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On the growth of meromorphic solutions of some higher order differential equations

Dedicated to Professor Y. Kusunoki on his 60th birthday

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1. Introduction.

About fifty years ago, Yosida ([**19**]) generalized a Malmquist's theorem ([8]) with the aid of the Nevanlinna theory of meromorphic functions.

THEOREM OF YOSIDA. *If the differential equation*

(1) $\qquad (w')^m = R(z, w),\qquad R$ *rational in* $z, w,$ *and* m *a positive integer,*

possesses a transcendental meromorphic solution $w=w(z)$ *in the complex plane, then* $R(z, w)$ *must be a polynomial in* w *of degree at most* $2m$. *Further, if* $w(z)$ *has only a finite number of poles, the degree is at most* m.

It is well-known that this is the starting-point of applying the Nevanlinna theory to the ordinary differential equation in the complex plane. Thereafter up to the present, there are many researches in this field (e. g. see the references in [**2**], [**18**]). Among them, there are many generalizations of this theorem ([**1**], [**3**], [**6**], [**7**], [**12**], [**13**], [**14**], [**16**]).

In this paper we shall consider a general differential equation and some higher order differential equations. We denote by $\mathcal{M}$ the set of meromorphic functions in the complex plane and $\mathcal{L}$ the set of $E\subset[0, \infty)$ for which $\operatorname{meas} E<\infty$. Further, the term "meromorphic" will mean meromorphic in the complex plane.

Let P be a polynomial of $w, w', \cdots, w^{(n)}$ $(n\geqq 1)$ with coefficients in $\mathcal{M}$:

$$P=P(z, w, w', \cdots, w^{(n)})=\sum_{\lambda\in I} c_\lambda(z) w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n},$$

where $c_\lambda\in\mathcal{M}$ and I is a finite set of multi-indices $\lambda=(i_0, i_1, \cdots, i_n)$ for which $c_\lambda\neq 0$ and $i_0, i_1, \cdots, i_n$ are non-negative integers, and let $A(z, w)$, $B(z, w)$ be polynomials in w with coefficients in $\mathcal{M}$ and mutually prime in $\mathcal{M}$:

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$$A(z, w)=\sum_{j=0}^{p} a_j(z)w^j, \qquad B(z, w)=\sum_{k=0}^{q} b_k(z)w^k,$$

where $a_j, b_k \in \mathcal{M}$ such that $a_p \cdot b_q \neq 0$.

As a generalization of (1) we shall consider the differential equation

$$P^m=\frac{A(z, w)}{B(z, w)} \qquad (m \geqq 1). \tag{2}$$

We put

$$\Delta=\max_{\lambda \in I}(i_0+2i_1+\cdots+(n+1)i_n),$$

$$d=\max_{\lambda \in I}(i_0+i_1+\cdots+i_n),$$

$$\Delta_o=\max_{\lambda \in I}(i_1+2i_2+\cdots+ni_n).$$

Applying Theorem in [**16**] to this equation, we obtain the following.

PROPOSITION. *Let $w=w(z)$ be a nonconstant meromorphic solution of* (2).
(I) *When $q \neq 0$ or $p>m\Delta$,*

$$\max(q, p-m\Delta)T(r, w) \leqq m\sum_{\lambda \in I} T(r, c_\lambda) + O\Big(\sum_{j=0}^{p} T(r, a_j)+\sum_{k=0}^{q} T(r, b_k)\Big)+S_o(r, w);$$

(II) *When $q \neq 0$ or $p>md$,*

$$\max(q, p-md)T(r, w) \leqq m\Delta_o \bar{N}(r, w)+m\sum_{\lambda \in I} T(r, c_\lambda) + O\Big(\sum_{j=0}^{p} T(r, a_j)+\sum_{k=0}^{q} T(r, b_k)\Big)+S_o(r, w).$$

