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On the Martin boundary of Lipschitz strips

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§1. Introduction.

Let $n\geqq 1$ and $m\geqq 1$. We denote by $P=(X, Y)$ a point in $\boldsymbol{R}^{n+m}=\boldsymbol{R}^{n}\times\boldsymbol{R}^{m}$, where $X=(x_{1}, \cdots, x_{n})\in\boldsymbol{R}^{n}$ and $Y=(y_{1}, \cdots, y_{m})\in\boldsymbol{R}^{m}$. We write $|P|$, $|X|$ and $|Y|$ for $(\sum_{j=1}^{n}x_{j}^{2}+\sum_{j=1}^{m}y_{j}^{2})^{1/2}$, $(\sum_{j=1}^{n}x_{j}^{2})^{1/2}$ and $(\sum_{j=1}^{m}y_{j}^{2})^{1/2}$, respectively. We identify $\boldsymbol{R}^{n}$ and $\boldsymbol{R}^{m}$ with $\{(X, Y); Y=0\}$ and $\{(X, Y); X=0\}$, respectively. We denote by S^{n-1} the unit sphere $\{\alpha\in\boldsymbol{R}^{n}; |\alpha|=1\}$ with center at the origin in $\boldsymbol{R}^{n}$. Let D be a bounded domain in $\boldsymbol{R}^{m}$. We call $L=\boldsymbol{R}^{n}\times D=\{(X, Y); Y\in D\}$ a strip. If D is a Lipschitz domain, then L is said to be a Lipschitz strip. In this note we consider the Martin compactification of a Lipschitz strip.

We denote by $\bar{L}=\boldsymbol{R}^{n}\times\bar{D}$ the Euclidean closure of L in $\boldsymbol{R}^{n+m}$. Let M_{α}, $\alpha\in S^{n-1}$, be a point at infinity and let $\hat{L}=\bar{L}\cup\{M_{\alpha}; \alpha\in S^{n-1}\}$ be a compact topological space with open base $\mathcal{O}_{1}\cup\mathcal{O}_{2}$, where $\mathcal{O}_{1}=\{U\cap\bar{L}; U$ is an open set of $\boldsymbol{R}^{n+m}\}$ and $\mathcal{O}_{2}=\{U(\alpha, \varepsilon, R); \alpha\in S^{n-1}, 0<\varepsilon<1$ and $R>0\}$ with $U(\alpha, \varepsilon, R)=\{M_{\beta}; \beta\in S^{n-1}, \sum_{i=1}^{n}\alpha_{i}\beta_{i}>1-\varepsilon\}\cup\{(X, Y)\in\bar{L}; (1-\varepsilon)^{-1}\sum_{i=1}^{n}x_{i}\alpha_{i}>|X|>R\}$. We note that $P_{j}=(X_{j}, Y_{j})\in\bar{L}$ converges to M_{α} if and only if $\lim_{j\to\infty}|X_{j}|=+\infty$ and $\lim_{j\to\infty}X_{j}/|X_{j}|=\alpha$. We shall prove

THEOREM 1. *The Martin compactification of L is homeomorphic to $\hat{L}$.*

In case $m=1$ and $D=(0, 1)$, Brawn [2] proved Theorem 1 by using the exact formula for the Green function (see [1]). However it seems to be difficult to obtain such a formula if D is a general Lipschitz domain in $\boldsymbol{R}^{m}$, $m\geqq 2$.

In this paper we shall present a new proof based on the boundary Harnack principle (see Lemma 1) and the symmetric property of the Green function G for L, i.e., if $Y\in D$ and $X, X'\in\boldsymbol{R}^{n}$, then $G((X, Y), (X', Y))$ depends only on $|X-X'|$. We shall also consider the Martin boundary of the semi-strip $\{X\in\boldsymbol{R}^{n}; x_{1}>0\}\times D$ in §3, and give a generalization of [6; Example 3].

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§ 2. Proof of Theorem 1.

We shall use the following notation: Let $X_0=(0, \cdots, 0)\in \boldsymbol{R}^n$, $Y_0=(0, \cdots, 0)\in \boldsymbol{R}^m$ and $P_0=(X_0, Y_0)\in \boldsymbol{R}^{n+m}$. Without loss of generality we may assume that $Y_0\in D$, and hence that $P_0\in L$. We let $L_0=\boldsymbol{R}^n\times\{Y_0\}$. Denote by $B^n(X, r)$, $B^m(Y, r)$ and $B(P, r)$ the n-dimensional open ball with center at X and radius r, the m-dimensional open ball with center at Y and radius r and the $(n+m)$-dimensional open ball with center at P and radius r, respectively. We may assume that $D\supset B^m(Y_0, 5)$. Let $\pi: \boldsymbol{R}^{n+m}\to \boldsymbol{R}^n$ be the projection defined by $\pi((X, Y))=X$ and let $\pi_0(P)=(\pi(P), Y_0)$. We put $U_+(t)=\{P\in L; (\pi(P))_n>t\}$, $U_-(t)=\{P\in L; (\pi(P))_n<t\}$, $E(t)=\{P\in L; (\pi(P))_n=t\}$ and $\Delta(t)=\partial U_-(t)\cap\partial L$, where $(\pi(P))_n=x_n$ if $P=(X, Y)$ and $X=(x_1, \cdots, x_n)$. We observe that $\Omega(X)=B^n(X, 1)\times D$ and $\Omega^*(X)=B^n(X, 2)\times D$ are bounded Lipschitz domains in $\boldsymbol{R}^{n+m}$.

Unless otherwise specified, A will stand for a positive constant depending only on L, possibly changing from one occurrence to the next, even in the same string. If f and g are positive quantities such that $A^{-1}f\leqq g\leqq Af$, then we write $f\sim g$.

The boundary Harnack principle ([8; Theorem 1]) stated below is a useful tool.

