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Some arithmetic on Dedekind sums

Dedicated to Professor Tikao Tatsuzawa on his 70th birthday

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We shall consider the two problems on the arithmetical nature of the Dedekind sums. One is Rademacher's problem on the signs of the Dedekind sums for three consecutive Farey fractions, and the other is Salié's problem on the exceptional values of the Dedekind sums. For these purposes we shall first prepare some general lemmas with their own independent interest. Indeed some of them seem to give a new light on the reciprocity of the Dedekind sums and also on the multiplier theory of Dedekind's eta function.

1. Definition.

It is sometimes more convenient to deal with the function $D(h, k)$, following H. Salié and others, rather than $s(h, k)$ itself because of its integral valuedness. Our standpoint is this, so that we define the function $D(h, k)$ as follows. For each pair (h, k) of relatively prime integers with positive k, or equivalently, for each reduced fraction h/k $(k>0)$, we put

$$D(h, k)=12ks(h, k),$$

where $s(h, k)$ is the Dedekind sum, which is defined by

$$s(h, k)=\sum_{\mu (\mathrm{mod}\ k)}\left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

and $((x))=0$ or $x-[x]-1/2$ according as x is an integer or not. Besides we put for a convenience

$$D(h, 0)=2h \qquad \text{for} \quad h=\pm 1.$$

As we shall deal with the function $D(h, k)$ exclusively, we call it simply the Dedekind sum henceforth. We shall also understand that every pair (h, k) is always of relatively prime integers with non-negative k when it appears in the function $D(h, k)$ or as a reduced fraction h/k, which may happen to be $\pm 1/0$.

We can make reference to the monograph [4] by H. Rademacher and E. Grosswald for general facts and background of the Dedekind sums, and principally we need the following well-known formulas.

(1a) $$hD(h, k)+kD(k, h)=h^2+k^2-3hk+1,$$

(1b) $$D(h, k)=D(h', k) \quad \text{if} \quad h\equiv h' \pmod{k},$$

(2) $$D(-h, k)=-D(h, k),$$

(3) $$D(h, k)=D(\bar{h}, k) \quad \text{if} \quad h\bar{h}\equiv 1 \pmod{k}.$$

We should notice that the function $D(h, k)$ is completely determined by the properties (1) only. The formula (1a) will be often quoted as the reciprocity of the Dedekind sums.

A further investigation, especially on the values of $D(h, k)$, can be found in his very suggestive work [6] by H. Salié.

2. General lemmas.

We here give some lemmas about the fundamental properties of the Dedekind sums, not only for later use but also for their own considerable interest. The main and general results are Lemmas 5, 6, 7 and 8, but some particular cases of them, namely Lemmas 3, 4 and 9, will be frequently used.

Let Λ denote the semi-group consisting of all two by two matrices with non-negative integer entries and determinant one. If $\begin{pmatrix} h & H \\ k & K \end{pmatrix}$ is an element of Λ, then h/k and H/K are both non-negative reduced fractions, the former of which may happen to be $\infty=1/0$, and these $H/K<h/k$ are so-called adjacent Farey fractions, and vice-versa.

Lemma 1. *It holds*

(4a) $$D(h+k, k)=D(h, k),$$

(4b) $$D(h, h+k)=D(h, k)-D(k, h)+2k-2h.$$

Proof. Directly or easily derived from (1a) and (1b). In fact, the properties (1) and (4) are equivalent to each other, so that the function $D(h, k)$ is completely determined by (4) only, too. For instance we can deduce the reciprocity from (4) by mathematical induction.

Lemma 2. *If* $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is an element of* Λ *and* h/k *is a non-negative reduced fraction, then it holds*

$$D(ah+bk, ch+dk)$$
$$=dD(h, k)-cD(k, h)+hD(a, c)+kD(b, d)+2ck-2dh.$$

Proof. We can first see that the formula is nothing but (4) in the case

$\sigma=T_+=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ or $T_-=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$. Since the two elements T_+ and T_- generate the semi-group Λ, the general formula is obtained by mathematical induction method. In fact, we can easily deduce the formula of the case $T_+\sigma$ or $T_-\sigma$ from one of the case σ, where it must be noticed that the element $\begin{pmatrix}d & c\\ b & a\end{pmatrix}$ can be obtained by exchanging T_+ and T_- for each other in the word expression of $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ by them.