For nonconstant $f \in \mathcal{M}$, we denote by $S_o(r, f)$ any quantity satisfying

$$S_o(r, f)=\begin{cases} O(1) \ (r \to \infty), & \text{when } f \text{ is rational;} \\ O(\log r) \ (r \to \infty), & \text{when } f \text{ is transcendental and of finite order;} \\ O(\log rT(r, f)) \ (r \to \infty,\ r \notin E \in \mathcal{L}), & \text{when } f \text{ is of infinite order.} \end{cases}$$

In this paper we shall consider the differential equation (2) when $q=0$ and $0 \leqq p \leqq m\Delta$. We shall estimate the Nevanlinna characteristic $T(r, w)$ of meromorphic solution $w=w(z)$ of the differential equation

$$P^m=\sum_{j=0}^{p} a_j w^j \qquad (0 \leqq p \leqq m\Delta,\ a_p \neq 0,\ a_j \in \mathcal{M}). \tag{3}$$

A meromorphic solution $w=w(z)$ is said to be *admissible* when it satisfies

$$T(r, f)=o(T(r, w)) \qquad (r \to \infty,\ r \notin E \in \mathcal{L})$$

for all coefficients $f=a_j$ and c_λ in (3).

It is assumed that the reader is familiar with the standard notation of the Nevanlinna theory (see [4], [5] or [10]).

2. Lemmas.

We shall give some lemmas for later use.

LEMMA 1. *For nonconstant $f\in\mathcal{M}$ such that $f^{(j)}\neq 0$,*

$$m(r, f^{(j+k)}/f^{(j)})=S_0(r, f) \qquad (j\geqq 0,\ k\geqq 1)$$

(see [4] or [10]).

LEMMA 2. *Let g_0 and g_1 be meromorphic and linearly independent over $\boldsymbol{C}$, and put*

$$g_0+g_1=\phi\,. \tag{4}$$

Then, we have

$$T(r, g_0)\leqq T(r, \phi)+\bar{N}(r, \phi)+\bar{N}'(r, g_1)+N(r, g_1'/g_1-g_0'/g_0)+S_0(r, g_0)+S_0(r, g_1),$$

where $\bar{N}'(r, g_1)$ is the $\bar{N}$-function of the poles of g_1 other than poles of ϕ (cf. [15]).

PROOF. From (3) and $g_0'+g_1'=\phi'$, we have

$$g_0=\frac{(\phi g_1'/g_1-\phi')}{(g_1'/g_1-g_0'/g_0)}\,,$$

from which we obtain, putting $D=g_1'/g_1-g_0'/g_0$,

$$\begin{aligned} m(r, g_0) &\leqq m(r, \phi g_1'/g_1-\phi')+m(r, 1/D)+O(1) \\ &\leqq m(r, \phi)+m(r, g_1'/g_1)+m(r, \phi'/\phi)+m(r, D)+N(r, D)-N(r, 1/D)+O(1) \end{aligned} \tag{5}$$

and

$$N(r, g_0)\leqq N(r, \phi)+\bar{N}(r, \phi)+\bar{N}'(r, g_1)+N(r, 1/D)\,. \tag{6}$$

Using the inequality

$$m(r, D)\leqq m(r, g_0'/g_0)+m(r, g_1'/g_1)+O(1)$$

and Lemma 1, from (5) and (6) we have the desired inequality:

$$T(r, g_0)\leqq T(r, \phi)+\bar{N}(r, \phi)+\bar{N}'(r, g_1)+N(r, D)+S_0(r, g_0)+S_0(r, g_1)\,.$$

LEMMA 3. *Let $f,\ a_0, \cdots, a_t\,(\neq 0)$ be meromorphic, then we have the following inequalities:*

(i) $$t\Big(m(r, f)-m(r, 1/a_t)-\sum_{j=0}^{t-1} m(r, a_j)\Big)+O(1)$$
$$\leqq m\Big(r, \sum_{j=0}^{t} a_j f^j\Big) \leqq tm(r, f)+\sum_{j=0}^{t} m(r, a_j)+O(1);$$

(ii) $$t\Big(T(r, f)-\sum_{j=0}^{t} T(r, a_j)\Big)+O(1)$$
$$\leqq T\Big(r, \sum_{j=0}^{t} a_j f^j\Big) \leqq tT(r, f)+\sum_{j=0}^{t} T(r, a_j)+O(1)$$

(see [**9**]).