LEMMA 1. *Let $P\in L$. Let u and v be positive harmonic functions on $\Omega^*(\pi(P))$ which vanish continuously on $\partial\Omega^*(\pi(P))\cap\partial L$. If $u(\pi_0(P))\leqq v(\pi_0(P))$, then $u\leqq Av$ on $\Omega(\pi(P))$; in particular $u(P)\leqq Av(P)$.*

Let $\omega(P, E)$ be the harmonic measure at P of $E\subset\partial L$ in L. We observe that $\omega(\pi_0(P), \partial L\setminus\partial\Omega^*(\pi(P)))\geqq A$. Hence if u is as in Lemma 1, then we apply the lemma with u and $v=A^{-1}u(\pi_0(P))\omega(\cdot, \partial L\setminus\partial\Omega^*(\pi(P)))$, and obtain

$$u(P)\leqq Au(\pi_0(P)). \tag{1}$$

We need the following Phragmén-Lindelöf principle.

LEMMA 2. *Let L' be a subdomain of L. If u is subharmonic in L', bounded above in L' and $u\leqq 0$ on $\partial L'$, i.e.,*

$$\limsup_{P\to Q} u(P)\leqq 0 \qquad \textit{for any } Q\in\partial L',$$

then $u\leqq 0$ in L'.

PROOF. Since D is bounded, we can find a constant b such that $L\subset\{P=(p_1, \cdots, p_{n+m})\in\boldsymbol{R}^{n+m}; p_{n+m}>b\}$. Let $Q_0=(0, \cdots, 0, b)\in\boldsymbol{R}^{n+m}$ and $\Gamma=\{P=(p_1, \cdots, p_{n+m})\in\boldsymbol{R}^{n+m}; p_{n+m}-b>-2^{-1}|P-Q_0|\}$ be a cone with vertex at Q_0. Let v be a positive harmonic function on Γ vanishing on $\partial\Gamma$. We observe that $v(P)=|P-Q_0|^\delta v((P-Q_0)/|P-Q_0|+Q_0)$ with $\delta>0$. From the Harnack inequality we have $v(P) \geqq A|P-Q_0|^\delta$ on L with A independent of P.

Let $P_1 \in L'$ and $\varepsilon>0$ be given. Take $R>0$ such that $P_1 \in L' \cap B(Q_0, R)$ and $AR^\delta \varepsilon \geqq \sup_{P \in L'} u(P)$. By using the maximum principle in $L' \cap B(Q_0, R)$, we have $u(P_1) \leqq \varepsilon v(P_1)$. Since $\varepsilon>0$ is arbitrary, $u(P_1) \leqq 0$.

Let $G(\cdot, \cdot)$ be the Green function for L. From the symmetry of L it follows that if $P, Q \in L_0$, then $G(P, Q)$ depends only on $|P-Q|=|\pi(P)-\pi(Q)|$. We put $g(t)=G(P, Q)$ if $P, Q \in L_0$ and $t=|P-Q|$. Obviously $g(t)$ is a positive continuous decreasing function of $t>0$. Let $A(\rho)=\sup_{t \geqq 2} g(t)/g(t+\rho)$ for $0 \leqq \rho \leqq 1$. On account of the Harnack inequality, we have

$$\lim_{\rho \to +0} A(\rho) = 1 . \tag{2}$$

Hereafter we put $A_1=A(1)$.

LEMMA 3. *Let j be a positive integer. If $|\pi(Q)-X_0| \leqq |\pi(Q)-\pi(P)|+j$ and $|\pi(Q)-\pi(P)| \geqq 8$, then*

$$\frac{G(P, Q)}{G(P_0, Q)} \leqq AA_1^j .$$

PROOF. First suppose that $|\pi(Q)-X_0|<2$, i.e., $Q \in \Omega^*(X_0)$. Take any $Q' \in L$ such that $|\pi(Q')-X_0|=4$. We observe that

$$|\pi(P)-\pi(Q')| \geqq |\pi(P)-\pi(Q)|-|\pi(Q)-X_0|-|X_0-\pi(Q')| > 2 ,$$

so that

$$G(\pi_0(P), \pi_0(Q')) \leqq g(2) \leqq A_1^2 g(4) = A_1^2 G(P_0, \pi_0(Q')) .$$

Since $G(\pi_0(P), \cdot)$ and $G(P_0, \cdot)$ are positive and harmonic on $\Omega^*(\pi(Q'))$ and vanish on ∂L, it follows from Lemma 1 that

$$G(\pi_0(P), Q') \leqq AA_1^2 G(P_0, Q') .$$

Since $|\pi(P)-X_0| \geqq |\pi(P)-\pi(Q)|-|\pi(Q)-X_0| \geqq 6$, it follows that $G(\pi_0(P), \cdot)$ is harmonic on $B^n(X_0, 4) \times D$, so that the maximum principle leads to

$$G(\pi_0(P), Q) \leqq AA_1^2 G(P_0, Q) .$$

Noting that $G(\cdot, Q)$ is positive and harmonic on $\Omega^*(\pi(P))$ and vanishes on ∂L, we obtain from (1) that

$$G(P, Q) \leqq AG(\pi_0(P), Q) \leqq AA_1^2 G(P_0, Q) \leqq (AA_1)A_1^j G(P_0, Q) .$$

Next suppose that $|\pi(Q)-X_0| \geqq 2$. Since $G(\cdot, \pi_0(Q))$ is positive harmonic in $\Omega^*(\pi(P))$ and vanishes on ∂L, we have from (1)

$$G(P, \pi_0(Q)) \leqq AG(\pi_0(P), \pi_0(Q)) = Ag(|\pi(P)-\pi(Q)|)$$
$$\leqq AA_1^j g(|X_0-\pi(Q)|) = AA_1^j G(P_0, \pi_0(Q)) .$$

Since $G(P, \cdot)$ and $G(P_0, \cdot)$ are positive and harmonic on $\Omega^*(\pi(Q))$ and vanishes on ∂L, we have from Lemma 1

$$G(P, Q) \leqq AA_1^j G(P_0, Q).$$

Thus the lemma follows.

