J. Math. Soc. Japan
Vol. 39, No. 1, 1987

Remarks on real nilpotent orbits of a symmetric pair

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(Received Oct. 20, 1984)
(Revised July 16, 1985)

Introduction.

In this paper, we shall study the real nilpotent orbits of the vector space associated to a semisimple symmetric pair.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be its involution. Then we obtain the direct sum decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ for σ. The pair $(\mathfrak{g}, \mathfrak{h})$ is called a symmetric pair. The first purpose of this paper is to prove a theorem concerning the H-orbital structure of the nilpotent subvariety $\mathcal{N}(\mathfrak{q})$ of $\mathfrak{q}$. The second purpose is to determine the orbital structure of $\mathcal{N}(\mathfrak{q})$ when the pair $(\mathfrak{g}, \mathfrak{h})$ is of split rank one in the sense of [**OS**, Def. 2.5.1].

We are going to explain the contents of this paper in detail. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$ acting on $\mathfrak{q}$. Then H leaves $\mathcal{N}(\mathfrak{q})$ invariant. Let $[\mathcal{N}(\mathfrak{q})]$ be the totality of H-orbits of $\mathcal{N}(\mathfrak{q})$. One can define symmetric pairs $(\mathfrak{g}^a, \mathfrak{h}^a)$ and $(\mathfrak{g}^d, \mathfrak{h}^d)$ from $(\mathfrak{g}, \mathfrak{h})$ as we did in [**OS**, §1]. Using the notation there, we define $\mathcal{N}(\mathfrak{q}^a)$ and $\mathcal{N}(\mathfrak{q}^d)$. In §1, we shall show the following theorem.

THEOREM. $[\mathcal{N}(\mathfrak{q})] \cong [\mathcal{N}(\mathfrak{q}^a)] \cong [\mathcal{N}(\mathfrak{q}^d)]$.

Now suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are the complexifications of a real semisimple Lie algebra $\mathfrak{g}_0$ and its maximal compact subalgebra $\mathfrak{k}_0$, respectively. Then $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. In this case, one finds that $(\mathfrak{g}^d, \mathfrak{h}^d)\cong(\mathfrak{g}_0\oplus\mathfrak{g}_0, \mathfrak{g}_0)$ and therefore that $\mathcal{N}(\mathfrak{q}^d)$ is identified with the totality $[\mathcal{N}(\mathfrak{g}_0)]$ of the real nilpotent orbits of $\mathfrak{g}_0$. Then the theorem mentioned above implies that $[\mathcal{N}((\mathfrak{p}_0)_C)]\cong[\mathcal{N}(\mathfrak{g}_0)]$. Here $(\mathfrak{p}_0)_C$ is the complexification of $\mathfrak{p}_0$ which is the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to its Killing form. This is a modified version of an unpublished result of Kostant (see Remark 1.10 (ii)). D. Vogan pointed out the importance of this bijection in the study of irreducible representations of $\mathfrak{g}_0$. Note that in this case, $(\mathfrak{g}^a, \mathfrak{h}^a)\cong(\mathfrak{g}, \mathfrak{g}_0)$ and therefore $\mathcal{N}(\mathfrak{q}^a)\cong\mathcal{N}(\mathfrak{g}_0)$.

In [**OS**, Def. 2.5.1], the rank and the split rank of a symmetric pair were introduced. These notions correspond to the rank and split rank of a semisimple Lie algebra. In §2, we shall determine $[\mathcal{N}(\mathfrak{q})]$ in the case where $(\mathfrak{g}, \mathfrak{h})$ is irreducible and of split rank one. In this case, the structure of $[\mathcal{N}(\mathfrak{q})]$ is described in terms of the signature of roots and the result is summarized in Theorem 2.3. It is noted here that in the case where $\mathfrak{g}$ is of split rank one, the orbital structure of the nilpotent subvariety of $\mathfrak{g}$ is already determined (see, for example [**B**]). Theorem 2.3 plays a basic role in the study of invariant spherical hyperfunctions on $\mathfrak{q}$ when $(\mathfrak{g}, \mathfrak{h})$ is of split rank one (cf. [**vD**], [**S2**]). This application will be discussed elsewhere.

§1. The bijection.

First we prepare the notations which will be used in this paper. We mainly follow the ones in [**OS**].

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be its involution. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ be the direct sum for σ. In this paper, $(\mathfrak{g}, \mathfrak{h})$ is called a symmetric pair and $\mathfrak{q}$ is the vector space associated to $(\mathfrak{g}, \mathfrak{h})$. The pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible if the representation of $\mathfrak{h}$ on $\mathfrak{q}$ via the adjoint representation is irreducible. Let G be the adjoint group $\mathrm{Int}(\mathfrak{g})$ and let H be the analytic subgroup of G corresponding to $\mathfrak{h}$. At this stage, we recall the following theorem (cf. [**L**, Chap. IV, Th. 2.1]).

THEOREM 1.1. *There exists a Cartan involution θ of $\mathfrak{g}$ commuting with σ Moreover, any two such θ are conjugate by an automorphism of the form $e^{\mathrm{ad}X}$ with $X\in\mathfrak{h}$.*

Take a Cartan involution θ of $\mathfrak{g}$ commuting with σ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition for θ. Let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$.

An element X of $\mathfrak{g}$ is nilpotent if $\mathrm{ad}_{\mathfrak{g}}(X)$ is a nilpotent endomorphism of $\mathfrak{g}$. Jacobson-Morozov lemma shows that if $X\in\mathfrak{g}$ is nilpotent, there exist $A, Y\in\mathfrak{g}$ such that $[A, X]=2X$, $[A, Y]=-2Y$, $[X, Y]=A$. Such a triple (A, X, Y) is called an S-triple. An element X of $\mathfrak{q}$ is nilpotent for the pair $(\mathfrak{g}, \mathfrak{h})$ if X is nilpotent as an element of $\mathfrak{g}$. In the sequel, we frequently omit the term "for the pair $(\mathfrak{g}, \mathfrak{h})$" if there is no confusion. Let $\mathfrak{N}(\mathfrak{q})$ be the totality of nilpotent elements of $\mathfrak{q}$ which is called the nilpotent subvariety of $\mathfrak{q}$. By [**KR**, Prop. 4], for any $X\in\mathfrak{N}(\mathfrak{q})$, there exist $A\in\mathfrak{h}$ and $Y\in\mathfrak{q}$ such that (A, X, Y) is an S-triple. Such an S-triple is called a normal S-triple (for the pair $(\mathfrak{g}, \mathfrak{h})$).