LEMMA 4. *For nonconstant* $v\in \mathcal{M}$, *put*

$$\left(\frac{1}{v}\right)^{(n)} = \frac{H_n(v, v', \cdots, v^{(n)})}{v^{n+1}} \qquad (n\geqq 1).$$

Then,

(i) *H_n is a homogeneous polynomial of degree n in $v, v', \cdots, v^{(n)}$*

(ii) *For any term $cv^{i_0}(v')^{i_1}\cdots(v^{(n)})^{i_n}$ ($c\neq 0$, constant) of H_n,*

$$\nu i_0+(\nu+1)i_1+\cdots+(\nu+n)i_n = n(\nu+1) \qquad (\nu\geqq 1).$$

We can easily prove this lemma by induction.

3. Theorems — general case.

We rewrite (3) as follows as in [**15**], p. 241:

$$P^m = a_p(w+b)^p+\sum_{j=0}^{p-2} b_j w^j, \tag{7}$$

where $b=a_{p-1}/pa_p$, b_j is a rational function of a_j, a_{p-1} and a_p ($0\leqq j\leqq p-2$).

THEOREM 1. *Let $w=w(z)$ be any nonconstant meromorphic solution of* (7). *When $2\leqq p\leqq m-1$ and there is at least one j such that $b_j\neq 0$,*

$$T(r, w) \leqq K\Big\{\sum_{\lambda\in I} T(r, c_\lambda)+\sum_{j=0}^{p} T(r, a_j)\Big\}+\sum_{j=0}^{p} S_o(r, a_j)+S_o(r, w)$$

for some constant K.

PROOF. Let k be the largest number of j for which $b_j\neq 0$. Then, (7) becomes

$$P^m = a_p(w+b)^p+\sum_{j=0}^{k} b_j w^j \qquad (b_k\neq 0,\ 0\leqq k\leqq p-2). \tag{8}$$

Let $w=w(z)$ be any nonconstant meromorphic solution of (8) and put

$$g_0 = -a_p(w+b)^p, \qquad g_1 = P^m \qquad \text{and} \qquad \phi = \sum_{j=0}^{k} b_j w^j.$$

Case 1: $g_0=0$. In this case, $w=-b$ and we have

$$T(r, w)=T(r, b)\leqq T(r, a_{p})+T(r, a_{p-1})+O(1).$$

Case 2: $g_1=0$. In this case, we have from (3)

$$w^{p}=-a_{p}^{-1}\sum_{j=0}^{p-1}a_{j}w^{j},$$

from which we obtain by Lemma 3 (ii)

$$pT(r, w)\leqq(p-1)T(r, w)+\sum_{j=0}^{p}T(r, a_{j})+O(1),$$

that is,

$$T(r, w)\leqq\sum_{j=0}^{p}T(r, a_{j})+O(1).$$

Case 3: $\phi=0$. In this case, we have

$$w^{k}=-b_{k}^{-1}\sum_{j=0}^{k-1}b_{j}w^{j},$$

so that by Lemma 3 (ii) as in the case 2

$$T(r, w)\leqq\sum_{j=0}^{k}T(r, b_{j})+O(1)\leqq K_{1}\sum_{j=0}^{p}T(r, a_{j})+O(1)$$

as each b_j is rational in a_p, a_{p-1} and a_j, where K_1 is a constant.
Case 4: $g_0\cdot g_1\cdot\phi\neq 0$ and g_0, g_1 are linearly dependent. In this case, there are constants α, $\beta\in\boldsymbol{C}$ such that

$$\alpha g_{0}+\beta g_{1}=0\qquad(\alpha\cdot\beta\neq 0).$$

This means that

$$(\alpha/\beta-1)a_{p}(w+b)^{p}=\sum_{j=0}^{k}b_{j}w^{j}.$$

As $\phi\neq 0$, $\alpha/\beta\neq 1$. From this equality we have

$$\begin{aligned}T(r, w)&\leqq(p-k)^{-1}\Big(\sum_{j=0}^{k}T(r, b_{j})+pT(r, b)+T(r, a_{p})\Big)+O(1)\\&\leqq K_{2}\sum_{j=0}^{p}T(r, a_{j})+O(1)\end{aligned}$$