LEMMA 3. *If $H/K<h/k$ are both non-negative adjacent Farey fractions, then it holds*

$$D(H+h,\ K+k)=D(H,\ K)+D(h,\ k)+2k-2K.$$

PROOF. This is a special case of Lemma 2, namely, the case $h/k=1/1$, and $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is replaced anew by $\begin{pmatrix}h & H\\ k & K\end{pmatrix}$ for later convenience.

LEMMA 4. *If $h_1/k_1<h_2/k_2$ are both non-negative adjacent Farey fractions and n is a non-negative integer, then it holds*

$$D(h_1n+h_2,\ k_1n+k_2)=k_1n^2+(D(h_1,\ k_1)+2k_2-3k_1)n+D(h_2,\ k_2),$$

$$D(h_1+h_2n,\ k_1+k_2n)=-k_2n^2+(D(h_2,\ k_2)-2k_1+3k_2)n+D(h_1,\ k_1).$$

PROOF. This is also a special case of Lemma 2, namely, the case $h/k=1/n$ or $n/1$, and $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is replaced by $\begin{pmatrix}h_2 & h_1\\ k_2 & k_1\end{pmatrix}$. The needed formula $D(1,\ n)=n^2-3n+2$ is obvious by the reciprocity. Salié also obtained essentially the same formula by a different method ([**6**], p. 73, Satz 1′).

The nature of the formula in Lemma 2 can be much clarified if we use a matric expression. For each element $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of Λ let us put

$$D(\sigma)=\begin{pmatrix}D(a,\ c) & D(b,\ d)\\ D(c,\ a) & D(d,\ b)\end{pmatrix},$$

and

$$C(\sigma)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}(D(\sigma)\sigma^{-1}-2I),$$

where I denotes the identity matrix. Then we have

LEMMA 5. *If σ and τ are elements of Λ, then it holds*

$$C(\sigma\tau)=C(\sigma)+\sigma C(\tau)\sigma^{-1}.$$

PROOF. Let $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $\tau=\begin{pmatrix}h & H\\ k & K\end{pmatrix}$ and $\sigma\tau=\begin{pmatrix}h_1 & H_1\\ h_1 & K_1\end{pmatrix}$, then Lemma 2 says

$$D(h_1, k_1)$$
$$=(d, -c)\begin{pmatrix} D(h, k) \\ D(k, h) \end{pmatrix}+(D(a, c), D(b, d))\begin{pmatrix} h \\ k \end{pmatrix}-2(d, -c)\begin{pmatrix} h \\ k \end{pmatrix}.$$

At the same time it holds

$$D(k_1, h_1)$$
$$=(-b, a)\begin{pmatrix} D(h, k) \\ D(k, h) \end{pmatrix}+(D(c, a), D(d, b))\begin{pmatrix} h \\ k \end{pmatrix}-2(-b, a)\begin{pmatrix} h \\ k \end{pmatrix},$$

because $\begin{pmatrix} d & c \\ b & a \end{pmatrix}\begin{pmatrix} k \\ h \end{pmatrix}=\begin{pmatrix} k_1 \\ h_1 \end{pmatrix}$. Further if we replace h and k with H and K, respectively in these right-hand sides, then h_1 and k_1 are replaced by H_1 and K_1, respectively. Thus we have

$$D(\sigma\tau)={}^t\sigma^{-1}D(\tau)+D(\sigma)\tau-2\,{}^t\sigma^{-1}\tau,$$

which implies Lemma 5, since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}{}^t\sigma^{-1}=\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Lemma 6. *If σ and τ are elements of Λ, then it holds*

(5) $$v(\sigma\tau)=v(\sigma)+v(\tau),$$

where $v(\sigma)$ is the function on Λ defined by

(6) $$2v(\sigma)=dD(c, a)-cD(d, b)+bD(a, c)-aD(b, d)$$

for $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Lambda$.