The following lemma is shown by an argument parallel to [**KR**, Prop. 4] (cf. [**K1**, Th. 3.6]).

LEMMA 1.2. *Let (A_i, X_i, Y_i) $(i=1, 2)$ be two normal S-triples. If X_1 and X_2 are conjugate by an element of H, there is an $h\in H$ such that $(h\cdot A_1, h\cdot X_1, h\cdot Y_1)=(A_2, X_2, Y_2)$.*

DEFINITION 1.3. Let (A, X, Y) be a normal S-triple. Then (A, X, Y) is a strictly normal S-triple if $\theta A=-A$ and $\theta X=-Y$.

We are going to show some basic properties of strictly normal S-triples.

LEMMA 1.4. *Let (A, X, Y) be a normal S-triple. Then there exists an $h\in H$ such that $(h\cdot A, h\cdot X, h\cdot Y)$ is a strictly normal S-triple.*

PROOF. Let (A, X, Y) be a normal S-triple. We may assume that $X\neq 0$. Consider the subalgebra $\mathfrak{l}=\boldsymbol{R}A+\boldsymbol{R}X+\boldsymbol{R}Y$ isomorphic to $\mathfrak{sl}(2, \boldsymbol{R})$. Define a Cartan involution θ_1 of $\mathfrak{l}$ by $\theta_1 A=-A$, $\theta_1 X=-Y$. Since θ_1 commutes with σ on $\mathfrak{l}$, it follows from [**vD**, Lemma 1] that there is a Cartan involution $\tilde{\theta}_1$ of $\mathfrak{g}$ such that $\tilde{\theta}_1\sigma=\sigma\tilde{\theta}_1$ and that $\tilde{\theta}_1|\mathfrak{l}=\theta_1$. Then in virtue of Theorem 1.1, we find that there is an $h\in H$ such that $h\cdot\tilde{\theta}_1\cdot h^{-1}=\theta$. Accordingly $(h\cdot A, h\cdot X, h\cdot Y)$ is a normal S-triple such that $\theta(h\cdot A)=-h\cdot A$, $\theta(h\cdot X)=-h\cdot Y$. q. e. d.

LEMMA 1.5. *Let (A_i, X_i, Y_i) $(i=1, 2)$ be two strictly normal S-triples. If X_1 and X_2 are conjugate by an element of H, then there is a $k\in H\cap K$ such that $(k\cdot A_1, k\cdot X_1, k\cdot Y_1)=(A_2, X_2, Y_2)$.*

PROOF. It follows from the assumption and Lemma 1.2 that there exists an $h\in H$ such that $(h\cdot A_1, h\cdot X_1, h\cdot Y_1)=(A_2, X_2, Y_2)$. Note that $H\cap K$ is a maximal compact subgroup of H and $H=\exp(\mathfrak{h}\cap\mathfrak{p})\cdot(H\cap K)$ is a Cartan decomposition. Hence write $h=e^Z\cdot k$ with $Z\in\mathfrak{h}\cap\mathfrak{p}$ and $k\in K\cap H$. Then to prove the lemma, it suffices to show that Z commutes with $k\cdot A_1$, $k\cdot X_1$ and $k\cdot Y_1$. Write $(A_1', X_1', Y_1')=(k\cdot A_1, k\cdot X_1, k\cdot Y_1)$ which is, by definition, a strictly normal S-triple. Then $(e^Z A_1', e^Z X_1', e^Z Y_1')=(A_2, X_2, Y_2)$.

Since X_2-Y_2, $\operatorname{ch} Z(X_1'-Y_1')\in\mathfrak{k}\cap\mathfrak{q}$ and $\operatorname{sh} Z(X_1'-Y_1')\in\mathfrak{p}\cap\mathfrak{q}$, the equality $X_2-Y_2=\operatorname{ch} Z(X_1'-Y_1')+\operatorname{sh} Z(X_1'-Y_1')$ implies that $\operatorname{sh} Z(X_1'-Y_1')=0$. Since $\operatorname{ad} Z$ leaves $\mathfrak{p}\cap\mathfrak{q}$ invariant and since the eigenvalues of $\operatorname{ad} Z$ are real values, it follows that $\operatorname{ad} Z(X_1'-Y_1')=0$. By a similar argument, we also find that $\operatorname{ad} Z(A_1')=0$. Since $[A_1', X_1'-Y_1']=2(X_1'+Y_1')$, it follows that Z commutes with A_1', X_1' and Y_1'. Hence the lemma is proved.

Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$ and extend σ and θ as complex linear endomorphisms of $\mathfrak{g}_c$. Define

$$\mathfrak{g}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\mathfrak{p}\cap\mathfrak{q}\,.$$

Then $\mathfrak{g}^d$ is a real form of $\mathfrak{g}_c$ and σ is a Cartan involution of $\mathfrak{g}^d$. Let $\mathfrak{g}^d=\mathfrak{k}^d+\mathfrak{p}^d$ be the Cartan decomposition for σ. Then $\mathfrak{k}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{h}\cap\mathfrak{p})$ and $\mathfrak{p}^d=\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\mathfrak{p}\cap\mathfrak{q}$. Since θ is also an involution of $\mathfrak{g}^d$, let $\mathfrak{g}^d=\mathfrak{h}^d+\mathfrak{q}^d$ be the direct sum for θ. Then $\mathfrak{h}^d=\mathfrak{k}\cap\mathfrak{q}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})$ and $\mathfrak{q}^d=\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\mathfrak{p}\cap\mathfrak{q}$ (cf. [**OS**, p. 436]). The symmetric pair $(\mathfrak{g}^d, \mathfrak{h}^d)$ is called the dual of $(\mathfrak{g}, \mathfrak{h})$. Let G^d be the adjoint group $\mathrm{Int}(\mathfrak{g}^d)$, K^d the maximal compact subgroup of G^d with Lie algebra $\mathfrak{k}^d$ and H^d the analytic subgroup of G^d corresponding to $\mathfrak{h}^d$.