by Lemma 3 (ii), where K_2 is a constant.
Case 5: $\phi\neq 0$ and g_0, g_1 are linearly independent over $\boldsymbol{C}$. As $g_0+g_1=\phi$, we have by Lemma 2

$$T(r, g_{0})\leqq T(r,\phi)+\bar{N}(r,\phi)+\bar{N}'(r, g_{1})+N(r, g_{1}'/g_{1}-g_{0}'/g_{0})+S_{0}(r, g_{0})+S_{0}(r, g_{1}).\tag{9}$$

Here we estimate each term of (9).

$$pT(r, w)-pT(r, b)-T(r, a_{p})+O(1)\leqq T(r, g_{0});\tag{10}$$

$$T(r,\phi)\leqq kT(r, w)+\sum_{j=0}^{k}T(r, b_{j})+O(1);\tag{11}$$

(12) $\bar{N}(r, \phi) \leqq \bar{N}(r, w)+\sum_{j=0}^{k} \bar{N}(r, b_j)$;

(13) $\bar{N}'(r, g_1) \leqq \bar{N}(r, g_1)=\bar{N}(r, P) \leqq \bar{N}(r, w)+\sum_{\lambda \in I} \bar{N}(r, c_\lambda)$.

Let z_0 be a pole of w other than zeros or poles of a_j ($j=0, \cdots, p$), then it is easily seen that $g_1'/g_1-g_0'/g_0$ has no pole at z_0. This shows that the following inequality holds.

(14) $N(r, g_1'/g_1-g_0'/g_0) \leqq \bar{N}(r, 0, g_0)+\bar{N}(r, 0, g_1)+\sum_{j=0}^{p}(\bar{N}(r, 0, a_j)+\bar{N}(r, a_j))$;

(15) $\bar{N}(r, 0, g_0) \leqq \bar{N}(r, 0, w+b)+\bar{N}(r, 0, a_p) \leqq T(r, w)+T(r, b)+T(r, a_p)+O(1)$;

(16) $\bar{N}(r, 0, g_1)=\bar{N}(r, 0, P) \leqq T(r, P)+O(1) \leqq \frac{p}{m} T(r, w)+\frac{1}{m} \sum_{j=0}^{p} T(r, a_j)+O(1)$;

(17) $\bar{N}(r, w) \leqq \sum_{\lambda \in I}(\bar{N}(r, 0, c_\lambda)+\bar{N}(r, c_\lambda))+\sum_{j=0}^{p}(\bar{N}(r, 0, a_j)+\bar{N}(r, a_j))$.

This is because w has no poles other than zeros or poles of c_λ and a_j. In fact, if w has a pole of order ν at z_0 other than zeros or poles of c_λ and a_j, the right-hand side of (8) has a pole of order $p\nu$ at z_0 and P^m has a pole of order at least $m\nu$ at z_0. This is impossible as $m>p$.

(18) $S_0(r, g_0) \leqq S_0(r, w)+S_0(r, a_p)+S_0(r, b)$;

(19) $S_0(r, g_1)=S_0\left(r, \sum_{j=0}^{p} a_j w^j\right) \leqq S_0(r, w)+\sum_{j=0}^{p} S_0(r, a_j)$.

From (9)-(19), we have for a constant K_3

$$(p-(k+1+p/m))T(r, w) \leqq K_3\left(\sum_{\lambda \in I} T(r, c_\lambda)+\sum_{j=0}^{p} T(r, a_j)\right)+S_0(r, w)+\sum_{j=0}^{p} S_0(r, a_j).$$

As $p-(k+1+p/m)>0$, we have for a constant K_4

$$T(r, w) \leqq K_4\left(\sum_{\lambda \in I} T(r, c_\lambda)+\sum_{j=0}^{p} T(r, a_j)\right)+S_0(r, w)+\sum_{j=0}^{p} S_0(r, a_j).$$

Combining the cases $1 \sim 5$, we have the theorem.