Proof. Since $2v(\sigma)$ is nothing but the trace of the matrix $C(\sigma)$, the formula (5) is immediately derived from Lemma 5.

Lemma 7. *If $\sigma=\begin{pmatrix} h & H \\ k & K \end{pmatrix}$ is an element of Λ and $v(\sigma)=l$, then it holds*

(7a) $$H=lh+k-D(k, h),$$

(7b) $$K=(l+3)k-h+D(h, k),$$

(7c) $$h=(-l+3)H-K+D(K, H),$$

(7d) $$k=-lK+H-D(H, K).$$

Proof. By means of the reciprocity, $H_0=k-D(k, h)$ and $K_0=3k-h+D(h, k)$ satisfy the condition $K_0h-H_0k=1$. Hence we have $H=H_0+mh$ and $K=K_0+mk$ for some integer m, that is, $D(h, k)=h-(m+3)k+K$ and $D(k, h)=k+mh-H$. On the other hand, from the formulas (2) and (3) it follows that $D(K, k)=D(h, k)$ and $D(k, K)=-D(H, K)$, so that we have $KD(h, k)-kD(H, K)=K^2+k^2-3Kk+1$. Hence we get $kD(H, K)=k(H-k-mK)$, and so $D(H, K)=H-k-mK$, which is still valid when $k=0$. Similarly we have $D(K, H)=K-(3-m)H+h$. It follows therefore $2v(\sigma)=K(k+mh-H)-k(K-(3-m)H+h)+H(h-(m+3)k+K)-h(H-k-mK)=2(Kh-Hk)m=2m$ in view of the formula (6), so that $m=l$. This completes the proof.

LEMMA 8. *If σ is an element of Λ, then it holds*

$$C(\sigma)=v(\sigma)I+\partial A,$$

where $\partial A=\sigma A\sigma^{-1}-A$ and $A=\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$.

PROOF. It might sound strange that this lemma is equivalent to Lemma 7. Since $D(\sigma)=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}C(\sigma)\sigma+2\sigma$ by definition, it is sufficient to prove

$$D(\sigma)=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(v(\sigma)\sigma+\sigma A-A\sigma)+2\sigma,$$

or in other words

$$D(a, c)=-(l+3)c+a+d, \qquad D(b, d)=-ld+b-c,$$
$$D(c, a)=la-b+c, \qquad D(d, b)=(l-3)b+a+d,$$

where we put $l=v(\sigma)$. The latters are the same formulas as (7) of Lemma 7, which finishes the proof.

LEMMA 9. *Let h/k be a positive reduced fraction and let us put*

$$H=k-D(k, h), \qquad K=3k-h+D(h, k). \tag{8}$$

If H is positive, then $H/K<h/k$ are adjacent Farey fractions and it holds

$$D(H, K)=-D(k, h), \tag{9a}$$

$$D(K, H)=D(h, k)+3D(k, h). \tag{9b}$$

PROOF. This very useful lemma is a particular case of Lemma 7, namely, the case $l=0$. In fact, from our assumption it follows $\sigma=\begin{pmatrix} h & H \\ k & K \end{pmatrix}$ is an element of Λ and $v(\sigma)=0$. Hence we have $D(H, K)=H-k=-D(k, h)$ and $D(K, H)=K-3H+h=(K-3k+h)-3(H-k)=D(h, k)+3D(k, h)$. This completes the proof. The formula (9a) was already known by Salié ([**6**], p. 72, Satz 6).

