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Cohen-Macaulay normal local domains whose associated graded rings have no depth

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Introduction.

In this note we prove the following proposition by giving explicit examples

PROPOSITION. *For every integer* $r\geqq 2$, *there exists a Cohen-Macaulay normal local domain* $(B, \mathfrak{m})$ *of dimension* r *such that* $\mathrm{depth}_{\bar{\mathfrak{m}}}(\mathrm{Gr}^{\mathfrak{m}}(B))=0$, *where* $\bar{\mathfrak{m}}=\bigoplus_{r\geq 1}\mathfrak{m}^{i}/\mathfrak{m}^{i+1}$ *is the maximal ideal of* $\mathrm{Gr}^{\mathfrak{m}}(B)$.

One dimensional complete local domains with the analogous property were found some ten years ago by several authors ([**2**], [**3**] and [**5**]). The rings we present here are obtained by localizing the affine coordinate rings of normal determinantal schemes of codimension two at certain singular points which may be assumed to be isolated if $\dim B\leqq 4$. We see by these examples that, even if a given local domain has some fairly good properties such as normality or Cohen-Macaulayness, its depth provides no information on the depth of its own associated graded ring in general.

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Proof of the proposition.

Throughout this paper A denotes the polynomial ring $k[x_1, \cdots, x_n]$ $(n\geqq 4)$ over an algebraically closed field k of arbitrary characteristic. For an integer $r\geqq 2$, let n and m be integers satisfying $n=r+2$, $m\geqq n-1$. We introduce sets of parameters $t=\{t_{ij}^{a}\mid 1\leqq i\leqq m+1,\ 1\leqq j\leqq m,\ 1\leqq a\leqq n\}$, $u=\{u_{ij}^{ab}\mid 1\leqq i\leqq m+1,\ 1\leqq j\leqq m,\ 1\leqq a,\ b\leqq n\}$, and for a subset v of $t\cup u$, we will denote by $A[t, u\setminus v]$ the polynomial ring generated over A by the elements contained in $t\cup u\setminus v$, in particular $A[t, u]$ is the polynomial ring generated by all the elements of $t\cup u$. Let M_2 be an $(m+1)\times m$-matrix whose (i, j)-component is $h_{ij}^{\circ}:=\sum_{a=1}^{n}t_{ij}^{a}x_{a}^{2}+\sum_{a,b=1}^{n}u_{ij}^{ab}x_{a}^{2}x_{b}$,

$$M_1 = \underbrace{\begin{pmatrix} x_2 & x_3 & \cdots\cdots & x_n & 0 & \cdots & 0 \\ x_1 & & & & & & \\ & x_1 & & & 0 & & \\ & & \ddots & & & & \\ & 0 & & \ddots & & & \\ & & & & & & x_1 \end{pmatrix}}_{m}, \qquad M = M_1 + M_2,$$

and let h_{ij} denote the (i, j)-component of M. We will consider the ideal $\tilde{I}$ in $A[t, u]$ generated by the maximal minors of M and the family of affine schemes $\operatorname{Spec} A[t, u]/\tilde{I}$ over $\operatorname{Spec} k[t, u]$. Let $p: X \to S$ denote the morphism induced by the natural inclusion $k[t, u] \subsetneq A[t, u]$ and $q: \operatorname{Spec} A[t, u] \to \operatorname{Spec} A$ the morphism induced by $A \subsetneq A[t, u]$, where $X = \operatorname{Spec} A[t, u]/\tilde{I}$ and $S = \operatorname{Spec} k[t, u]$. From now on, the symbol o will denote the point of $\operatorname{Spec} A$ defined by $x_1 = x_2 = \cdots = x_n = 0$ and D_i will denote the divisor $x_i = 0$ for $1 \leqq i \leqq n$.