Since H acts on $\mathcal{N}(\mathfrak{q})$, denote by $[\mathcal{N}(\mathfrak{q})]$ the totality of H-orbits of $\mathcal{N}(\mathfrak{q})$. It is known (cf. [**KR**, Th. 2], [**V**, p. 14]) that $[\mathcal{N}(\mathfrak{q})]$ is a finite set. Similarly, let $[\mathcal{N}(\mathfrak{q}^d)]$ be the totality of H^d-orbits of $\mathcal{N}(\mathfrak{q}^d)$. We are going to define bijective mappings between $[\mathcal{N}(\mathfrak{q})]$ and $[\mathcal{N}(\mathfrak{q}^d)]$.

Take an H-orbit $\mathcal{O}$ of $[\mathcal{N}(\mathfrak{q})]$. Then it follows from Lemma 1.4 that there is a strictly normal S-triple (A, X, Y) for the pair $(\mathfrak{g}, \mathfrak{h})$ such that $X \in \mathcal{O}$. Then put

$$(1) \qquad A^d = \sqrt{-1}(X-Y), \quad X^d = \frac{1}{2}(X+Y+\sqrt{-1}A), \quad Y^d = \frac{1}{2}(X+Y-\sqrt{-1}A).$$

It follows from the definition that (A^d, X^d, Y^d) is a strictly normal S-triple for the pair $(\mathfrak{g}^d, \mathfrak{h}^d)$, or equivalently, the following conditions hold:

$$\begin{aligned} &A^d \in \mathfrak{h}^d, \quad X^d,\ Y^d \in \mathfrak{q}^d, \\ &\theta X^d = -X^d, \quad \theta A^d = A^d, \quad \theta Y^d = -Y^d, \\ &\sigma X^d = -Y^d, \quad \sigma A^d = -A^d, \quad \sigma Y^d = -X^d. \end{aligned}$$

Lemma 1.5 implies that the H^d-orbits $H^d \cdot X^d$ and $H^d \cdot Y^d$ depend only on the H-orbit $\mathcal{O}$. Noting this, we define

$$\Phi_+(\mathcal{O}) = H^d \cdot X^d, \qquad \Phi_-(\mathcal{O}) = H^d \cdot Y^d.$$

Then Φ_+ and Φ_- are mappings of $[\mathcal{N}(\mathfrak{q})]$ to $[\mathcal{N}(\mathfrak{q}^d)]$.

Similarly, define mappings Φ_+^d, Φ_-^d of $[\mathcal{N}(\mathfrak{q}^d)]$ to $[\mathcal{N}(\mathfrak{q})]$. Namely, for an H^d-orbit $\mathcal{O}$ of $[\mathcal{N}(\mathfrak{q}^d)]$, take a strictly normal S-triple (A, X, Y) for the pair $(\mathfrak{g}^d, \mathfrak{h}^d)$ such that $X \in \mathcal{O}$. Then define A^d, X^d and Y^d by formula (1), and put $\Phi_+^d(\mathcal{O}) = H \cdot X^d$ and $\Phi_-^d(\mathcal{O}) = H \cdot Y^d$.

THEOREM 1.6. (i) *Φ_- and Φ_+ give bijective mappings between $[\mathcal{N}(\mathfrak{q})]$ and $[\mathcal{N}(\mathfrak{q}^d)]$.*

(ii) *Take a strictly normal S-triple (A, X, Y) for the pair $(\mathfrak{g}, \mathfrak{h})$. Then $\Phi_-^d \Phi_+(H \cdot X) = H \cdot X$ and $\Phi_+^d \Phi_+(H \cdot X) = H \cdot Y$.*

PROOF. The claim (ii) follows from the definitions of $\Phi_\pm$ and $\Phi_\pm^d$, and this implies (i).

We recall the definition of $\mathfrak{h}^a$ and $\mathfrak{q}^a$ (cf. [**OS**, p. 436]), namely, $\mathfrak{h}^a = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{q}^a = \mathfrak{k} \cap \mathfrak{q} + \mathfrak{h} \cap \mathfrak{p}$. Then $\mathfrak{g} = \mathfrak{h}^a + \mathfrak{q}^a$ and $(\mathfrak{g}^a, \mathfrak{h}^a)$ is a symmetric pair, where $\mathfrak{g}^a = \mathfrak{g}$. This is called the associated pair to $(\mathfrak{g}, \mathfrak{h})$. Let (A, X, Y) be a strictly normal S-triple for the pair $(\mathfrak{g}, \mathfrak{h})$. As in (1), we put

$$(2) \qquad A' = X+Y, \quad X' = \frac{1}{2}(-A+X-Y), \quad Y' = \frac{1}{2}(-A-X+Y).$$

Then (A', X', Y') is a strictly normal S-triple for the pair $(\mathfrak{g}^a, \mathfrak{h}^a)$. Following the construction of bijections between $[\mathcal{N}(\mathfrak{q})]$ and $[\mathcal{N}(\mathfrak{q}^d)]$, we are going to define mappings between $[\mathcal{N}(\mathfrak{q})]$ and $[\mathcal{N}(\mathfrak{q}^a)]$. Namely, take an H-orbit $\mathcal{O}$ of $\mathcal{N}(\mathfrak{q})$. Then there is a strictly normal S-triple (A, X, Y) for the pair $(\mathfrak{g}, \mathfrak{h})$ such that $X \in \mathcal{O}$. Define A', X', Y' by formula (2) by using (A, X, Y) and put $\Psi_+(\mathcal{O}) = H^a \cdot X'$ and $\Psi_-(\mathcal{O}) = H^a \cdot Y'$, where H^a is the analytic subgroup of G corresponding to $\mathfrak{h}^a$. Then it follows from Lemma 1.5 that the H^a-orbits $H^a \cdot X'$ and $H^a \cdot Y'$ of $\mathcal{N}(\mathfrak{q}^a)$ depend only on the H-orbit $\mathcal{O}$. By the same reason as in the case $\Phi_\pm : [\mathcal{N}(\mathfrak{q})] \to [\mathcal{N}(\mathfrak{q}^d)]$, we conclude that $\Psi_\pm : [\mathcal{N}(\mathfrak{q})] \to [\mathcal{N}(\mathfrak{q}^a)]$ are bijective mappings. Thus we obtain the following theorem.

