, and we can choose a decomposition such that

$$f_1=\sum_{j=1}^{N_\Delta}f_1^j,\qquad \text{S.S. } f_1^j\Subset\overline{X_1^j}.$$

Then put

$$u_1 = \sum_{j=1}^{N_\Delta} E^{\Delta,j} * f_1^j .$$

(In the case we assume (2)′, it suffices to prepare a pair of "good" micro-local fundamental solutions $E^{\Delta,\pm}$, put

$$(7)' \quad X_1^\pm = \{(x, \xi) \in \partial\Omega \times \Delta \ ; \ P_m(\xi) \neq 0 \text{ or } (\{x\} \pm K_\xi) \cap \Omega = \emptyset\} ,$$

choose a corresponding decomposition $f_1 = f_1^+ + f_1^-$, and put

$$u_1 = E^{\Delta,+} * f_1^+ + E^{\Delta,-} * f_1^- .)$$

Then by the estimate of Theorem 5, and (7), (3), we have

$$\text{S.S.}\, u_1 \Subset (\boldsymbol{R}^n \setminus \Omega) \times (\overline{\Xi_1} \cap \Delta) \cup \bar{\Omega} \times (\overline{\Xi}_2 \cap \Delta) .$$

Thus choose a corresponding decomposition by S.S.

$$u_1 = v_1 + w_1 ,$$

and put

$$f_2 = f_1 - P(D) v_1 = P(D) w_1 .$$

Then we have

$$\text{S.S.}\, f_2 \Subset (\boldsymbol{R}^n \setminus \Omega) \times (\overline{\Xi}_1 \cap \Delta) \cap \bar{\Omega} \times (\overline{\Xi}_2 \cap \Delta) = \partial\Omega \times (\overline{\Xi}_2 \cap \Delta) .$$

From now on the proof works similarly and we finally obtain a solution $u = v_0 + v_1 + \cdots + v_s$ of $P(D)u = f_0$ modulo $\mathcal{A}(\boldsymbol{R}^n)$, which is real analytic in Ω.

q. e. d.

3. We give here an example of stratification to Zampieri's operator:

$$P(D) = D_1^2 D_2^2 - D_2^2 D_3^2 - D_3^4 - D_4^4 - D_3^2 D_4^2 .$$

We have

$$\boldsymbol{S}^{n-1} = \Xi_0 \sqcup \Xi_1 \sqcup \Xi_2 ,$$

where $\Xi_0 = \{P(\xi) \neq 0\}$, $\Xi_1 = \{P(\xi) = 0, \nabla_\xi P(\xi) \neq 0\}$ and $\Xi_2 = \{(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0)\}$. As is shown by Zampieri, the pointwise geometric condition (2)′ does not imply (1) if we consider e. g.

$$\Omega = \{|x_1 + x_3| < 1, \ |x_1 - x_3| < 1, \ |x_2| < 1, \ |x_4| < 1\} .$$

(See Example 3.1 of Kaneko [**6**] for detailed calculus.) Hence the stratification is really necessary for this example. (In fact this example has stimulated our present work.)

A typical case where the stratification is unnecessary is the operators whose local propagation cones are all half lines, hence especially operators with simple characteristics. For such operators the global solvability under pointwise geometric condition (2)′ is already obtained by Kawai [7]. For general locally hyperbolic operators Kawai formulates his main theorems with some concrete geometric assumptions which assure the decomposition (1). As is seen from the above example, however, these assumptions are in any case not definitive.

We suppose that our condition (2) is even necessary for the existence of real analytic solutions on a bounded domain. In fact, when we consider a bounded domain Ω, there seems to be no practical difference between ours and Zampieri's refined one in terms of the local propagation cones of local irreducible factors of P_m. An approach from the micro-local analysis to the necessary condition will be discussed in the forthcoming paper.

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