LEMMA 1. *Let Y be the subscheme of X defined by $(m-1)\times(m-1)$-minors of M. Then, there exists a nonempty Zariski open set U_1 of S such that, for every $s \in U_1$, we have*

$$\dim(Y_s \setminus q^{-1}(o)) \leqq n-6, \qquad \textit{where } Y_s = p^{-1}(s) \cap Y.$$

PROOF. (The idea is due to [6].) Let y_{ij} $(1 \leqq i \leqq m+1,\ 1 \leqq j \leqq m)$ be algebraically independent elements over $A[t, u]$ and $k[y]$ the polynomial ring generated by all these y_{ij}. For each c $(1 \leqq c \leqq n)$, set $P_c = A[t, u \setminus \{t_{ij}^c \mid 1 \leqq i \leqq m+1, 1 \leqq j \leqq m\}] \otimes_k k[y]$ and define a map $F_c: P_c \to A[t, u]$ by $F_c(t_{ij}^a) = t_{ij}^a$ for $a \neq c$, $F_c(y_{ij}) = h_{ij}$ and $F_c(u_{ij}^{ab}) = u_{ij}^{ab}$. Let $I^{m-1}(y)$ (resp. $I^{m-1}(M)$) denote the ideal in P_c (resp. $A[t, u]$) generated by $(m-1)\times(m-1)$-minors of the matrix (y_{ij}) (resp. M). The ring homomorphism

$$\bar{F}_c : (P_c/I^{m-1}(y))_{x_c} \longrightarrow (A[t, u]/I^{m-1}(M))_{x_c},$$

induced by F_c has the inverse satisfying $\bar{F}_c^{-1}(t_{ij}^a) = t_{ij}^a$ for $a \neq c$, $\bar{F}_c^{-1}(t_{ij}^c) = \{y_{ij} - (h_{ij} - t_{ij}^c x_c^2)\}/x_c^2$ and $\bar{F}_c^{-1}(u_{ij}^{ab}) = u_{ij}^{ab}$, hence it is an isomorphism. The height of $I^{m-1}(y)$ is 6 (see [**6**; p. 679] for example), so it follows that $\dim(Y \setminus q^{-1}(D_c)) = n + \dim S - 6$, and since $Y \setminus q^{-1}(o) = \bigcup_{c=1}^n (Y \setminus q^{-1}(D_c))$, we have $\dim(Y \setminus q^{-1}(o)) = n + \dim S - 6$. The existence of U_1 in the statement is now obvious. QED

LEMMA 2. *There exists a nonempty Zariski open set U_2 of S such that, for every $s \in U_2$, the scheme $X_s \setminus (Y \cup q^{-1}(o))$ is smooth, where $X_s = p^{-1}(s)$.*

PROOF. (The idea is due to [**6**].) Let w be a closed point of $\operatorname{Spec} A[t, u]$ not contained in $Y \cup q^{-1}(D_c)$ for some c $(1 \leqq c \leqq n)$. Then, by the definition of Y, there exists an affine open neighborhood W of w in $(\operatorname{Spec} A[t, u]) \setminus q^{-1}(D_c)$ such that one of the $(m-1)\times(m-1)$-minors does not vanish at any points of W. We may therefore assume, by renumbering the rows and columns suitably, that M

is of the form $\left(\begin{array}{c|c} h_{11} & * \\ h_{12} & \\ \hline * & M' \end{array}\right)$, with $d:=\det M'$ not vanishing at any points of W. Multiply M by a suitable matrix $N=\left(\begin{array}{c|c} 1 & 0 \\ \hline * & (M')^{-1} \end{array}\right)\in GL(m, A[t, u]_d)$ on the right so that MN takes the form $\left(\begin{array}{c|c} g_1 & * \\ g_2 & \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right)$. In this expression, one sees by Cramer's formula that $g_i=h_{i1}^c+h_i$ $(i=1, 2)$, where h_1, h_2 are elements of $A[t, u]_d$ and none of the parameters t_{i1}^a, u_{i1}^{ab} $(i=1, 2, 1\leqq a, b\leqq n)$ occur in them. Observe that X is defined in W by the equation $g_1=g_2=0$ and that the singularity of $X_s\cap W$ coincides with the zero locus of the maximal minors of the Jacobian matrix $(\partial g_i/\partial x_j)$ $(i=1, 2, 1\leqq j\leqq n)$. Let Z denote the subscheme of W defined by the ideal J generated by g_1, g_2 and $\det\begin{pmatrix}\partial g_1/\partial x_{i_1} & \partial g_1/\partial x_{i_2} \\ \partial g_2/\partial x_{i_1} & \partial g_2/\partial x_{i_2}\end{pmatrix}$ $(1\leqq i_1<i_2\leqq n)$. We want to show $\dim Z=\dim S-1$. Let z_{ij} $(i=1, 2, 0\leqq j\leqq n)$ be algebraically independent elements over $A[t, u]$ and $k[z]$ the polynomial ring generated by all these z_{ij}. Set