THEOREM 1.7. *$\Psi_\pm : [\mathcal{N}(\mathfrak{q})] \to [\mathcal{N}(\mathfrak{q}^a)]$ are bijective mappings.*

In order to state the main theorem of this section, we use the notation $(\mathfrak{g}, \mathfrak{h})^a = (\mathfrak{g}^a, \mathfrak{h}^a)$ and $(\mathfrak{g}, \mathfrak{h})^d = (\mathfrak{g}^d, \mathfrak{h}^d)$. Moreover $(\mathfrak{g}, \mathfrak{h})^{ad} = (\mathfrak{g}^{ad}, \mathfrak{h}^{ad})$ and $\mathfrak{q}^{ad}$ denote the dual of $(\mathfrak{g}, \mathfrak{h})^a$ and the vector space associated to $(\mathfrak{g}, \mathfrak{h})^{ad}$, respectively and so on. Then it follows that $(\mathfrak{g}, \mathfrak{h})^{ada} \cong (\mathfrak{g}, \mathfrak{h})^{dad}$. Hence there are at most six symmetric pairs obtained from the given $(\mathfrak{g}, \mathfrak{h})$ by taking dual pair and associated pair, succesively (cf. [**OS**, p. 436]). Then Theorems 1.6 and 1.7 imply the following theorem.

THEOREM 1.8.
$$[\mathcal{N}(\mathfrak{q})] \cong [\mathcal{N}(\mathfrak{q}^d)] \cong [\mathcal{N}(\mathfrak{q}^a)] \cong [\mathcal{N}(\mathfrak{q}^{ad})]$$
$$\cong [\mathcal{N}(\mathfrak{q}^{da})] \cong [\mathcal{N}(\mathfrak{q}^{ada})].$$

Theorem 1.6 has an important consequence which we are going to state. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its Cartan decomposition. Let $\mathfrak{g}_c$, $\mathfrak{k}_c$ and $\mathfrak{p}_c$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Then $(\mathfrak{g}_c, \mathfrak{k}_c)$ is an example of a symmetric pair and its dual $(\mathfrak{g}_c, \mathfrak{k}_c)^d$ is isomorphic to $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$ (cf. [**OS**, Lemma 1.13.1]). Let K_c be the analytic subgroup of the adjoint group G_c of $\mathfrak{g}_c$ corresponding to $\mathfrak{k}_c$ and let G be the adjoint group of $\mathfrak{g}$. In this case, the nilpotent subvariety of the vector space associated to the pair $(\mathfrak{g}_c, \mathfrak{k}_c)^d$ is identified with the totality $\mathcal{N}_\mathfrak{g}$ of the nilpotent elements of $\mathfrak{g}$.

THEOREM 1.9. (i) *Let $\mathcal{O}$ be a K_c-orbit of $\mathcal{N}(\mathfrak{p}_c)$. Then we have the following.*

(a) *There are $A \in \mathfrak{k}_c$ and $X, Y \in \mathfrak{p}_c$ such that*
$$K_c \cdot X = \mathcal{O}, \qquad [A, X] = 2X, \qquad [A, Y] = -2Y, \qquad [X, Y] = A,$$
$$X + Y, \ \sqrt{-1}(X - Y) \in \mathfrak{p}, \qquad \sqrt{-1}A \in \mathfrak{k}.$$

(b) *$X^d = (1/2)(X + Y + \sqrt{-1}A)$ and $Y^d = (1/2)(X + Y - \sqrt{-1}A)$ are contained in $\mathfrak{g}$.*

(c) *X^d, Y^d and X are contained in the same G_c-orbit of $\mathfrak{g}_c$.*

(ii) *With the notation in* (i), *put* $\Phi_+(\mathcal{O})=G\cdot X^d$ *and* $\Phi_-(\mathcal{O})=G\cdot Y^d$ *for any* K_c*-orbit* $\mathcal{O}$ *of* $\mathcal{N}(\mathfrak{p}_c)$. *Then* Φ_+ *and* Φ_- *are bijective mappings of the set of* K_c*-orbits of* $\mathcal{N}(\mathfrak{p}_c)$ *onto that of* G*-orbits of* $\mathcal{N}_{\mathfrak{g}}$.

PROOF. (i) Take $A\in\mathfrak{k}_c$ and $X, Y\in\mathfrak{p}_c$. Then it follows from the definition that (A, X, Y) is a strictly normal S-triple for the pair $(\mathfrak{g}_c, \mathfrak{k}_c)$ if and only if A, X, Y satisfy the conditions in (a). Hence (a) is a consequence of Lemma 1.4. Since $X+Y\in\mathfrak{p}$ and $\sqrt{-1}A\in\mathfrak{k}$, (b) follows. Reducing to the case $\mathfrak{sl}(2, \boldsymbol{C})$, we easily find that X^d, Y^d and X are G_c-conjugate.

The statement (ii) is a consequence of Theorem 1.6. q. e. d.

REMARK 1.10. (i) In this case, $(\mathfrak{g}^a, \mathfrak{h}^a)\cong(\mathfrak{g}, \mathfrak{g}_0)$ and therefore $\mathcal{N}(\mathfrak{q}^a)\cong\mathcal{N}_{\mathfrak{g}}$.

(ii) We give here the original version of Theorem 1.9 due to B. Kostant.