It should be remarked that the function $v(\sigma)$ is uniquely determined by the property (5) and the conditions $v(T_+)=1$ and $v(T_-)=-1$. Hence $v(\sigma)$ is essentially the same as the addend of $\log\eta(z)$, and so the formula (6) can be regarded as a new description of this for the modular substitution $\sigma\in\Lambda$. Moreover, in view of Lemma 7, it is possible to extend the matric function $C(\sigma)$ on Λ to one on the group $SL_2(\boldsymbol{Z})$, and we can see that its coboundary coincides with $-12w(\sigma, \tau)I$, where $w(\sigma, \tau)$ is the well-known cocycle which defines the universal covering group of $SL_2(\boldsymbol{R})$. This viewpoint will give some improvement of our early discussion [**1**], but we do not touch on this topic.

3. Rademacher's pair.

H. Rademacher once posed the following problem ([**3**], p. 626): If $h_1/k_1 < h_2/k_2$ are adjacent Farey fractions and if the Dedekind sums $s(h_1, k_1)$ and $s(h_2, k_2)$ are both positive, is it necessarily true that $s(h_1+h_2, k_1+k_2)$ is non-negative? Unfortunately there are counter examples of this, e.g. $13/23<4/7$: $s(13, 23)=1/46$, $s(4, 7)=1/14$ and $s(17, 30)=-1/18$. In fact L. Pinzur ([**2**]) and K. H. Rosen ([**5**]) already answered to the question by giving some infinite class of such examples, independently. We here consider the same problem again but by different approach from theirs, and obtain some new and much clearer results, especially which will show that the problem is closely related to "$v(\sigma)=0$" problem.

If $h_1/k_1<h_2/k_2$ are adjacent Farey fractions of some order, the mediant $h/k=(h_1+h_2)/(k_1+k_2)$ gives rise to a new adjacent Farey sequence $h_1/k_1<h/k<h_2/k_2$ of higher order, and then we will say that the left (right, resp.) parent of h/k is h_1/k_1 (h_2/k_2, resp.). Every reduced fraction $0<h/k<1$ has its unique left parent and its unique right parent. For convenience' sake we call a pair of adjacent Farey fractions $H/K<h/k$ a Rademacher's pair if it satisfies the condition that $D(H, K)>0$, $D(h, k)>0$ and $D(H+h, K+k)<0$. In view of the property (1a) we may assume $0<H/K<h/k<1$ without loss of generality.

THEOREM 1. *For each reduced fraction $0<h/k<1$ with $D(h, k)>0$ there exists a unique reduced fraction H/K such that $H/K<h/k$ is a Rademacher's pair, unless $k^2\equiv-1 \pmod h$. If $k^2\equiv-1 \pmod h$, there are no such pairs.*

PROOF. It can be easily seen by Lemma 3 that $k<K$ if $H/K<h/k$ is a Rademacher's pair, so that h/k is necessarily the right parent of H/K. Any fraction whose right parent is h/k can be expressed as H_n/K_n with $H_n=h_0+nh$ and $K_n=k_0+nk$ by some positive integer n, where h_0/k_0 denotes the left parent of h/k. We first prove that there exists a positive integer n such that $D(H_n, K_n)=0$ if and only if $k^2\equiv-1 \pmod h$. Assume that $D(H_n, K_n)=0$, then $D(k, K_n)=0$ hence $k^2\equiv-1 \pmod{K_n}$ by the reciprocity. Therefore either fraction h/H_n or $((k^2+1)/K_n)/k$ is the right parent of k/K_n, so that they coincide with each other, thus we have $k^2+1=K_nh$. If $k^2\equiv-1 \pmod h$ conversely, then $k^2\equiv kh_0\equiv-1 \pmod h$ hence $k\equiv h_0 \pmod h$. This combined with the assumption $0<h/k<1$ gives a positive integer $n=(k-h_0)/h=(k^2+1-k_0h)/kh$, so that $D(H_n, K_n)=D(k, (k^2+1)/h)=0$ (cf. [**4**], p. 28).