Let $\alpha_0=(0, \cdots, 0, -1)\in S^{n-1}$ and $\{Q_j\}_j$ be a sequence of points which approach M_{α_0} in L. We observe that if $\pi(Q_j)=X_j=(x_1^j, \cdots, x_n^j)$, then

$$\lim_{j\to\infty} x_n^j = -\infty,$$

$$\lim_{j\to\infty}\left|\frac{x_i^j}{x_n^j}\right| = 0 \qquad \text{for} \quad i,\ 1\leqq i\leqq n-1.$$

Let $X=(x_1, \cdots, x_n)$ and $X'=(x_1', \cdots, x_n')$. By a simple calculation we obtain that

$$\lim_{j\to\infty}\{|X-X_j|-|X'-X_j|\}= x_n-x_n'. \tag{3}$$

Let $P, Q\in L$ and $(\pi(P))_n=(\pi(Q))_n$. From (2) and (3) we find an integer $j_1=j(\pi(P), \pi(Q), \{\pi(Q_j)\})$ such that if $j\geqq j_1$, then $Q_j\in U_-((\pi(P))_n-3)$ and

$$2^{-1}\leqq \frac{G(\pi_0(P), \pi_0(Q_j))}{G(\pi_0(Q), \pi_0(Q_j))}\leqq 2. \tag{4}$$

LEMMA 4. *Let $r\leqq s-3$. If $P, Q\in E(s)$ and $j\geqq j(\pi(P), \pi(Q), \{\pi(Q_j)\})$, then*

$$\frac{G(P, Q_j)}{G(Q, Q_j)}\sim\frac{\omega(P, \Delta(r))}{\omega(Q, \Delta(r))}.$$

PROOF. First we prove

$$\frac{G(P, \pi_0(Q_j))}{G(\pi_0(Q), \pi_0(Q_j))}\sim\frac{\omega(P, \Delta(r))}{\omega(\pi_0(Q), \Delta(r))}. \tag{5}$$

Let $u_j(\cdot)=G(\cdot, \pi_0(Q_j))/G(\pi_0(Q), \pi_0(Q_j))$ and $v_r(\cdot)=\omega(\cdot, \Delta(r))/\omega(\pi_0(Q), \Delta(r))$. We observe that u_j and v_r are positive and harmonic on $\Omega^*(\pi(P))$ and vanish on $\partial\Omega^*(\pi(P))\cap\partial L$. Since $\omega(\pi_0(P), \Delta(r))=\omega(\pi_0(Q), \Delta(r))$, it follows from (4) that

$$2^{-1}\leqq\frac{u_j(\pi_0(P))}{v_r(\pi_0(P))}=u_j(\pi_0(P))\leqq 2.$$

On account of Lemma 1, we have $u_j(P)\sim v_r(P)$, and hence (5).

Next we observe that $G(\cdot, \pi_0(Q_j))/G(P, \pi_0(Q_j))$ and $\omega(\cdot, \Delta(r))/\omega(P, \Delta(r))$ are positive and harmonic on $\Omega^*(\pi(Q))$ and vanish on $\partial\Omega^*(\pi(Q))\cap\partial L$. We infer from Lemma 1 and (5) that

$$\frac{G(P, \pi_0(Q_j))}{G(Q, \pi_0(Q_j))}\sim\frac{\omega(P, \Delta(r))}{\omega(Q, \Delta(r))}.$$

Finally we observe that $G(P, \cdot)$ and $G(Q, \cdot)$ are positive and harmonic on

$\Omega^*(\pi(Q_j))$ and vanish on ∂L. From Lemma 1 and the above estimate we obtain the lemma.

It is well known that $\{G(P, Q_j)/G(P_0, Q_j)\}_j$ has a subsequence which converges to a positive harmonic function h uniformly on every compact subset of L. Without loss of generality we may assume $G(\cdot, Q_j)/G(P_0, Q_j)\rightarrow h$. The function h is called a kernel function at M_{α_0} determined by $\{Q_j\}_j$.

LEMMA 5. *For each s, h is bounded on* $U_+(s)$.

PROOF. We observe from (3) that

$$\lim_{j\rightarrow\infty}\{|\pi(Q_j)-\pi(P)|-|\pi(Q_j)-X_0|\}=(\pi(P))_n,$$

so that if j is large, then

$$|\pi(Q_j)-X_0|\leqq|\pi(Q_j)-\pi(P)|-(\pi(P))_n+1.$$

Since $\lim_{j\rightarrow\infty}|\pi(Q_j)-\pi(P)|=\infty$, it follows from Lemma 3 that $h(P)\leqq AA_1^{\eta}$, where η is the least positive integer greater than $1-(\pi(P))_n$. Hence $h(P)$ is bounded on $U_+(s)$.

It follows from Lemma 4 that if $r\leqq s-3$, then

$$\frac{h(P)}{h(Q)}\sim\frac{\omega(P, \varDelta(r))}{\omega(Q, \varDelta(r))}\qquad\text{for } P, Q\in E(s).$$

On account of Lemmas 2 and 5, we have

$$\frac{h(P)}{h(Q)}\sim\frac{\omega(P, \varDelta(r))}{\omega(Q, \varDelta(r))}\qquad\text{for } P\in U_+(s) \text{ and } Q\in E(s),$$

and hence

$$\frac{h(P)}{h(Q)}\sim\frac{\omega(P, \varDelta(r))}{\omega(Q, \varDelta(r))}\qquad\text{for } P, Q\in U_+(s).$$

Note that $\omega(\cdot, \varDelta(r))$ vanishes on $\partial L\cap\partial U_+(s)$ and so does h. Since s is arbitrary, h vanishes on ∂L. Letting $s\rightarrow-\infty$, we have $r\rightarrow-\infty$ and

$$A^{-1}\limsup_{r\rightarrow-\infty}\frac{\omega(P, \varDelta(r))}{\omega(Q, \varDelta(r))}\leqq\frac{h(P)}{h(Q)}\leqq A\liminf_{r\rightarrow-\infty}\frac{\omega(P, \varDelta(r))}{\omega(Q, \varDelta(r))}$$

for all $P, Q\in L$. In particular

$$A^{-1}\limsup_{r\rightarrow-\infty}\frac{\omega(P, \varDelta(r))}{\omega(P_0, \varDelta(r))}\leqq h(P)\leqq A\liminf_{r\rightarrow-\infty}\frac{\omega(P, \varDelta(r))}{\omega(P_0, \varDelta(r))}.$$