Let $\tilde{K}_c$ and $\tilde{G}$ be the normalizers of K_c and G in G_c, respectively. Let $[\mathcal{N}(\mathfrak{p}_c)]^{\sim}$ be the set of $\tilde{K}_c$-orbits of $\mathcal{N}(\mathfrak{p}_c)$ and let $[\mathcal{N}_{\mathfrak{g}}]^{\sim}$ be the set of $\tilde{G}$-orbits of $\mathcal{N}_{\mathfrak{g}}$. Then

THEOREM. *Keep the notation in Theorem* 1.9. *If* $\tilde{\mathcal{O}}$ *is a* $\tilde{K}_c$*-orbit of* $\mathcal{N}(\mathfrak{p}_c)$, *take* A, X, Y *and define* X^d, Y^d *as in Theorem* 1.9 (i). *Put* $\tilde{\Phi}_+(\tilde{\mathcal{O}})=\tilde{G}\cdot X^d$ *and* $\tilde{\Phi}_-(\tilde{\mathcal{O}})=\tilde{G}\cdot Y^d$. *Then* $\tilde{G}\cdot X^d$ *and* $\tilde{G}\cdot Y^d$ *depend only on* $\tilde{\mathcal{O}}$ *and* $\tilde{\Phi}_\pm$ *are bijective mappings of* $[\mathcal{N}(\mathfrak{p}_c)]^{\sim}$ *onto* $[\mathcal{N}_{\mathfrak{g}}]^{\sim}$.

This is an unpublished result of B. Kostant.

(iii) D. Vogan pointed out the importance of the bijections of Theorem 1.9 in the study of unitary representation of semisimple Lie groups. Inspired by Vogan's lecture, the author was led to formulate Theorem 1.6.

(iv) In the case where $\mathfrak{g}_c$ is simple of classical type, the orbital structure of $\mathcal{N}_{\mathfrak{g}}$ is completely determined by Bourgoyne and Cushman [**BC**]. This combined with Theorem 1.9 gives a classification of K_c-orbital structure of $\mathcal{N}(\mathfrak{p}_c)$ when $\mathfrak{g}_c$ is simple of classical type.

Theorem 1.9 combined with a result of L. Antonyon (cf. [**S1**, Prop. 1.18′]) implies the following.

PROPOSITION 1.11. *Keep the notation above. Let* (A, X, Y) *be an* S*-triple with* $A, X, Y\in\mathfrak{g}_c$. *Then the following conditions are equivalent.*

(i) $G_c\cdot X\cap\mathfrak{p}_c\neq\emptyset$.

(ii) $G_c\cdot X\cap\mathfrak{g}\neq\emptyset$.

(iii) $G_c\cdot A\cap\mathfrak{p}_c\neq\emptyset$.

PROOF. The equivalence of (i) and (ii) is a consequence of Theorem 1.9. On the other hand, a result of Antonyon (cf. [**S1**, Prop. 1.18′]) implies the equivalence of (i) and (iii). But there seems to be no reference to its proof. For this reason, we give it here for the sake of completeness.

The implication (i)$\Rightarrow$(iii) is easy (cf. [**S1**, Lemma 1.15]). We are going to show the implication (iii)$\Rightarrow$(ii). Let (A, X, Y) be an S-triple with $A, X, Y \in \mathfrak{g}_c$. Assume that $G_c \cdot A \cap \mathfrak{p}_c \neq \emptyset$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}_c$ its complexification. Due to [**KR**, Th. 1], we may assume that $A \in \mathfrak{a}_c$. Since all the eigenvalues of $\mathrm{ad}_{\mathfrak{g}}(A)$ are real, it follows that $A \in \mathfrak{g}$. Hence $A \in \mathfrak{a}$. Now put $\mathfrak{g}_c(j) = \{Z \in \mathfrak{g}_c\,; [A, Z] = jZ\}$ $(j \in \boldsymbol{Z})$ and $V = \{Z \in \mathfrak{g}_c\,; [\mathfrak{g}_c(0), Z] = \mathfrak{g}_c(2)\}$. Then V is a non-empty Zariski open subset of $\mathfrak{g}_c(2)$ (cf. [**K1**, Lemma 4.2B]). On the other hand, it is clear that $\mathfrak{g}_c(2) \cap \mathfrak{g}$ is an everywhere Zariski dense subset of $\mathfrak{g}_c(2)$. These imply that $\mathfrak{g}_c(2) \cap \mathfrak{g} \cap V \neq \emptyset$. Take an element X' of $\mathfrak{g}_c(2) \cap \mathfrak{g} \cap V$. Since X' and X are G_c-conjugate (cf. [**K1**, Th. 4.2]), we have thus shown $G_c \cdot X \cap \mathfrak{g} \neq \emptyset$. q. e. d.

REMARK 1.12. The proof of (iii)$\Rightarrow$(ii) employed here is due to T. Tanisaki.

Now let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be a direct sum as before. Let X be a nilpotent element of $\mathfrak{g}_c$. We give here a sufficient condition in order that $G_c \cdot X \cap \mathfrak{q} \neq \emptyset$.

PROPOSITION 1.13. *Retain the notation above. Assume that $G_c \cdot X$ intersects with $\mathfrak{q}$. Then the weighted Dynkin diagram of X satisfies the following condition.*

In the weighted Dynkin diagram of X, let n_i $(=0, 1$ or $2)$ be the number written in the node i. Then, in the Satake diagram for the real form $\mathfrak{g}$ of $\mathfrak{g}_c$, the following holds:

$$\overset{\curvearrowleft\curvearrowright}{\underset{i}{\circ}\quad\underset{i'}{\circ}} \Longrightarrow n_i = n_{i'}\,,$$

$$\underset{i}{\bullet} \qquad \Longrightarrow n_i = 0\,.$$

Moreover the same condition holds for the Satake diagram of the real form $\mathfrak{g}^d$ of $\mathfrak{g}_c$.

PROOF. Let X be a nilpotent element of $\mathfrak{q}$. Take a normal S-triple (A, X, Y) for the pair $(\mathfrak{g}, \mathfrak{h})$. Due to Lemma 1.4, we may assume that (A, X, Y) is strictly normal. Then A and $X+Y$ are G-conjugate and $X+Y \in \mathfrak{p} \cap \mathfrak{q}$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$ containing $X+Y$. Then there exists a Cartan subalgebra $\tilde{\mathfrak{j}}$ of $\mathfrak{g}$ such that $\tilde{\mathfrak{j}} \cap \mathfrak{p}$ (resp. $\tilde{\mathfrak{j}} \cap \mathfrak{q}$) is maximal abelian in $\mathfrak{p}$ (resp. $\mathfrak{q}$) (cf. [**OS**, Lemma 2.4 (i)]). The rest of the proof is accomplished by an argument similar to the one in [**S1**, Prop. 1.16]. q. e. d.

