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A recursive calculation of the Arf invariant of a link

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The Arf invariant of a knot was introduced in [**11**], and it can be calculated from its Alexander polynomial or its Conway polynomial [**6**]. The Arf invariant of a proper link (a link L is proper if $\mathrm{lk}(K, L-K)$ is even for every component K in L, where lk means a linking number) is defined to be that of a knot which is related to it (a knot K is related to a link L if there is a smoothly and properly embedded disk with holes D in $\boldsymbol{R}^3\times[0, 1]$ with $D\cap\boldsymbol{R}^3\times\{0\}=K$ and $D\cap\boldsymbol{R}^3\times\{1\}=-L$ [**11**]). K. Murasugi found a relation between the Arf invariants and the Alexander polynomials of two-component links [**10**]. The author showed in [**9**] that for some classes of proper links the Arf invariants can be expressed in terms of their Conway polynomials. See also [**3**].

In this paper we consider $V_L(i)$, where $V_L(t)$ is V. F. R. Jones' trace invariant [**5**] and $i=\sqrt{-1}$. He proposed there that one is allowed to define *an* Arf invariant of L as $V_L(i)$, and here we show that

$$V_L(i)=\begin{cases} (\sqrt{2})^{\#(L)-1}, & \text{if } L \text{ is proper and } \mathrm{Arf}(L)=0, \\ -(\sqrt{2})^{\#(L)-1}, & \text{if } L \text{ is proper and } \mathrm{Arf}(L)=1, \text{ and} \\ 0, & \text{if } L \text{ is not proper,} \end{cases}$$

where $\#(L)$ is the number of components in L, $\mathrm{Arf}(L)$ is the Arf invariant of L, and $\sqrt{i}$ is chosen to be $e^{(5/8)\cdot 2\pi i}$ in $V_L(i)$. This gives an answer to the Problem 12 in [**2**].

Using a recursive definition of $V_L(t)$ introduced by several people ([**4**], [**8**]), we can calculate the Arf invariant of any proper link recursively as follows.

DEFINITION. For any oriented link L, a numerical link type invariant $I(L)$ is defined so that it satisfies the following two axioms.

(i) For the trivial knot O, $I(O)=1$, and

(ii) If three links L, L', and l are related as in Figure 1 (the other parts are identical), then

$$I(L)+I(L')=\sqrt{2}\cdot I(l).$$

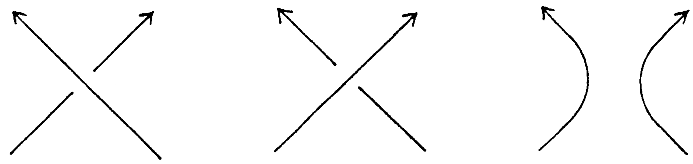

Figure 1.

REMARK. $V_L(t)$ is defined so that $1/t \cdot V_L(t) - t \cdot V_{L'}(t) = (\sqrt{t} - 1/\sqrt{t}) V_l(t)$ with $V_O(t)=1$. A simple calculation shows that $I(L)=V_L(i)$, and so the above definition is well-defined. For another proof of well-definedness see [4], [8].

Then we have

THEOREM.

$$I(L)=\begin{cases} (\sqrt{2})^{\#(L)-1}, & \textit{if } L \textit{ is proper and } \mathrm{Arf}(L)=0, \\ -(\sqrt{2})^{\#(L)-1}, & \textit{if } L \textit{ is proper and } \mathrm{Arf}(L)=1, \textit{ and} \\ 0, & \textit{if } L \textit{ is not proper.} \end{cases}$$

Before proving the theorem, we show the following.

LEMMA. *Suppose that L, L', and l are given as in Figure* 1 *and that* $\#(L)=\#(L')=\#(l)-1$. *If L and L' are proper and l is not proper, then* $\mathrm{Arf}(L)\neq\mathrm{Arf}(L')$.

PROOF. Let K, K', k_1, and k_2 be knots in L, L', or l as indicated in Figure 2.

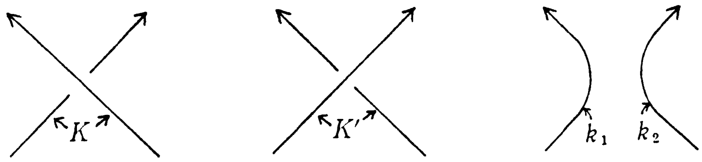

Figure 2.