COROLLARY. *When* $0 \leqq p \leqq m-1$, *the differential equation* (3) *has no admissible solutions except the following form:*

$$P^m=a(w+b)^p \qquad (a, b \in \mathcal{M}, a \neq 0).$$

It is uncertain whether the excluded differential equation has an admissible solution. We shall discuss some excluded differential equations in § 4.

EXAMPLE. Let $P=ww''-(w')^2$. Then the differential equation

$$P=1$$

has an admissible solution $w=\sinh z$.

Next we consider the differential equation when $p\geqq m$.

THEOREM 2. *Let $w=w(z)$ be any nonconstant meromorphic solution of the differential equation*

$$(20)\qquad P^{m}=aw^{m+k}+\sum_{j=0}^{s}a_{j}w^{j}\qquad(a\neq 0,\ a_{s}\neq 0,\ 0\leqq k\leqq(\Delta-1)m).$$

If $m+k-s-3-k/m>0$, then there exists a constant K for which

$$T(r, w)\leqq K\left\{\sum_{\lambda\in I}T(r, c_{\lambda})+\sum_{j=0}^{s}T(r, a_{j})+T(r, a)\right\}+S_{o}(r, w)+S_{o}(r, a)+\sum_{j=0}^{s}S_{o}(r, a_{j}),$$

PROOF. Putting

$$g_{0}=-aw^{m+k},\qquad g_{1}=P^{m}\qquad\text{and}\qquad\phi=\sum_{j=0}^{s}a_{j}w^{j},$$

we can prove this theorem similarly to the case of Theorem 1 except the followings.

(i) Case 1 does not occur in this case.

(ii) In stead of (12) and (13), we use

$$\bar{N}(r, \phi)+\bar{N}'(r, g_{1})\leqq\bar{N}(r, w)+\sum_{\lambda\in I}(\bar{N}(r, 0, c_{\lambda})+\bar{N}(r, c_{\lambda}))+\sum_{j=0}^{s}(\bar{N}(r, 0, a_{j})+\bar{N}(r, a_{j})).$$

(iii) In stead of (17), we use

$$\bar{N}(r, w)\leqq T(r, w)+O(1).$$

NOTE. This theorem is a generalization of Theorem 2 ([**17**]).

4. Theorems — special case.

It is difficult to estimate the growth of meromorphic solutions of the differential equation (3) excluded from Theorems 1 and 2 in the general case. Therefore, we shall study the growth of meromorphic solutions of (3) when $P=w^{(n)}$ $(n\geqq 2)$ in some excluded cases from Theorems 1 and 2. (See [**11**], [**15**], [**17**] when $n=1$.)

THEOREM 3. *Let $w=w(z)$ be a meromorphic solution of the differential equation*

$$(21)\qquad (w^{(n)})^{m}=a(w+b)^{p}\qquad(a,\ b\in\mathcal{M},\ a\neq 0\,;\ 1\leqq p\leqq m-1)$$

such that $w+b\neq 0$.

(I) *When $b^{(n)}\neq 0$, if m is not a divisor of jp $(2\leqq j\leqq n)$, for a constant K,*

$$T(r, w)\leqq K(T(r, a)+T(r, b))+S_{o}(r, a)+S_{o}(r, b)+S_{o}(r, w).$$

(II) *When* $b^{(n)}=0$, *if* $m-p$ *is not a divisor of* mn *and if* m *is not a divisor of* jp $(2\leqq j\leqq n-1)$,

$$T(r, w)\leqq \frac{1}{m-p}T(r, a)+T(r, b)+\frac{mn}{m-p}(\bar{N}(r, 0, a)+\bar{N}(r, a))$$
$$+S_0(r, b)+S_0(r, w).$$

PROOF. From (21), we have

$$(22)\qquad (w^{(n)})^m\left(\frac{Q_k}{a^k(w^{(n)})^k}\right)^p=p^{kp}a(w^{(k)}+b^{(k)})^p \qquad (k\geqq 1),$$

where

(23) Q_k is a polynomial in $w^{(n)}, \cdots, w^{(n+k)}, a, a', \cdots, a^{(k)}$, homogeneous of degree k with respect to $w^{(n)}, \cdots, w^{(n+k)}$ and $a, a', \cdots, a^{(k)}$ respectively.