§ 2. The case of split rank one.

Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair and let σ be the involution for $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition for a Cartan involution θ of $\mathfrak{g}$ commuting with σ. If $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, $r = \dim \mathfrak{a}$ is, by definition, the split rank of $(\mathfrak{g}, \mathfrak{h})$ (cf. [**OS**, Def. 2.5.1]). In this section, we always assume that

$(\mathfrak{g}, \mathfrak{h})$ is irreducible and of split rank one. A classification of such pairs is given in [**OS**, §5]. The purpose of this section is to determine the H-orbital structure of $\mathfrak{N}(\mathfrak{q})$ in this case.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. Let Σ be the root system of $(\mathfrak{g}, \mathfrak{a})$ (cf. [**OS**, §2]). Since Σ is of rank one, we may assume that $\Sigma=\{\alpha, -\alpha\}$ or $\Sigma=\{\alpha, -\alpha, 2\alpha, -2\alpha\}$. Take the unique element $A_0\in\mathfrak{a}$ such that $\alpha(A_0)=1$. Then $\mathfrak{a}=\boldsymbol{R}A_0$. For any $\lambda\in\Sigma$, put $\mathfrak{g}^{\pm}(\mathfrak{a}, \lambda)=\{X\in\mathfrak{g}\,;[Y, X]=\lambda(Y)X$ for any $Y\in\mathfrak{a}$, $\sigma\theta X=\pm X\}$ and $m^{\pm}(\lambda)=\dim_{\boldsymbol{R}}\mathfrak{g}^{\pm}(\mathfrak{a}, \lambda)$.

LEMMA 2.1. *If* $\mathfrak{g}^{+}(\mathfrak{a}, \alpha)\neq 0$ *or* $\mathfrak{g}^{+}(\mathfrak{a}, 2\alpha)\neq 0$, *there is an* $h\in H\cap K$ *such that* $h\cdot A_0=-A_0$.

PROOF. By definition, $\mathfrak{g}^{+}(\mathfrak{a}, \alpha)$ and $\mathfrak{g}^{+}(\mathfrak{a}, 2\alpha)$ are contained in $\mathfrak{h}^{a}=\mathfrak{h}\cap\mathfrak{k}+\mathfrak{p}\cap\mathfrak{q}$ which is a reductive Lie algebra. Then the assumption implies that $\mathfrak{h}^{a}$ is not abelian and therefore $[\mathfrak{h}^{a}, \mathfrak{h}^{a}]$ is semisimple of split rank one. Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$ and since $H\cap K$ is a maximal compact subgroup of H^{a}, the lemma follows.

LEMMA 2.2. *Let* $(\mathfrak{g}, \mathfrak{h})$ *be an irreducible symmetric pair of split rank one and assume that* $\mathfrak{h}$ *is not compact.*

(i) *If* $m^{+}(\alpha)=m^{+}(2\alpha)=0$, *then* $m^{-}(2\alpha)=0$ *and* $(\mathfrak{g}, \mathfrak{h})$ *is isomorphic to* $(\mathfrak{so}(p+1, 1), \mathfrak{so}(p, 1))$ *for some* $p>0$.

(ii) *If* $m^{+}(\alpha)=m^{-}(\alpha)=1$, *then* $m^{+}(2\alpha)=0$ *and* $m^{-}(2\alpha)=0$ *or* 1. *In the case where* $m^{-}(2\alpha)=0$, $(\mathfrak{g}, \mathfrak{h})$ *is isomorphic to one of the pairs* $(\mathfrak{so}(2, 2), \mathfrak{so}(2, 1))$, $(\mathfrak{so}(3, 1), \mathfrak{so}(2)+\mathfrak{so}(1, 1))$, *and in the case where* $m^{-}(2\alpha)=1$, $(\mathfrak{g}, \mathfrak{h})$ *is isomorphic to one of the pairs* $(\mathfrak{sl}(3, \boldsymbol{R}), \mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R})$, $(\mathfrak{su}(2, 1), \mathfrak{so}(2, 1))$.

This lemma follows from the classification in [**OS**, §5].

If $\mathfrak{g}^{-}(\mathfrak{a}, \alpha)\neq 0$, take an element X_{α} of $\mathfrak{g}^{-}(\mathfrak{a}, \alpha)$ such that $(2A_0, X_{\alpha}, -\theta X_{\alpha})$ is an S-triple. Similarly, if $\mathfrak{g}^{-}(\mathfrak{a}, 2\alpha)\neq 0$, take an element $X_{2\alpha}$ of $\mathfrak{g}^{-}(\mathfrak{a}, 2\alpha)$ such that $(A_0, X_{2\alpha}, -\theta X_{2\alpha})$ is an S-triple. Now fix such X_{α} and $X_{2\alpha}$ and define

$$\begin{aligned} A_i&=-X_{i\alpha}+\theta X_{i\alpha}\,, \\ X_i&=\frac{1}{2}((3-i)A_0+X_{i\alpha}+\theta X_{i\alpha}) \qquad (i=1, 2)\,, \\ Y_i&=\frac{1}{2}((3-i)A_0-X_{i\alpha}-\theta X_{i\alpha})\,. \end{aligned}$$

By definition, (A_i, X_i, Y_i) $(i=1, 2)$ are strictly normal S-triples and in particular, $X_i, Y_i\in\mathfrak{N}(\mathfrak{q})$.

THEOREM 2.3. *Assume that* $(\mathfrak{g}, \mathfrak{h})$ *is an irreducible symmetric pair of split rank one and that* $\mathfrak{h}$ *is not compact. Then* $(\mathfrak{g}, \mathfrak{h})$ *satisfies one of the following conditions* (a), (b), (c):

(a) $m^+(\alpha)=m^+(2\alpha)=m^-(2\alpha)=0$.

