J. Math. Soc. Japan
Vol. 38, No. 3, 1986

On removability of sets for holomorphic and harmonic functions

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

By Yoshihiro MIZUTA

(Received July 30, 1984)
(Revised Jan. 25, 1985)

1. Introduction.

Let W be an open set in the complex plane $\boldsymbol{C}$. For a function f on W, denote by $S(f)$ the set of all points at which f fails to admit a complex derivative; as noted in Kaufman [4], $S(f)$ is a Borel subset of W if f is a Borel measurable function on W.

We say that a function h on the interval $[0, \infty)$ is a measure function if $h(0)=0$, $h(r)>0$ for $r>0$, h is nondecreasing on $[0, \infty)$ and further

$$h(2r) \leqq \text{const. } h(r) \qquad \text{for } r>0$$

(cf. Carleson [2]). We denote by Λ_h the Hausdorff measure associated with the measure function h, which is defined by

$$\Lambda_h(E) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_{j=1}^{\infty} h(r_j) \ ; \ r_j \leqq \delta, \ \bigcup_{j=1}^{\infty} B(z_j, r_j) \supset E \right\}$$

for a set E, where $B(z, r)$ denotes the open disc with center at z and radius r. If $h(r)=r^{\alpha}$, $\alpha>0$, then we shall write Λ_{α} for Λ_h.

Let $1 \leqq p \leqq \infty$ and $1/p+1/p'=1$. For a measure function h and a locally integrable (Borel) function f on W, define

$$F(z) = \sup_{B} r^{-1-2/p} h(r)^{-1/p'} \inf_{g} \int_{B} | f(w)-g(w) | \, d\Lambda_2(w) ,$$

where the supremum is taken over all open discs B with radius r such that $z \in B \subset W$ and the infimum is taken over all functions g which is holomorphic in B.

Our first aim is to establish the following result.

THEOREM 1. *Suppose* $F \in L^p(W)$.

(i) *If* $p<\infty$, $\lim_{r \downarrow 0} r^{-2} h(r)=\infty$ *and* $\Lambda_h(S(f))<\infty$, *then* f *can be corrected on a set of measure zero to be holomorphic in* W.

(ii) *If* $p=1$ *and* $\Lambda_2(S(f))=0$ *or if* $p>1$ *and* $\Lambda_h(S(f))=0$, *then the same conclusion as above holds.*

This result gives a generalization of Kaufman [4], in which the case $p=\infty$ and $h(r)=r$ was dealt with.

We next extend Theorem 1 to the higher dimensional case. We are concerned with harmonic, or more generally subharmonic, functions in the n-dimensional euclidean space $\boldsymbol{R}^n$. Let U be an open set in $\boldsymbol{R}^n$. For a locally integrable function f on U, we define

$$F(x) = \sup_B r^{-2-n/p} h(r)^{-1/p'} \inf_v \int_B |f(y)-v(y)|dy,$$

where the supremum is taken over all open balls B with radius r such that $x \in B \subset U$ and the infimum is taken over all functions v which is subharmonic in B. Denote by $S^*(f)$ the set of all points x such that

$$\limsup_{r \downarrow 0} r^{-n-2} \int_{B(x,r)} |f(y)-v(y)|dy > 0$$

for any function v which is subharmonic in a neighborhood of x, where $B(x, r)$ denotes the open ball with center at x and radius r. As before, let Λ_h denote the Hausdorff measure associated with a measure function h.

THEOREM 2. *Suppose* $F \in L^p(U)$.

(i) *If* $p<\infty$, $\lim_{r\downarrow 0} r^{-n}h(r)=\infty$ *and* $\Lambda_h(S^*(f))<\infty$, *then* f *can be corrected on a set of measure zero to be subharmonic in* U.

(ii) *If* $p=1$ *and* $\Lambda_n(S^*(f))=0$ *or if* $p>1$ *and* $\Lambda_h(S^*(f))=0$, *then the same conclusion as above holds.*

The proofs of Theorems 1 and 2 can be carried out along the same lines as Kaufman [4] and Kaufman-Wu [5]; the proof of Theorem 2 will be omitted, since it is similar to the proof of Theorem 1.