(24) The orders of poles of $Q_k/a^{(k)}(w^{(n)})^k$ are at most k;

(25) $m(r, Q_k/a^k(w^{(n)})^k)=S_0(r, a)+S_0(r, w)$.

In fact, using

$$Q_1=maw^{(n+1)}-a'w^{(n)};$$
$$Q_{k+1}=((m-pk)aw^{(n+1)}-(pk+1)a'w^{(n)})Q_k+paw^{(n)}Q_k',$$

we can easily prove (23), (24) by induction and (25) by (23) and by Lemma 1.

(I) First we estimate $m(r, 1/w^{(n)})$. From (22) for $k=n$, we have

$$p^{-np}a^{-1}\left(\frac{Q_n}{a^n(w^{(n)})^n}\right)^p=\frac{(w^{(n)}+b^{(n)})^p}{(w^{(n)})^m}=\frac{(b^{(n)})^p}{(w^{(n)})^m}+\cdots+\frac{1}{(w^{(n)})^{m-p}},$$

from which we obtain by Lemma 3 (i) and (25)

$$m(m(r, 1/w^{(n)})-pm(r, 1/b^{(n)})-p^2m(r, b^{(n)}))\leqq m(r, 1/a)+S_0(r, a)+S_0(r, w),$$

that is,

$$(26)\qquad m(r, 1/w^{(n)})\leqq\frac{1}{m}m(r, 1/a)+pm(r, 1/b^{(n)})+p^2m(r, b^{(n)})+S_0(r, a)+S_0(r, w).$$

Next, we estimate $N(r, 1/w^{(n)})$. We use (22) for $k=n$:

$$(27)\qquad (w^{(n)})^m\left(\frac{Q_n}{a^n(w^{(n)})^n}\right)^p=p^{np}a(w^{(n)}+b^{(n)})^p.$$

We have only to estimate $N(r, 1/w^{(n)})$ when $w^{(n)}$ has at least one zero. Let z_0 be a zero of $w^{(n)}$ of order ν $(\geqq 1)$, μ $(\geqq 0)$ be the order of zero of $b^{(n)}$ at z_0 and s $(\geqq 0)$ that of a at z_0. If z_0 is a pole of $b^{(n)}$ or a, we consider that $b^{(n)}$ (or a) has a zero of order $-\mu$ (or $-s$) at z_0. Further, let t be the order of pole of $Q_n/a^n(w^{(n)})^n$ at z_0, then $t\leqq n$ by (24).

Case 1: $a(z_0)\neq 0$, ∞ and $b(z_0)\neq\infty$. At z_0, the right-hand side of (27) is finite, therefore the left-hand side of (27) is finite. The order of zero of the left-hand side of (27) at z_0 is $m\nu-tp$, which is not equal to zero by the hypothesis. As it has no pole at z_0, $m\nu-tp>0$, this shows that $\mu>0$ and from (27) we have