(b) $m^+(\alpha)=m^-(\alpha)=1$.

(c) $m^+(\alpha)+m^+(2\alpha)>0$ *but* (b) *does not hold.*

Moreover the structure of $[\mathcal{N}(\mathfrak{q})]$ *is given as follows.*

(I) *The case* (a).

$$[\mathcal{N}(\mathfrak{q})]=\begin{cases}\{H\cdot X_1,\ H\cdot Y_1,\ H\cdot(-X_1),\ H\cdot(-Y_1),\ \{0\}\} & \text{if } m^-(\alpha)=1\,,\\ \{H\cdot X_1,\ H\cdot Y_1,\ \{0\}\} & \text{if } m^-(\alpha)>1\,.\end{cases}$$

(II) *The case* (b).

$$[\mathcal{N}(\mathfrak{q})]=\begin{cases}\{H\cdot X_1,\ H\cdot Y_1,\ \{0\}\} & \text{if } m^-(2\alpha)=0\,,\\ \{H\cdot X_1,\ H\cdot Y_1,\ H\cdot X_2,\ H\cdot Y_2,\ \{0\}\} & \text{if } m^-(2\alpha)=1\,.\end{cases}$$

(III) *The case* (c).

(III. i) *If* $m^-(2\alpha)=0$, *then*

$$[\mathcal{N}(\mathfrak{q})]=\begin{cases}\{H\cdot X_1,\ H\cdot Y_1,\ \{0\}\} & \text{if } m^-(\alpha)=1\,,\\ \{H\cdot X_1,\ \{0\}\} & \text{if } m^-(\alpha)>1\,.\end{cases}$$

(III. ii) *If* $m^-(2\alpha)>0$, *then*

$$[\mathcal{N}(\mathfrak{q})]=\begin{cases}\{H\cdot X_1,\ H\cdot X_2,\ H\cdot Y_2,\ \{0\}\} & \text{if } m^-(2\alpha)=1\,,\\ \{H\cdot X_1,\ H\cdot X_2,\ \{0\}\} & \text{if } m^-(2\alpha)>1\,.\end{cases}$$

PROOF. By the classification in [**OS**, § 5], we find that one of the conditions (a), (b), (c) holds for $(\mathfrak{g}, \mathfrak{h})$.

Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair satisfying one of the conditions (a) and (b). Then it follows from Lemma 2.2 that $(\mathfrak{g}, \mathfrak{h})$ is isomorphic to one of the pairs

$$(\mathfrak{so}(p+1, 1),\ \mathfrak{so}(p, 1))\,,$$
$$(\mathfrak{so}(2, 2),\ \mathfrak{so}(2, 1))\,,\qquad (\mathfrak{so}(3, 1),\ \mathfrak{so}(2)+\mathfrak{so}(1, 1))\,,$$
$$(\mathfrak{sl}(3, \boldsymbol{R}),\ \mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R})\,,\qquad (\mathfrak{su}(2, 1),\ \mathfrak{so}(2, 1))\,.$$

It is easy to determine $[\mathcal{N}(\mathfrak{q})]$ in these cases. So (I) and (II) follow.

We are going to prove (III). Hence we assume that $m^+(\alpha)+m^+(2\alpha)>0$ and that $m^+(\alpha)=m^-(\alpha)=1$ does not hold. Take $X\in\mathcal{N}(\mathfrak{q})$ $(X\neq 0)$ and consider its orbit $H\cdot X$. In virtue of Lemma 1.4, we may assume that there is a strictly normal S-triple (A, X, Y). As in § 1, (2), put

$$A'=X+Y\,,\qquad X'=\frac{1}{2}(-A+X-Y)\,,\qquad Y'=\frac{1}{2}(-A-X+Y)\,.$$

Since $A'\in\mathfrak{p}\cap\mathfrak{q}$, there exists an $h\in H\cap K$ such that $h\cdot A'\in\mathfrak{a}$. Therefore we may assume from the first that A' is contained in $\mathfrak{a}$. In virtue of Lemma 2.1, we may also assume that $A'=jA_0$ for a positive constant j. Since (A', X', Y') is an S-triple, X' is contained in the eigenspace of $\operatorname{ad} A_0$. On the other hand, it follows from the assumption that $\theta\sigma X'=-X'$. These imply that X' is contained in $\mathfrak{g}^-(\mathfrak{a}, \alpha)+\mathfrak{g}^-(\mathfrak{a}, 2\alpha)$. Hence $j=1$ or $j=2$ and

(1) $X'\in\mathfrak{g}^-(\mathfrak{a}, \alpha)$ if $A'=2A_0$,

(2) $X'\in\mathfrak{g}^-(\mathfrak{a}, 2\alpha)$ if $A'=A_0$.

Now consider the case (1). Let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^-(\mathfrak{a}, \alpha)$ and $\mathfrak{g}^-(\mathfrak{a}, -\alpha)$. Then $\mathfrak{g}_0$ is semisimple and is invariant by θ and σ. Moreover $\mathfrak{g}_0=\mathfrak{g}_0\cap\mathfrak{k}+\mathfrak{g}_0\cap\mathfrak{p}$ is the Cartan decomposition for θ and $\mathfrak{g}_0=\mathfrak{g}_0\cap\mathfrak{h}+\mathfrak{g}_0\cap\mathfrak{q}$ is the direct sum for σ. We are now going to show that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}_0\cap\mathfrak{p}$. Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}_0\cap\mathfrak{p}\cap\mathfrak{q}$, let Σ_0 be the root system of $(\mathfrak{g}_0, \mathfrak{a})$. From the definition of $\mathfrak{g}_0$, we find that $\mathfrak{g}(\mathfrak{a}, \alpha)\cap\mathfrak{g}_0=\mathfrak{g}^-(\mathfrak{a}, \alpha)\cap\mathfrak{g}_0$. On the other hand, $(\mathfrak{g}_0, \mathfrak{h}\cap\mathfrak{g}_0)$ is an irreducible symmetric pair of split rank one. Noting these, we conclude by the classification that $(\mathfrak{g}_0, \mathfrak{h}\cap\mathfrak{g}_0)$ is of Type $(\mathfrak{k}_\varepsilon)$ in the sense of [**OS**, § 1]. In this case, every maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}\cap\mathfrak{g}_0$ is also that of $\mathfrak{p}\cap\mathfrak{g}_0$. Therefore we find that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{g}_0$. Let G_0 be the analytic subgroup of G corresponding to $\mathfrak{g}_0$. Then $K_0=K\cap G_0$ is a maximal compact subgroup of G_0. Let M_0 be the centralizer of $\mathfrak{a}$ in K_0. For any positive number p, let S_p be the sphere $\{Z\in\mathfrak{g}^-(\mathfrak{a}, \alpha)\ ;\ -B(Z, \theta Z)=p\}$ ($B(\ ,\ )$ denotes the Killing form). If $m^-(\alpha)>1$, it follows from [**K2**, Th. 2.1.7] that M_0 acts on S_p in a transitive way. On the other hand, in the case when $m^-(\alpha)=1$, S_p consists of just two points. So we arrive at the following results.