$$Q_w=(A[t, u\setminus\{t_{11}^c, t_{21}^c\}\cup\{u_{i1}^{cb}\mid i=1, 2, 1\leqq b\leqq n\}]\otimes_k k[z])_d$$

and define a map $G_w: Q_w\to A[t, u]_d$ by

$$G_w(z_{10})=g_1,\qquad G_w(z_{20})=g_2,\qquad G_w(z_{ij})=\partial g_i/\partial x_j\ (i=1, 2, 1\leqq j\leqq n)$$
$$\text{and}\qquad G_w(t_{ij}^a)=t_{ij}^a,\qquad G_w(u_{ij}^{ab})=u_{ij}^{ab}$$

for all parameters contained in Q_w. We have

$$\begin{cases} g_i=t_{i1}^c x_c^2+u_{i1}^{cc}x_c^3+\sum_{b\neq c}u_{i1}^{cb}x_c^2x_b+g_{i0} \\ \partial g_i/\partial x_c=2t_{i1}^c x_c+3u_{i1}^{cc}x_c^2+\sum_{b\neq c}2u_{i1}^{cb}x_cx_b+\partial g_{i0}/\partial x_c \\ \partial g_i/\partial x_b=u_{i1}^{cb}x_c^2+\partial g_{i0}/\partial x_b \qquad \text{for } b\neq c\,, \end{cases}$$

with appropriate elements g_{10}, g_{20} of $A[t, u\setminus\{t_{11}^c, t_{21}^c\}\cup\{u_{i1}^{cb}\mid i=1, 2, 1\leqq b\leqq n\}]_d$. Since $\det\begin{pmatrix}x_c^2 & x_c^3 \\ 2x_c & 3x_c^2\end{pmatrix}=x_c^4\neq 0$ for $x_c\neq 0$, it is easily seen that G_w induces the isomorphism

$$\bar{G}_w:(Q_w/J(z))_{x_c}\longrightarrow(A[t, u]_d/J)_{x_c}\,,$$

where $J(z)$ denotes the ideal generated by z_{10}, z_{20} and $\det\begin{pmatrix}z_{1i_1} & z_{1i_2} \\ z_{2i_1} & z_{2i_2}\end{pmatrix}$ $(1\leqq i_1<i_2\leqq n)$. The ideal $J(z)$ is of height $n+1$ [**6**; pp. 679 and 683], therefore we have $\dim\operatorname{Spec}(A[t, u]_d/J)_{x_c}=\dim Z=\dim S-1$. Since

$$X\smallsetminus(Y\cup q^{-1}(o)) = \bigcup_{c=1}^{n}(X\smallsetminus(Y\cup q^{-1}(D_c)))$$

is quasi-compact, the existence of the open set U_2 in the statement is now clear. QED