$$m\nu-tp=p\min(\nu,\mu)\leqq p\mu,$$

that is,

$$\nu\leqq\frac{pn}{m}+\frac{p}{m}\mu.$$

Case 2: $a(z_0)=0$ and $b(z_0)\neq\infty$. From (27), we obtain

$$m\nu-tp=s+p\min(\nu,\mu)\leqq s+p\mu,$$

so that

$$\nu\leqq\frac{pn}{m}+\frac{s}{m}+\frac{p}{m}\mu.$$

Case 3: $a(z_0)=\infty$ and $b(z_0)\neq\infty$. Similarly, we obtain

$$m\nu-tp=-s+p\min(\nu,\mu)\leqq -s+p\mu,$$

that is,

$$\nu\leqq\frac{pn}{m}-\frac{s}{m}+\frac{p}{m}\mu.$$

Case 4: $a(z_0)=0$ and $b(z_0)=\infty$. In this case,

$$m\nu-tp=s-p\mu,$$

that is,

$$\nu\leqq\frac{pn}{m}+\frac{s}{m}-\frac{p}{m}\mu.$$

Case 5: $a(z_0)=\infty$ and $b(z_0)=\infty$. In this case,

$$m\nu-tp=-s-p\mu,$$

that is,

$$\nu\leqq\frac{pn}{m}-\frac{s}{m}-\frac{p}{m}\mu.$$

From the cases 1~5, we obtain

$$N(r,1/w^{(n)})\leqq\frac{pn}{m}(\overline{N}(r,1/a)+\overline{N}(r,a)+\overline{N}(r,1/b^{(n)}))$$
$$+\frac{1}{m}N(r,1/a)+\frac{p}{m}N(r,1/b^{(n)}). \tag{28}$$

Using the inequality

$$T(r,b^{(n)})\leqq K_1T(r,b)+S_0(r,b)\qquad (K_1\text{: constant})$$

and from (26), (28) for a constant K_2,

(29) $$T(r, w^{(n)}) \leqq K_2(T(r, a)+T(r, b))+S_0(r, a)+S_0(r, b)+S_0(r, w).$$

Further from (21)

(30) $$pT(r, w)-T(r, a)-pT(r, b)+O(1) \leqq mT(r, w^{(n)}).$$

Combining (29) and (30), we have for a constant K

$$T(r, w) \leqq K(T(r, a)+T(r, b))+S_0(r, a)+S_0(r, b)+S_0(r, w).$$

$(\mathrm{II})_1$ First we consider the differential equation (21) when $b=0$:

(31) $$(w^{(n)})^m = aw^p \qquad (a\neq 0\in \mathcal{M},\ 1\leqq p\leqq m-1).$$

Let $w=w(z)$ be a non-zero meromorphic solution of (31). We estimate $m(r, 1/w)$. From (31)

$$w^{-(m-p)} = a^{-1}(w^{(n)}/w)^m$$

and by Lemma 1 we have

$$(m-p)m(r, 1/w) \leqq m(r, 1/a)+S_0(r, w),$$

that is,

(32) $$m(r, 1/w) \leqq \frac{1}{m-p}m(r, 1/a)+S_0(r, w).$$

Next we estimate $N(r, 1/w)$. We have only to do it when w has at least one zero. Let z_0 be a zero of w of order ν ($\geqq 1$).

(i) $a(z_0)\neq 0, \infty$. From (31), $w^{(n)}(z_0)=0$ and let μ be the order.
Case 1: $1\leqq \nu\leqq n-1$. From (31), it must be $m\mu=p\nu$, which is impossible by the hypothesis that m is not a divisor of jp $(2\leqq j\leqq n-1)$ and $m>p$.
Case 2: $\nu\geqq n$. In this case, $\mu=\nu-n$ and from (31), $m(\nu-n)=p\nu$ or $\nu(m-p)=mn$, which is impossible by the hypothesis that $m-p$ is not a divisor of mn. That is, w has no zeros other than zeros or poles of a.

(ii) $a(z_0)=0$. Let s be the order of zero of a at z_0. Put $v=1/w$ and using Lemma 4, we obtain from (31)

(33) $$(H_n(v, v', \cdots, v^{(n)}))^m = av^{(n+1)m-p}, \qquad \text{where } H_n=v^{n+1}(1/v)^{(n)}.$$

As w has a zero of order ν at z_0, v has a pole of order ν at z_0. By Lemma 4(ii), H_n has a pole of order at most $n(\nu+1)$ at z_0. From (33), we have

$$mn(\nu+1) \geqq ((n+1)m-p)\nu-s,$$

that is,

$$\nu \leqq \frac{mn}{m-p}+\frac{s}{m-p}.$$

(iii) $a(z_0)=\infty$. Let t be the order of pole of a at z_0. As in the case (ii), from (33), we have

$$mn(\nu+1) \geqq (m(n+1)-p)\nu+t,$$

that is,

$$\nu \leqq \frac{mn}{m-p} - \frac{t}{m-p}.$$

From (i), (ii) and (iii), we obtain

$$N(r, 1/w) \leqq \frac{mn}{m-p}(\bar{N}(r, 1/a)+\bar{N}(r, a))+\frac{1}{m-p}N(r, 1/a). \tag{34}$$