(1.1) If $m^-(\alpha)>1$, there exist an $m\in M_0$ and a constant $c>0$ such that $m\cdot X'=cX_\alpha$.

(1.2) If $m^-(\alpha)=1$, there exist an $m\in M_0$ and a constant $c\neq0$ such that $m\cdot X'=cX_\alpha$.

If $m\cdot X'=cX_\alpha$, then $(2A_0, m\cdot X', m\cdot Y')=(2A_0, cX_\alpha, -c\theta X_\alpha)$ is an S-triple. This implies that $c^2=1$. Hence, if $c>0$, then $c=1$ and if $c<0$, then $c=-1$. Therefore we find that

(1.1)′ If $m^-(\alpha)>1$, then (A', X', Y') and $(2A_0, X_\alpha, -\theta X_\alpha)$ are $(H\cap K)$-conjugate.

(1.2)′ If $m^-(\alpha)=1$, then (A', X', Y') is $(H\cap K)$-conjugate to one of $(2A_0, X_\alpha, -\theta X_\alpha)$ and $(2A_0, -X_\alpha, \theta X_\alpha)$.

We are going to show that if $m^-(\alpha)=1$, then $H\cdot X_1$ and $H\cdot Y_1$ are disjoint. Consider the strictly normal S-triple $(-A_1, Y_1, X_1)$. Then $(2A_0, -X_\alpha, \theta X_\alpha)$ is the S-triple obtained from formula (2) of § 1 if one applies this to $(-A_1, Y_1, X_1)$ instead of (A_1, X_1, Y_1). Now assume that X_1 and Y_1 are H-conjugate. Then it follows from Lemma 1.4 that there exists an $h\in H\cap K$ such that $(h\cdot A_1, h\cdot X_1, h\cdot Y_1)$

$=(-A_1, Y_1, X_1)$. Then we find that $h\cdot X_\alpha=-X_\alpha$ and $h\cdot A_0=A_0$. Define $\mathfrak{h}^a=\mathfrak{k}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{q}$. Then, by definition, $\mathfrak{h}^a$ is a reductive Lie algebra and its derived algebra is of split rank one. Let H^a be the analytic subgroup of G corresponding to $\mathfrak{h}^a$. Then $H\cap K$ is a maximal compact subgroup of H^a. At this stage, we note that $m^+(\alpha)>1$ holds. In fact, we assumed that $m^-(\alpha)=1$ and $m^+(\alpha)+m^+(2\alpha)>0$. These assumptions combined with the classification in [**OS**, § 5] imply that $m^+(\alpha)>1$ (cf. Lemma 2.2). Then it follows that $M^a=Z_{H\cap K}(\mathfrak{a})$ is connected (cf. [**H**, p. 435]). On the other hand, it is clear that $h\in M^a$. Since $m^-(\alpha)=1$, we find that $\mathfrak{g}^-(\mathfrak{a}, \alpha)=\boldsymbol{R}X_\alpha$. Then $M^a\cdot X_\alpha\subseteqq\{cX_\alpha\,;\,c>0\}$. This contradicts that $hX_\alpha=-X_\alpha$. Hence X_1 and Y_1 are not H-conjugate.

Next consider the case (2). Note that if $m^-(2\alpha)>0$, then $m^+(\alpha)=m^-(\alpha)>0$ (cf. [**OS**, Lemma 2.17.1]). By an argument similar to the previous case, we can prove the following results.

(2.1) If $m^-(2\alpha)>1$, then (A', X', Y') is $(H\cap K)$-conjugate to $(A_0, X_{2\alpha}, -\theta X_{2\alpha})$.

(2.2) If $m^-(2\alpha)=1$, then (A', X', Y') is $(H\cap K)$-conjugate to one of $(A_0, X_{2\alpha}, -\theta X_{2\alpha})$ and $(A_0, -X_{2\alpha}, \theta X_{2\alpha})$.

Now consider the case when $m^-(2\alpha)=1$. In this case, it follows that $m^+(\alpha)>0$. Since we exclude the case $m^+(\alpha)=m^-(\alpha)=1$, we may assume that $m^+(\alpha)>1$. Then we can prove by an argument similar to the previous case that X_2 and Y_2 are not H-conjugate. We have thus shown (III). q. e. d.

REMARK 2.4. (i) If $(\mathfrak{g}, \mathfrak{h})$ is of rank one, that is, $\mathfrak{g}^d$ is of split rank one, the H-orbital structure of $\mathcal{N}(\mathfrak{q})$ is investigated by several authors (cf. [**vD**] and its references).

(ii) In the case where $(\mathfrak{g}, \mathfrak{h})\cong(\mathfrak{g}'\oplus\mathfrak{g}', \mathfrak{g}')$ with $\mathfrak{g}'$ semisimple of split rank one, the orbital structure is already determined (cf. [**B**], [**BC**]).

We shall give an application of Theorem 2.3 to the study of invariant spherical hyperfunctions elsewhere (cf. [**vD**], [**S2**]).

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