The above lemmas imply that, for every point $s\in U_1\cap U_2$, the scheme X_s is not empty (containing at least the point o), $\dim X_s\leqq n-2$ and $\dim \mathrm{Sing}(X_s)\leqq n-6$. Fix a closed point $s\in U_1\cap U_2$ ($s=\mathfrak{n}_s$ a maximal ideal) and denote by L the matrix M (mod $\mathfrak{n}_s$). Let $L^{(i)}$ be the $m\times m$-matrix obtained by deleting the i-th row from L and put $f_i=(-1)^i\det L^{(i)}$ $(1\leqq i\leqq m+1)$. The ideal $I:=\tilde{I}$ (mod $\mathfrak{n}_s$), then, coincides with $(f_1, \cdots, f_{m+1})A$. We now turn our attention to the ring $A/I=A[t, u]/\tilde{I}\otimes_{k[t,u]}k[t, u]/\mathfrak{n}_s$ and its localization by $\tilde{\mathfrak{m}}:=(x_1, \cdots, x_n)A$. Since $\dim A/I\leqq n-2$, we find by [**1**; Theorem 5.1] that A/I is Cohen-Macaulay of dimension $n-2$, therefore A/I satisfies the condition S_2 of Serre's criterion of normality (see [**4**; p. 125]). On the other hand, since $X_s\smallsetminus(Y_s\cup\{o\})$ is smooth with $\dim(Y_s\cup\{o\})\leqq n-6$ and since X_s is pure dimensional of dimension $n-2$, the condition R_1 is also satisfied by A/I. Hence A/I is a normal ring of dimension $n-2=r$. Let B denote the normal local ring $A_{\tilde{\mathfrak{m}}}/IA_{\tilde{\mathfrak{m}}}$. Then, B is a domain (see the proof of [**4**; Theorem 39]), and its associated graded ring actually has no depth, which we will prove below. Let $\mathfrak{m}$ denote the maximal ideal of B and $\mathrm{Gr}^{\mathfrak{m}}(B)$ the associated graded ring $\bigoplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$.

LEMMA 3. $\mathrm{Gr}^{\mathfrak{m}}(B)$ *has no depth, namely* $\mathrm{depth}_{\bar{\mathfrak{m}}}(\mathrm{Gr}^{\mathfrak{m}}(B))=0$, *where* $\bar{\mathfrak{m}}$ *denotes the maximal ideal* $\bigoplus_{i\geq 1}\mathfrak{m}^i/\mathfrak{m}^{i+1}$ *in* $\mathrm{Gr}^{\mathfrak{m}}(B)$.

PROOF. Let H be the homogeneous ideal in A generated by the initial forms of the elements of $IA_{\tilde{\mathfrak{m}}}$. By definition, for each $f\in\tilde{\mathfrak{m}}^iA_{\tilde{\mathfrak{m}}}\smallsetminus\tilde{\mathfrak{m}}^{i+1}A_{\tilde{\mathfrak{m}}}$, $\mathrm{in}(f)$ denotes the homogeneous polynomial $f^{(i)}$ of degree i such that $f-f^{(i)}\in\tilde{\mathfrak{m}}^{i+1}A_{\tilde{\mathfrak{m}}}$ and we have

$$\begin{cases} H_i = \{g \mid \deg g=i \text{ and } g=\mathrm{in}(f) \text{ for some } f\in IA_{\tilde{\mathfrak{m}}}\} \\ H=\bigoplus_{i\geq 0} H_i\subset A\,. \end{cases}$$

$\mathrm{Gr}^{\mathfrak{m}}(B)$ is canonically isomorphic to A/H and under this isomorphism $\bar{\mathfrak{m}}$ corresponds to $\tilde{\mathfrak{m}}$ (mod H). It is therefore sufficient to prove $\mathrm{depth}_{\tilde{\mathfrak{m}}}(A/H)=0$. Recall that L is of the form M_1+L_2 ($L_2:=M_2$ (mod $\mathfrak{n}_s$)), in particular, that every entry of L_2 belongs to $\tilde{\mathfrak{m}}^2$. One sees immediately $\mathrm{in}(f_1)=-x_1^m$, $\mathrm{in}(f_i)=x_ix_1^{m-1}$ for $2\leqq i\leqq n$, and $H_i=0$ for $i<m$. Hence the nonzero element x_1^{m-1} (mod H) of A/H is annihilated by $\tilde{\mathfrak{m}}$, which means $\mathrm{depth}_{\tilde{\mathfrak{m}}}(A/H)=0$. QED

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