Let $r_j\rightarrow-\infty$ and $\{\omega(\cdot, \varDelta(r_j))/\omega(P_0, \varDelta(r_j))\}_j$ converge to a positive harmonic function f on L. We have

$$h\sim f\qquad\text{on } L \tag{6}$$

Since the constant of comparison in (6) is independent of $\{Q_j\}_j$, we have

LEMMA 6. *Every kernel function h at* M_{α_0} *determined by* $\{Q_j\}_j$ *satisfies* (6) *and vanishes on* ∂L. *Accordingly, if* h' *is a kernel function at* M_{α_0} *determined by* $\{Q'_j\}_j$, *then* $h\sim h'$. *Furthermore for each s, h and f are constant on* $L_0\cap E(s)$, *i.e.*,

$$h(P)=h(Q) \quad \text{and} \quad f(P)=f(Q) \qquad \text{for } P, Q\in L_0\cap E(s). \tag{7}$$

PROOF. The proof of the last assertion remains. We note that if $P, Q\in L_0$ and $|P-Q_j|=|Q-Q_j|$, then $G(Q_j, P)=G(Q_j, Q)$. Hence it follows from the Harnack inequality and (3) that if $P, Q\in L_0\cap E(s)$, then

$$\lim_{j\to\infty}\frac{G(Q_j, Q)}{G(Q_j, P)}=1,$$

so that $h(P)=h(Q)$. We infer from the symmetry that $\omega(P, \Delta(r_j))=\omega(Q, \Delta(r_j))$ for $P, Q\in L_0\cap E(s)$ and have (7) for f.

Now we shall prove the uniqueness of kernel function in a way similar to [**5**; §3]. For a sequence $\{t_j\}_j$, $t_j\to-\infty$, and r_0, $0<r_0\leqq 4$, we put $M_j=((0, \cdots, 0, t_j), Y_0)$, $F_j=F(t_j, r_0)=\{X\in \boldsymbol{R}^n ; t_j-r_0<x_n<t_j+r_0\}\times B^m(Y_0, r_0)$, and $C_j=\bigcup_{k=j}^{\infty}F_k$. We recall $B^m(Y_0, 5)\subset D$ and obtain $\bar{F}_j\subset L$.

LEMMA 7. *Let* $0<r_0<1$ *and let* C_j *be as above. There is a positive constant A depending only on* r_0 *such that if h is a kernel function at* M_{α_0}, *then*

$$\lim_{j\to\infty}\hat{R}_h^{C_j}(P)\geqq Ah(P) \qquad \text{for } P\in L,$$

where $\hat{R}_h^{C_j}$ *is the regularized reduced function of h relative to* C_j (see [**3**; p. 49]).

PROOF. In this proof A depends on r_0. Let $h_j=\hat{R}_h^{C_j}$. The Harnack inequality and (7) yield that

$$h(P)\geqq Ah(M_j) \qquad \text{for } P\in F_j,$$

so that

$$h_j\geqq Ah(M_j)\hat{R}_1^{F_j} \qquad \text{on } L. \tag{8}$$

We observe that $\hat{R}_1^{F_j}$ is positive and harmonic on $L\setminus\bar{F}_j$, and that

$$\hat{R}_1^{F_j}(P)\geqq A \qquad \text{for } P\in F(t_j, 4).$$

Let $r\leqq t_j$, $u_j(\cdot)=\omega(\cdot, \Delta(r))/\omega(M_j, \Delta(r))$ and $v_j(\cdot)=\hat{R}_1^{F_j}(\cdot)$. Take $P\in E(t_j+3)$. We infer from the Harnack inequality and the symmetry of L that

$$u_j(\pi_0(P))\sim v_j(\pi_0(P))\sim 1.$$

Since u_j and v_j are positive and harmonic on $\Omega^*(\pi(P))$ and vanish on $\partial\Omega^*(\pi(P))\cap\partial L$, it follows from Lemma 1 that $u_j(P)\sim v_j(P)$. Since u_j and v_j are bounded on $U_+(t_j+3)$, Lemma 2 leads to

$$u_j(P) \sim v_j(P) \qquad \text{for } P \in U_+(t_j+3)\,.$$

On account of (8), we have

$$h_j(P) \geqq Ah(M_j)\frac{\omega(P_0,\, \Delta(r))\, \omega(P,\, \Delta(r))}{\omega(M_j,\, \Delta(r))\, \omega(P_0,\, \Delta(r))} \qquad \text{for } P \in U_+(t_j+3)\,.$$

Letting $r \to -\infty$, we have from Lemma 6

$$h_j(P) \geqq Ah(P) \qquad \text{for } P \in U_+(t_j+3)\,.$$

Letting $j \to \infty$, we obtain the lemma.

The next lemma is proved by Hunt and Wheeden [**5**; Lemma (3.4)].

LEMMA 8. *Suppose that v is harmonic in L, u is superharmonic in L, $0 \leqq v \leqq u$ and $E \subset L$. If $\hat{R}_u^E = u$, then $\hat{R}_v^E = v$.*

LEMMA 9. *Let h be a kernel function at M_{α_0} and $C = C_1$ be defined as before Lemma 7. Then*

$$\hat{R}_h^C = h \qquad \text{on } L\,.$$

PROOF. Let r_0, M_j, F_j, C_j and h_j be as in Lemma 7. In this proof A depends on r_0. We observe that h_j converges decreasingly to a harmonic function h_0. On account of Lemma 7 $h_0 \geqq Ah$ on L. Since $\hat{R}_h^C = h_1 = h$ on C, we infer that $\hat{R}_{h_1}^C = \hat{R}_h^C = h_1$. Applying Lemma 8 with $v = h_0$ and $u = h_1$, we have $\hat{R}_{h_0}^C = h_0$. Again using Lemma 8 with $v = Ah$ and $u = h_0$, we obtain

$$\hat{R}_h^C = h \qquad \text{on } L\,.$$

Thus the proof is complete.

LEMMA 10. *Let h be a kernel function at M_{α_0}. Suppose that v is a positive harmonic function on L and for each s*

(9) $$v(P) = v(Q) \qquad \text{for } P, Q \in L_0 \cap E(s)\,.$$

If $v(P_0) = 1$ and $v \sim h$ on L, then $v \equiv h$ on L.