Let $\bar{k}$ be a knot obtained from $l-k_1$ after a fusion, $\bar{l}$ be the resulting two-component link obtained from l, and $\bar{L}$ and $\bar{L}'$ be the corresponding knots obtained from L and L' respectively.

We will show that $\mathrm{lk}(k_1, \bar{k})$ is odd. Since L is proper, $0\equiv\mathrm{lk}(K, L-K)\equiv \mathrm{lk}(k_1, l-k_1)+\mathrm{lk}(k_2, l-k_2) \pmod 2$. Thus we have $\mathrm{lk}(k_1, \bar{k})\equiv 1 \pmod 2$ since otherwise $\mathrm{lk}(k_1, l-k_1)\equiv\mathrm{lk}(k_2, l-k_2)\equiv 0 \pmod 2$ and l cannot be non-proper.

Now it follows from Theorem 10.7 in [7] (see also Lemma 3.1 in [12]) that $\mathrm{Arf}(L)+\mathrm{Arf}(L')\equiv\mathrm{Arf}(\bar{L})+\mathrm{Arf}(\bar{L}')\equiv\mathrm{lk}(k_1, \bar{k})\equiv 1 \pmod 2$. Thus $\mathrm{Arf}(L)\neq\mathrm{Arf}(L')$, completing the proof.

PROOF OF THE THEOREM. If L, L', and l are given as in Figure 1, then we write $L=L'\oplus l$ and also $L'=L\oplus l$. Continuing this, we can write $L=L_1\oplus L_2\oplus\cdots\oplus L_m$ (here we omit parentheses), where L_j is a trivial link ($j=1, 2, \cdots, m$) [7]. We define $d(L)$ to be the minimum number of such m's ($d(L)\geqq 1$).

We will induct on $d(L)$. If $d(L)=1$, L is a trivial link.

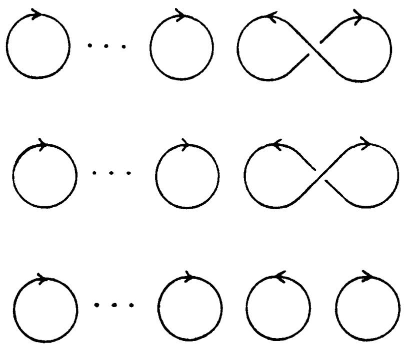

Figure 3.

Figure 3 and a simple induction will show that $I(L)=(\sqrt{2})^{\#(L)-1}$ in this case, while $\mathrm{Arf}(L)=0$.

Now suppose that the theorem is proved for every link L' with $d(L')<m$ and consider a link L with $d(L)=m$. We may assume that L, L', and l are as in Figure 1 and that $d(L')<m$ and $d(l)<m$. There are two cases.

Case I. Suppose that $\#(L)=\#(L')=\#(l)-1$.

(A) First assume that L is proper. Then L' is also proper. If l is proper, then $\mathrm{Arf}(L)=\mathrm{Arf}(L')=\mathrm{Arf}(l)$. So from the inductive hypothesis $I(L)=\sqrt{2}\cdot\pm(\sqrt{2})^{\#(l)-1}-(\pm(\sqrt{2})^{\#(L')-1})=\pm(\sqrt{2})^{\#(L)-1}$ according to whether $\mathrm{Arf}(L)$ is 0 or 1. If l is not proper, then from the above lemma $\mathrm{Arf}(L)\neq\mathrm{Arf}(L')$. So $I(L)=\mp(\sqrt{2})^{\#(L)-1}$ according to whether $\mathrm{Arf}(L)$ is 1 or 0.

(B) Next assume that L is non-proper. Then L' is also non-proper. It is easily shown that l is non-proper and so $I(L)=0$.

Case II. Suppose that $\#(L)=\#(L')=\#(l)+1$.

(A) Assume that L is proper. Then L' is non-proper and l is proper. Since $\mathrm{Arf}(L)=\mathrm{Arf}(l)$, $I(L)=\sqrt{2}\cdot\pm(\sqrt{2})^{\#(l)-1}=\pm(\sqrt{2})^{\#(L)-1}$ according to whether $\mathrm{Arf}(L)$ is 0 or 1.

(B) Assume that L is non-proper. If L' is proper, then l is proper and $\mathrm{Arf}(L')=\mathrm{Arf}(l)$. So $I(L)=\sqrt{2}\cdot I(l)-I(L')=0$. If L' is non-proper, then it is easily proved that l is also non-proper and so $I(L)=0$.

Now the proof is complete.

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