2. Proof of Theorem 1.

For a proof of Theorem 1, we need the following lemma, which can be proved in a way similar to the proof of Harvey-Polking [3; Lemma 3.1].

LEMMA. *Let* $\{B(z_j, r_j)\}$ *be a finite collection of discs such that* $\{B(z_j, r_j/5)\}$ *is mutually disjoint. Then there exists a family* $\{\psi_j\} \subset C_0^\infty(\boldsymbol{C})$ *with the following properties:*

(a) $\psi_j = 0$ *outside* $B(z_j, 2r_j)$;

(b) $\psi_j \geqq 0$ *on* $\boldsymbol{C}$;

(c) $\sum_j \psi_j \leqq 1$ *on* $\boldsymbol{C}$;

(d) $\sum_j \psi_j = 1$ *on* $\bigcup_j B(z_j, r_j)$;

(e) $\left|\dfrac{\partial^{k+l}}{\partial x^k \partial y^l}\psi_j(z)\right| \leqq A_{k,l} r_j^{-k-l}$ *on* $\boldsymbol{C}$,

where $z=x+\sqrt{-1}y$ *and* $A_{k,l}$ *are positive constants independent of* j *and* z.

PROOF OF THEOREM 1. We shall prove (i) only, because (ii) can be proved similarly. Suppose $F\in L^p(W)$, $p<\infty$, $\lim_{r\downarrow 0} r^{-2}h(r)=\infty$ and $\Lambda_h(S(f))<\infty$.

Let $\varepsilon>0$ and $\Lambda_h(S(f))<M<\infty$. By the definition of Λ_h, there exists a countable covering $\{B(z_i, r_i)\}$ of $S(f)$ such that

$$\sum_i h(r_i) < M$$

and

$$\sum_i r_i^2 < \varepsilon$$

because $\lim_{r\downarrow 0} r^{-2}h(r)=\infty$. For each $z\in W-S(f)$, take $r(z)>0$ such that

$$|f(w)-f(z)-(w-z)f'(z)| \leqq \varepsilon r(z)$$

whenever $w\in B(z, 10r(z))$.

Let $\phi\in C_0^\infty(W)$ and denote the support of ϕ by K. Since $K\subset(\bigcup_i B(z_i, r_i))\cup(\bigcup_{z\in W-S(f)} B(z, r(z)))$, there exists a finite family $\{B_i\}\subset\{B(z_i, r_i)\}\cup\{B(z, r(z));\ z\in W-S(f)\}$ such that $\bigcup_i B_i\supset K$. Further we can find a subfamily $\{B_{i_j}\}$ of $\{B_i\}$ such that $\{B_{i_j}\}$ is mutually disjoint and $K\subset\bigcup_j B_{i_j}^*$, where $B_{i_j}^*$ is the open disc whose center is that of B_{i_j} and whose radius is 5 times that of B_{i_j}. We write $\{B_{i_j}\}=\{B(z_{j'}, r_{j'})\}\cup\{B(z_{j''}, r(z_{j''}))\}$ and assume that all $B_{i_j}^*$ are included in W. Now we take $\{\psi_j\}$ in the lemma for the collection of discs $\{B_{i_j}^*\}$. Since $\int g(w)(\partial/\partial\bar{w})(\psi_j\phi)(w)d\Lambda_2(w)=0$ for g holomorphic in a neighborhood of the support of $\psi_j\phi$, we have

$$\begin{aligned}
&\left|\int f(w)(\partial/\partial\bar{w})(\psi_j\phi)(w)d\Lambda_2(w)\right| \\
&\leqq A_1 r_{j'}^{2/p} h(r_{j'})^{1/p'} \inf_{w\in B(z_{j'}, r_{j'})} F(w) \\
&\leqq A_2 h(r_{j'})^{1/p'}\left\{\int_{B(z_{j'}, r_{j'})} F(w)^p d\Lambda_2(w)\right\}^{1/p}
\end{aligned}$$

for ψ_j vanishing outside $B(z_{j'}, 10r_{j'})$, where A_1 and A_2 are positive constants which may depend on ϕ. For ψ_j vanishing outside $B(z_{j''}, 10r(z_{j''}))$, the left hand side is dominated by $A_3\varepsilon r(z_{j''})^2$ with a positive constant A_3. Hence it follows from Hölder's inequality that