PROOF. Let M_j be defined as before Lemma 7. We claim that

$$\lim_{j\to\infty} \frac{h(M_j)}{v(M_j)} = 1\,.$$

For suppose

$$\limsup_{j\to\infty} \frac{h(M_j)}{v(M_j)} > 1\,.$$

We can choose $\varepsilon > 0$ and subsequence $\{M_{ji}\}_i$ such that $h(M_{ji}) > (1+2\varepsilon)v(M_{ji})$ for all i. Using the Harnack inequality, (7) and (9), we obtain that if $r_0>0$ is small enough, then

$$h(P)>(1+\varepsilon)v(P) \qquad \text{for } P\in C'=\bigcup_{i=1}^{\infty}F(t_{ji}, r_0).$$

Since $v\sim h$ on L, we infer from Lemmas 8 and 9 with $E=C=C'$ that

$$h=\hat{R}_h^{C'}\geqq(1+\varepsilon)\hat{R}_v^{C'}=(1+\varepsilon)v;$$

in particular $1=h(P_0)\geqq(1+\varepsilon)v(P_0)=1+\varepsilon$. This is a contradiction. Changing h and v, we have $\liminf_{j\to\infty}h(M_j)/v(M_j)\geqq 1$. Thus $\lim_{j\to\infty}h(M_j)/v(M_j)=1$.

Given $\varepsilon>0$, there is an integer N such that

$$1-\frac{\varepsilon}{2}\leqq\frac{h(M_j)}{v(M_j)}\leqq 1+\frac{\varepsilon}{2} \qquad \text{for all } j\geqq N.$$

By the aid of the Harnack inequality, (7) and (9), we can find a constant r_0', $0<r_0'<1$, such that

$$(1-\varepsilon)h(P)\leqq v(P)\leqq(1+\varepsilon)h(P) \qquad \text{for } P\in C''=\bigcup_{j=1}^{\infty}F(t_j, r_0').$$

Since $v\sim h$ on L, we infer from Lemmas 8 and 9 with $E=C=C''$ that

$$(1-\varepsilon)h=(1-\varepsilon)\hat{R}_h^{C''}\leqq\hat{R}_v^{C''}=v\leqq(1+\varepsilon)\hat{R}_h^{C''}=(1+\varepsilon)h.$$

Since $\varepsilon>0$ is arbitrary, $v\equiv h$ on L.

We readily obtain from Lemmas 6 and 10

LEMMA 11. *There exists only one kernel function at M_{α_0}. Furthermore as $r\to-\infty$, $\omega(P, \Delta(r))/\omega(P_0, \Delta(r))$ converges to the kernel function h at M_{α_0}.*

Now we determine the form of h.

LEMMA 12. *There are a positive constant λ_D and a positive function $f_D(Y)$ on D vanishing on ∂D such that $f_D(Y_0)=1$ and*

$$h((X, Y))=f_D(Y)\exp(-\lambda_D x_n).$$

PROOF. Let $f_D(Y)=h((X_0, Y))$. Since $h(P_0)=1$, $h>0$ on L and h vanishes on ∂L, it follows that $f_D(Y_0)=1$, $f>0$ on D and f vanishes on ∂D. We infer from (7) that $h((X, Y))$ does not depend on $x_1, \cdots, x_{n-1}$. Put $P(s)=((0, \cdots, 0, s), Y_0)$ and $\phi(s)=h(P(s))$. Noting that

$$\frac{\omega(P(s+t), \Delta(r))}{\omega(P_0, \Delta(r))}=\frac{\omega(P(s), \Delta(r-t))}{\omega(P_0, \Delta(r-t))}\cdot\frac{\omega(P(t), \Delta(r))}{\omega(P_0, \Delta(r))},$$

we obtain from Lemma 11 that $\phi(s+t)=\phi(s)\phi(t)$. Since ϕ is continuous, there is a constant λ_D such that $\phi(s)=\exp(-\lambda_D s)$. Observe that

$$\frac{\omega((X, Y), \Delta(r))}{\omega(P_0, \Delta(r))} = \frac{\omega((X, Y), \Delta(r))}{\omega((X, Y_0), \Delta(r))} \cdot \frac{\omega((X, Y_0), \Delta(r))}{\omega(P_0, \Delta(r))}$$
$$= \frac{\omega((X_0, Y), \Delta(r-x_n))}{\omega(P_0, \Delta(r-x_n))} \cdot \frac{\omega(P(x_n), \Delta(r))}{\omega(P_0, \Delta(r))}.$$

Using Lemma 11 again, we have

$$h((X, Y)) = h((X_0, Y))\psi(x_n) = f_D(Y)\exp(-\lambda_D x_n).$$

It follows from Lemma 5 that h is bounded on $U_+(0)$, so that $\lambda_D \geqq 0$. If $\lambda_D=0$, then h is bounded on L, and hence $h=0$ by Lemma 2. This is a contradiction. Thus $\lambda_D>0$. The proof is complete.

From the symmetry of L, there exists exactly one kernel function $K(\cdot, M_\alpha)$ at M_α for every $\alpha \in S^{n-1}$, and $K(\cdot, M_\alpha)$ is of the form

$$K(P, M_\alpha) = f_D(Y)\exp\left(\lambda_D \sum_{i=1}^{n} \alpha_i x_i\right), \tag{10}$$

where $P=(X, Y)$, $X=(x_1, \cdots, x_n)$ and $\alpha=(\alpha_1, \cdots, \alpha_n)$. In view of (10), if $\alpha \neq \alpha'$, then $K(\cdot, M_\alpha) \neq K(\cdot, M_{\alpha'})$.

PROOF OF THEOREM 1. First note that L is dense in $\hat{L}$. Secondly let $Q \in \partial L$ and $\{M_j\}_j \subset L$ converge to Q. In the same way as in Hunt and Wheeden [**5**; §3] we can prove that $\{G(\cdot, M_j)/G(P_0, M_j)\}_j$ converges to a positive harmonic function $K(\cdot, Q)$. This together with Lemma 11 yields that $G(P, \cdot)/G(P_0, \cdot)$ has a continuous extension on $\hat{L}$.

Thirdly we see that $K(\cdot, Q)$ vanishes on $\partial L \setminus \{Q\}$ and is unbounded on $B(Q, r)$ for any $r>0$. Hence if $Q, Q' \in \partial L \cup \{M_\alpha; \alpha \in S^{n-1}\}$ and $Q \neq Q'$, then $K(\cdot, Q) \neq K(\cdot, Q')$, i.e. $\{K(P, \cdot); P \in L\}$ separates $\hat{L} \setminus L$. On account of [**3**; Theorems XIII, 1 and XIV, 1] or [**4**; pp. 240-243], the Martin compactification of L is homeomorphic to $\hat{L}$.