Combining (32) and (34), we have the inequality

$$T(r, w) \leqq \frac{1}{m-p}T(r, a)+\frac{mn}{m-p}(\bar{N}(r, 1/a)+\bar{N}(r, a))+S_0(r, w). \tag{35}$$

$(\mathrm{II})_2$ When $b \neq 0$, let $W=w+b$, then from (21) $(W^{(n)})^m=aW^p$ as $b^{(n)}=0$. Applying $(\mathrm{II})_1$ to this case, from (35) and the inequality

$$T(r, W) \geqq T(r, w)-T(r, b)+O(1),$$

we have the desired inequality.

EXAMPLE. The differential equation

$$(w'')^m = (w+\alpha z^2+\beta z+\gamma)^p \qquad (1 \leqq p<m,\ (m, p)=1;\ \alpha, \beta, \gamma \in \boldsymbol{C})$$

has no transcendental meromorphic solution.

PROOF. (i) $\alpha \neq 0$. By Theorem 3 (I), we have only to prove this example when $m=2$, $p=1$. Suppose that this equation has a transcendental meromorphic solution $w=w(z)$. Put $u=w+\alpha z^2+\beta z+\gamma$, then $u''=u^{1/2}+2\alpha$. As $u' \neq 0$, the relation $u'u''=u'u^{1/2}+2\alpha u'$ reduces to $(u')^2=4u^{3/2}/3+4\alpha u+c$ (c: any constant). Put $u^{1/2}=v$, then v is transcendental meromorphic and satisfies $v^2(v')^2=v^3/3+\alpha v^2+c/4$. When $c \neq 0$, v cannot be transcendental by Theorem of Yosida in § 1, and when $c=0$, v must be $(3^{-1/2}z+c_0)^2-3\alpha$ (c_0: any constant), which is not transcendental. This is a contradiction.

(ii) $\alpha=0$. By Theorem 3 (II), we have only to prove this example when $(m-p)|2m$. As in the case (i), we can also prove that this equation has no transcendental meromorphic solution in this case.

According to Gackstatter and Laine ([**3**], Satz 8), the differential equation

$$(w^{(n)})^m = \sum_{j=0}^{m+k} a_j w^j \qquad (1 \leqq k \leqq mn,\ a_j \in \mathcal{M},\ a_{m+k} \neq 0) \tag{36}$$

has no admissible solution when k is not a divisor of mn. We shall generalize this as follows.

THEOREM 4. *Let $w=w(z)$ be a nonconstant meromorphic solution of* (36). *When k is not a divisor of mn, for a constant K*

$$T(r, w) \leqq K\sum_{j=0}^{m+k} T(r, a_j)+S_o(r, w).$$

PROOF. When $n=1$, we proved this theorem in [**17**] (Theorem 1). For $n\geqq 2$, as w has no poles other than zeros or poles of a_j, we can prove this as in the case of $n=1$.

As to Theorem 2 for $k=0$, we can prove the following

THEOREM 5. *Let $w=w(z)$ be a nonconstant meromorphic solution of the differential equation*

$$(w^{(n)})^m = aw^m+\sum_{j=0}^{s} a_j w^j \qquad (a\neq 0,\ a_s\neq 0).$$

Then, if $s\leqq m-3$, for a constant K

$$T(r, w) \leqq K\Big(\sum_{j=0}^{s} T(r, a_j)+T(r, a)\Big)+S_o(r, w).$$

PROOF. When $n=1$, we proved this theorem in [**17**] (Theorem 2). For $n\geqq 2$, contrary to Theorem 2, as w has no poles other than zeros or poles of a_j, a, we can prove this as in the case of $n=1$.

NOTE. The condition "$s\leqq m-3$" is sharp as the following example shows.

EXAMPLE. The differential equation

$$(w''')^2 = w^2+1$$

has an admissible solution $w=\sinh z$.

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