$$\left|\int f(w)(\partial/\partial\bar{w})\phi(w)dA_2(w)\right| = \left|\sum_j \int f(w)(\partial/\partial\bar{w})(\psi_j\phi)(w)dA_2(w)\right|$$

$$\leqq A_4\left\{M^{1/p'}\left(\int_{\cup B(z_{j'},r_{j'})} F(w)^p dA_2(w)\right)^{1/p} + \varepsilon\sum r(z_{j'})^2\right\}$$

for a positive constant A_4. This implies that

$$\int f(w)(\partial/\partial\bar{w})\phi(w)dA_2(w) = 0\,,$$

since $A_2(\bigcup B(z_{j'}, r_{j'}))=\sum r_{j'}^2<\varepsilon$. We see from Weyl's lemma that f is equal a.e. to a function holomorphic in W. Thus the proof is complete.

3. Remarks.

REMARK 1. The same conclusion as Theorem 1 remains true if we replace $S(f)$ by the set of all z such that

$$\limsup_{r\downarrow 0} r^{-3}\int_{B(z,r)} |f(w)-g(w)|dA_2(w) > 0$$

for any function g which is holomorphic at z.

REMARK 2. Let $\alpha>0$, $2/p-1<\alpha<1$ and f be equal in W to the potential $\int|z-\zeta|^{\alpha-2}g(\zeta)dA_2(\zeta)$, where g is a function in $L^p(\boldsymbol{C})$ such that $\int(1+|\zeta|)^{\alpha-2}\cdot|g(\zeta)|dA_2(\zeta)<\infty$. Then

$$\sup_B r^{-\alpha-2}\int_B |f(w)-A_{z,B}|dA_2(w) \leqq \text{const.}\, Mg(z),$$

where the supremum is taken over all open discs B with radius r such that $z\in B\subset W$, Mg denotes the usual Hardy-Littlewood maximal function of g and

$$A_{z,B} = \int_{C-B^*}|z-\zeta|^{\alpha-2}g(\zeta)dA_2(\zeta)\,,$$

B^* denoting the open disc whose center is that of B and whose radius is 2 times that of B. Hence, as a consequence of Theorem 1, if $A_{p'(\alpha+1-2/p)}(S(f))<\infty$, then f is equal a.e. to a function holomorphic in W.

REMARK 3. Let $\alpha>0$, $2-n/p'<\alpha<2$ and f be equal in an open set $U\subset\boldsymbol{R}^n$ to the potential $\int|x-y|^{\alpha-n}g(y)dy$, where g is a function in $L^p(\boldsymbol{R}^n)$ such that $\int(1+|y|)^{\alpha-n}|g(y)|dy<\infty$. By Theorem 2, if $A_{n-(2-\alpha)p'}(S^*(f))<\infty$, then f is equal a.e. to a function subharmonic in U. On the other hand, it can be proved that if $B_{2-\alpha,p'}(\overline{S^*(f)})=0$, then f is equal a.e. to a function subharmonic in U (cf. Adams-Polking [1]), where $B_{\beta,q}$ denotes the Bessel capacity of index (β, q) (see Meyers [6]) and $\bar{E}$ denotes the closure of a set $E\subset\boldsymbol{R}^n$.

References

[1] D.R. Adams and J.C. Polking, The equivalence of two definitions of capacity, Proc. Amer. Math. Soc., **37** (1973), 529-534.

[2] L. Carleson, Selected problems on exceptional sets, Van Nostrand, Princeton, 1967.

[3] R. Harvey and J.C. Polking, A notion of capacity which characterizes removable singularities, Trans. Amer. Math. Soc., **169** (1972), 183-195.

[4] R. Kaufman, Hausdorff measure, BMO, and analytic functions, Pacific J. Math., **102** (1982), 369-371.

[5] R. Kaufman and J.-M. G. Wu, Removable singularities for analytic and harmonic functions, Ark. Mat., **18** (1980), 107-116.

[6] N.G. Meyers, A theory of capacities for potentials in Lebesgue classes, Math. Scand., **26** (1970), 255-292.

Yoshihiro MIZUTA

Department of Mathematics
Faculty of Integrated Arts and Sciences
Hiroshima University
Hiroshima 730
Japan