It is well known that there are two types of boundary points, minimal and not minimal ([**6**; p. 155], [**4**; p. 254]). We shall prove

THEOREM 2. *Every point on* $\partial L \cup \{M_\alpha; \alpha \in S^{n-1}\}$ *is a minimal boundary point.*

PROOF. It is easy to see that if $M \in \partial L$, then $K(\cdot, M)$ vanishes on $\partial L \setminus \{M\}$ and is bounded on $L \setminus B(M, r)$ for each $r>0$. We infer from the symmetry of L that

$$C(r) = \sup_{M \in \partial L} \sup_{P \in L \setminus B(M, r)} K(P, M)$$
$$= \sup_{M \in \{X_0\} \times \partial D} \sup_{P \in L \setminus B(M, r)} K(P, M) < \infty.$$

Take $Q=(X, Y) \in \partial L$. We shall prove that $K(\cdot, Q)$ is minimal. Suppose that u is a positive harmonic function on L such that $u \leqq K(\cdot, Q)$ on L. We have to prove $u=u(P_0)K(\cdot, Q)$. By the aid of [**6**; §3] or [**4**; Lemma 12.9], we can find measures μ on ∂L and ν on S^{n-1} such that

$$u = \int_{\partial L} K(\cdot, M)d\mu(M)+\int_{S^{n-1}} K(\cdot, M_\alpha)d\nu(\alpha),$$
$$\mu(\partial L)+\nu(S^{n-1}) = u(P_0). \tag{11}$$

We remark that μ and ν are not necessarily unique. Since $K(\cdot, Q)$ is bounded on $L \setminus B(Q, r)$ for any $r>0$, so is u. If $\nu \neq 0$, then there are a point $\alpha' \in S^{n-1}$ and ε, $0<\varepsilon<1$, such that $\nu(\{\alpha \in S^{n-1}; (\alpha, \alpha')>\varepsilon\})>0$, where (α, α') denotes the inner product of α and α'. We may assume that $E=\{\alpha \in S^{n-1}; \alpha_n<-\varepsilon\}$ has positive ν-measure. We obtain from (10) that if $t>0$, then

$$u((0, \cdots, 0, -t), Y_0) \geqq \int_{S^{n-1}} \exp(-\lambda_D \alpha_n t)d\nu(\alpha)$$
$$\geqq \int_E \exp(\lambda_D \varepsilon t)d\nu(\alpha) = \nu(E)\exp(\lambda_D \varepsilon t).$$

The last term tends to ∞ as $t \to \infty$, which contradicts the boundedness of u. Hence $\nu=0$ and $\mu(\partial L)=u(P_0)$. If $u \neq u(P_0)K(\cdot, Q)$, then there is $r>0$ such that $0<\mu(\partial L \setminus B(Q, 2r)) \leqq u(P_0)$. Let $u'=\int_{\partial L \setminus B(Q, 2r)} K(\cdot, M)d\mu(M)$. We see that u' vanishes on $\partial L \cap B(Q, r)$ and hence on ∂L. Note

$$\sup_{P \in L} u'(P) \leqq \max\{\sup_{P \in L \setminus B(Q, r)} K(P, Q), \sup_{P \in L \cap B(Q, r)} u'(P)\}$$
$$= \max\left\{C(r), \int_{\partial L \setminus B(Q, 2r)} C(r)d\mu(M)\right\} \leqq C(r).$$

Hence Lemma 2 leads to $u'=0$ on L. This is a contradiction. Therefore $u=u(P_0)K(\cdot, Q)$, so that $K(\cdot, Q)$ is minimal.

Now we shall prove that $K(\cdot, M_\alpha)$ is a minimal harmonic function for every $\alpha \in S^{n-1}$. From the symmetry it is sufficient to show that $h(\cdot)=K(\cdot, M_{\alpha_0})$ is minimal. Suppose that u is a positive harmonic function on L such that $u \leqq h$. We can find measures μ on ∂L and ν on S^{n-1} for which (11) holds.

Suppose that $\mu \neq 0$. Then there is $r>0$ such that $\mu(B^n(X_0, r)\times \partial D)>0$. Put

$$u'' = \int_{B^n(X_0, r)\times \partial D} K(\cdot, M)d\mu(M).$$

From Lemma 3 observe that u'' is bounded on $(\boldsymbol{R}^n \setminus B^n(X_0, r+8))\times D$, and hence on L. Since $u'' \leqq u \leqq h$, u'' vanishes on ∂L, so that Lemma 2 leads to $u'' \equiv 0$ on L. This is a contradiction. Hence $\mu=0$.

Next suppose $\nu(S^{n-1}\setminus\{\alpha_0\})>0$. Then there is a positive constant ε such that $\nu(F)>0$ with $F=\{\alpha=(\alpha_1, \cdots, \alpha_n)\in S^{n-1};\ \alpha_n\geqq\varepsilon-1\}$. Hence we obtain from (10) that if $t>0$, then

$$u((0, \cdots, 0, t), Y_0)=\int_{S^{n-1}}\exp(\lambda_D\alpha_n t)d\nu(\alpha)$$

$$\geqq\int_F\exp(\lambda_D(\varepsilon-1)t)d\nu(\alpha)=\nu(F)\exp(\lambda_D(\varepsilon-1)t).$$

Letting t tend to ∞, we have

$$1\geqq\frac{u((0, \cdots, 0, t), Y_0)}{h((0, \cdots, 0, t), Y_0)}\geqq\nu(F)\exp(\lambda_D\varepsilon t)\quad\rightarrow\quad\infty,$$

which is a contradiction. Thus the theorem follows.

§3. The Martin boundary of a semi-strip.

Let $L^+=\{X\in\boldsymbol{R}^n;\ x_1>0\}\times D$ be a Lipschitz semi-strip and let $\hat{L}^+=\bar{L}^+\cup\{M_\alpha;\ \alpha\in S^{n-1},\ \alpha_1\geqq0\}$ be a compact space with the relative topology induced from $\hat{L}$. We observe that $\hat{L}^+\setminus L^+$ consists of the Euclidean boundary of L^+ and $\{M_\alpha;\ \alpha\in S_+^{n-1}\cup S_0^{n-1}\}$, where $S_+^{n-1}=\{\alpha\in S^{n-1};\ \alpha_1>0\}$ and $S_0^{n-1}=\{\alpha\in S^{n-1};\ \alpha_1=0\}$. We shall show

THEOREM 3 (cf. [**6**; Example 3]). *The Martin compactification of L^+ is homeomorphic to $\hat{L}^+$ and every point on $\hat{L}^+\setminus L^+$ is a minimal boundary point.*

Let $\mathcal{G}$ be the Green function for L^+. For $P=(X, Y)\in\boldsymbol{R}^{n+m}$, we put $d(P)=|(\pi(P))_1|=|x_1|$ and $\bar{P}=((-x_1, x_2, \cdots, x_n), Y)$. From the symmetry we have

$$\mathcal{G}(P, Q)=G(P, Q)-G(P, \bar{Q})=G(P, Q)-G(\bar{P}, Q)\qquad\text{for } P, Q\in L^+. \tag{12}$$

By an elementary calculation we obtain that if $P, Q\in L^+$, then

$$|P-\bar{Q}|=|P-Q|\left\{1+\frac{4d(P)d(Q)}{|P-Q|^2}\right\}^{1/2}. \tag{13}$$

LEMMA 13. *Let $\alpha\in S_+^{n-1}$ and let $Q_j\in L^+$ tend to M_α. If $P\in L^+$, then*

$$\liminf_{j\to\infty}\frac{\mathcal{G}(P, Q_j)}{G(P, Q_j)}>0.$$

PROOF. First we assume that $P, Q_j\in L_0\cap L^+$. We observe

$$\lim_{j\to\infty}\frac{d(Q_j)}{|P-Q_j|}=\alpha_1>0.$$

Hence (13) yields that if j is large, then

$$|P-\bar{Q}_j| \geqq |P-Q_j|\left\{1+\frac{3\alpha_1 d(P)}{|P-Q_j|}\right\}^{1/2} \geqq |P-Q_j|+\alpha_1 d(P).$$

Since $g(t)$ is decreasing and $\lim_{r\to\infty} g(r+t)/g(r)=\exp(-\lambda_D t)$, it follows from (12) that

$$\mathcal{G}(P, Q_j) = g(|P-Q_j|)-g(|P-\bar{Q}_j|)$$
$$\geqq A'g(|P-Q_j|) = A'G(P, Q_j),$$

where A' depends only on α_1 and $d(P)$.

Now we assume that P and Q_j are general. We may assume that $|\pi(P)-\pi(Q_j)|\geqq 2$, $d(P)\geqq 2$ and $d(Q_j)\geqq 2$ by the Harnack principle. Then $\mathcal{G}(\pi_0(P), \cdot)$ and $G(\pi_0(P), \cdot)$ are positive and harmonic on $\Omega^*(\pi(Q_j))$ and vanish on $\partial\Omega^*(\pi(Q_j))\cap\partial L$. From Lemma 1 and the first case we have

$$\mathcal{G}(\pi_0(P), Q_j) \geqq A''G(\pi_0(P), Q_j).$$

Again applying Lemma 1 to $\mathcal{G}(\cdot, Q_j)$ and $G(\cdot, Q_j)$, we obtain

$$\mathcal{G}(P, Q_j) \geqq A''G(P, Q_j).$$

The lemma follows.

Let $P_1=((1, 0, \cdots, 0), Y_0)\in L_0\cap L^+$. On account of Lemma 13 and [7; Théorème 13], we have

LEMMA 14. *Let M_α and Q_j be as in Lemma 13. Then $\{\mathcal{G}(\cdot, Q_j)/\mathcal{G}(P_1, Q_j)\}_j$ is convergent.*

In order to consider the behavior of $\mathcal{G}(\cdot, Q_j)/\mathcal{G}(P_1, Q_j)$ as Q_j tends to M_α, $\alpha\in S_0^{n-1}$, we need an estimate of the Green function G for L.

LEMMA 15. *Let $\varepsilon>0$ and $Y\in D$. There are constants t_1, $0<t_1<1$, and $r_1>0$ such that if $Q\in L$, $r\geqq r_1$ and $0\leqq t\leqq t_1$, then*

$$(1-\varepsilon)\lambda_D t G(P, Q) \leqq G(P, Q)-G(P', Q) \leqq (1+\varepsilon)\lambda_D t G(P, Q)$$

for $P\in\partial B^n(\pi(Q), r)\times\{Y\}$ and $P'\in\partial B^n(\pi(Q), r+t)\times\{Y\}$.

PROOF. From the symmetry we may assume that $Q=Q_r=((0, \cdots, 0, -r), Y')$, $P=(X_0, Y)$ and $P'=P_t=((0, \cdots, 0, t), Y)$. Since $G(\cdot, Q_r)/G(P, Q_r)=[G(\cdot, Q_r)/G(P_0, Q_r)]/[G(P, Q_r)/G(P_0, Q_r)]$ converges to $K(\cdot, M_{\alpha_0})/K(P, M_{\alpha_0})$ uniformly on every compact subset of L as $r\to\infty$, we infer from the Poisson integral that each derivative of $G(\cdot, Q_r)/G(P, Q_r)$ converges to that of $K(\cdot, M_{\alpha_0})/K(P, M_{\alpha_0})$ uniformly on every compact subset of L as $r\to\infty$. Letting $\varphi_r(t)=G(P_t, Q_r)/G(P, Q_r)$, we have from (10)

$$\lim_{r\to\infty}\varphi_r(t) = \exp(-\lambda_D t),$$

$$\lim_{r\to\infty}\varphi_r'(t)=-\lambda_D\exp(-\lambda_D t),$$

uniformly for t, $0\leqq t\leqq 1$. Hence for $\varepsilon>0$, there are $r_1>0$ and t_1, $0<t_1<1$, such that if $r\geqq r_1$ and $0\leqq t\leqq t_1$, then

$$-(1+\varepsilon)\lambda_D\leqq\varphi_r'(t)\leqq-(1-\varepsilon)\lambda_D,$$

so that

$$-(1+\varepsilon)\lambda_D t\leqq\varphi_r(t)-\varphi_r(0)=\int_0^t\varphi_r'(\tau)d\tau\leqq-(1-\varepsilon)\lambda_D t.$$

Therefore

$$(1-\varepsilon)\lambda_D t\leqq 1-\frac{G(P_t,\,Q_r)}{G(P,\,Q_r)}\leqq(1+\varepsilon)\lambda_D t.$$

Multiplying each term by $G(P,\,Q_r)$, we have the lemma.

LEMMA 16. *Let $\alpha\in S_0^{n-1}$ and let $Q_j\in L^+$ tend to M_α. If $P=(X,\,Y)\in L^+$, then*

$$\lim_{j\to\infty}\frac{\mathcal{G}(P,\,Q_j)}{\mathcal{G}(P_1,\,Q_j)}=\frac{d(P)K(P,\,M_\alpha)}{K(P_1,\,M_\alpha)}$$

$$=f_D(Y)x_1\exp\Big(\lambda_D\sum_{i=2}^n\alpha_i x_i\Big)\qquad \textit{for } P\in L^+.$$

PROOF. Let $P\in L^+$. We observe that

$$\lim_{j\to\infty}\frac{d(Q_j)}{|P-Q_j|}=0.$$

Let $\varepsilon>0$ be given. It follows from (13) that if j is large, then

$$(2-\varepsilon)\frac{d(P)d(Q_j)}{|P-Q_j|}\leqq|\bar P-Q_j|-|P-Q_j|$$

$$\leqq(2+\varepsilon)\frac{d(P)d(Q_j)}{|P-Q_j|},$$

so that (12) and Lemma 15 with $Q=Q_j$ and $P'=\bar P$ yield

$$2(1-2\varepsilon)\lambda_D G(P,\,Q_j)\frac{d(P)d(Q_j)}{|P-Q_j|}\leqq\mathcal{G}(P,\,Q_j)$$

$$\leqq 2(1+2\varepsilon)\lambda_D G(P,\,Q_j)\frac{d(P)d(Q_j)}{|P-Q_j|}.$$

Hence

$$\frac{(1-2\varepsilon)G(P,\,Q_j)}{(1+2\varepsilon)G(P_1,\,Q_j)}\cdot d(P)\cdot\frac{|P_1-Q_j|}{|P-Q_j|}\leqq\frac{\mathcal{G}(P,\,Q_j)}{\mathcal{G}(P_1,\,Q_j)}$$

$$\leqq\frac{(1+2\varepsilon)G(P,\,Q_j)}{(1-2\varepsilon)G(P_1,\,Q_j)}\cdot d(P)\cdot\frac{|P_1-Q_j|}{|P-Q_j|}.$$

Letting $j\to\infty$, we have the lemma from the arbitrariness of ε.

PROOF OF THEOREM 3. On account of Lemmas 14 and 16, we observe that $\mathcal{G}(P,\cdot)/\mathcal{G}(P_0,\cdot)$ has a continuous extension $\mathcal{K}(P,\cdot)$ on $\hat{L}^+$. Further we have

$$(14)\qquad \lim_{j\to\infty}\frac{\mathcal{G}(P,Q_j)}{\mathcal{G}(P_1,Q_j)}=\frac{K(P,M_\alpha)-K(\bar{P},M_\alpha)}{K(P_1,M_\alpha)-K(\bar{P}_1,M_\alpha)}$$

$$=f_D(Y)\frac{\sinh(\lambda_D\alpha_1x_1)}{\sinh(\lambda_D\alpha_1)}\left(\exp\lambda_D\sum_{i=2}^n\alpha_ix_i\right)$$

if $Q_j\to M_\alpha$, $\alpha\in S_+^{n-1}$ and $P\in L^+$. It is easy to see that $\{\mathcal{K}(P,\cdot);\ P\in L^+\}$ separates $\hat{L}^+\setminus L^+$. Hence we obtain from [**3**; Theorems XIII, 1 and XIV, 1] or [**4**; pp. 240-243] that the Martin compactification of L^+ is homeomorphic to $\hat{L}^+$.

It follows from [**7**; Théorème 12] that every point of $\partial L\cup\{M_\alpha;\ \alpha\in S_+^{n-1}\}$ is minimal. We prove that M_α, $\alpha\in S_0^{n-1}$ is a minimal boundary point. From the symmetry we may assume that $\alpha=\alpha^*=(0,\cdots,0,1)\in S_0^{n-1}$. Let u be a positive harmonic function on L^+ such that $u\leqq\mathcal{K}(\cdot,M_{\alpha^*})$ on L^+. Since u vanishes on ∂L^+, we can, in the same way as in the proof of Theorem 2, find measures μ and ν on S_+^{n-1} and S_0^{n-1} such that

$$u=\int_{S_+^{n-1}}\mathcal{K}(\cdot,M_\alpha)d\mu(\alpha)+\int_{S_0^{n-1}}\mathcal{K}(\cdot,M_\alpha)d\nu(\alpha).$$

If $\mu\neq0$, then there is $\varepsilon>0$ such that $E=\{\alpha\in S_+^{n-1};\ \alpha_1>\varepsilon\}$ has positive μ measure. By Lemma 16 and (14) we have

$$1\geqq\frac{u(((t,0,\cdots,0),Y_0))}{\mathcal{K}(((t,0,\cdots,0),Y_0),M_{\alpha^*})}$$

$$\geqq\frac{[\sinh(\lambda_D\varepsilon t)/\sinh\lambda_D]}{t}\mu(E)\ \to\ \infty,$$

as $t\to\infty$, a contradiction. If $\nu(S_0^{n-1}\setminus\{\alpha^*\})>0$, then there is $\delta>0$ such that $F=\{\alpha\in S_0^{n-1};\ \alpha_n<1-\delta\}$ has positive ν measure. In the same way as above, we have

$$1\geqq\frac{u(((1,0,\cdots,0,-t),Y_0))}{\mathcal{K}(((1,0,\cdots,0,-t),Y_0),M_{\alpha^*})}$$

$$\geqq\frac{\exp(\lambda_D(\delta-1)t)}{\exp(-\lambda_Dt)}\nu(F)\ \to\ \infty,$$

as $t\to\infty$, a contradiction. Hence $u=\nu(\{\alpha^*\})\mathcal{K}(\cdot,M_{\alpha^*})$. Thus the theorem is completely proved.

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