Now we set

$$F(x)=-kx^2+(D(h,\ k)-2k_0+3k)x+D(h_0,\ k_0),$$

so that $D(H_n,\ K_n)=F(n)$ by Lemma 4. Suppose that $D(h,\ k)>0$, then we can see $F(1)>0$. In fact, $kF(1)=kD(h_0,\ k_0)+kD(h,\ k)+2k(k-k_0)=(k_0D(h,\ k)-k^2-k_0^2+3kk_0-1)+kD(h,\ k)+2k^2-2kk_0>k^2-k_0^2+kk_0-1>0$. Hence the equation $F(x)=0$ has a real solution β greater than 1, and β is not an integer unless $k^2\equiv-1 \pmod h$. Put $n=[\beta]\geqq 1$, then $D(H_n,\ K_n)=F(n)>0$ and $D(H_n+h,\ K_n+k)=F(n+1)<0$. Thus we have a Rademacher's pair $H_n/K_n<h/k$ unless $k^2\equiv-1 \pmod h$. Uniqueness of such n is obvious. On the contrary if $k^2\equiv-1 \pmod h$, then $D(H_n,\ K_n)=0$ for some n, which means that there are no Rademacher's pairs with the right fraction h/k. We have thus finished the proof of Theorem 1.

EXAMPLE. Suppose that a fraction $h/k=3/13$ is given. We see that $D(3,\ 13)=12>0$ and 3 does not divide 13^2+1. Since the left parent of 3/13 is 2/9 and $D(2,\ 9)=16$, we have $F(x)=-13x^2+33x+16$. So the integral part of the greater root of $F(x)=0$ is 2. Hence $H=2+2\cdot 3=8$ and $K=9+2\cdot 13=35$. Thus we obtain a Rademacher's pair $8/35<3/13$, and we can make certain that $D(3,\ 13)=12>0$, $D(8,\ 35)=30>0$ and $D(11,\ 48)=-2<0$.

THEOREM 2. *For each of almost all fractions h/k whose common left parent is a fixed fraction $0<h_0/k_0<1$, the Rademacher's pair $H/K<h/k$ is given by*

$$H=k-D(k,\ h),\qquad K=3k-h+D(h,\ k).$$

Further, at the same time it holds

$$h=3H-K+D(K,\ H),\qquad k=H-D(H,\ K).$$

PROOF. We first prove that the above defined pair $H/K<h/k$ is really a Rademacher's pair if both the conditions $D(h,\ k)>0$ and $D(k,\ h)<0$ are satisfied. By means of the reciprocity and Lemma 9, we can see that $H/K<h/k$ are both positive and adjacent Farey fractions and $D(H,\ K)=-D(k,\ h)>0$. In view of Lemma 3, it, further, follows that $D(H+h,\ K+k)=(H-k)+(K-3k+h)+2k-2K=(H-K)+(h-k)-k<0$. Next we have to show that almost all fractions h/k whose left parent is h_0/k_0 satisfy the conditions $D(h,\ k)>0$ and $D(k,\ h)<0$. If we denote the right parent of h_0/k_0 by h_∞/k_∞, we have $h=h_0n+h_\infty$ and $k=k_0n+k_\infty$ for some positive integer n. Since $h_0/k_0<h_\infty/k_\infty$ and $k_\infty/h_\infty<k_0/h_0$ are both adjacent Farey fractions, we can deduce from Lemma 4 that $D(h,\ k)\sim k_0n^2$ and $D(k,\ h)\sim -h_0n^2$ when $n\to\infty$, so that we have $D(h,\ k)>0$ and $D(k,\ h)<0$ with a finite number of possible exceptions of n or h/k. This completes the proof. In practice the evaluation can be done as follows. For given $0<h_0/k_0<1$, put first $h_\infty=h_0\left\{\dfrac{D(k_0,\ h_0)-k_0}{h_0}\right\}$ and $k_\infty=k_0\left\{\dfrac{h_0-D(h_0,\ k_0)}{k_0}\right\}$, where $\{x\}=x-[x]$ denotes the fractional part of x, so that h_∞/k_∞ is the right parent of h_0/k_0. Put next

$h=h_0n+h_\infty$ and $k=k_0n+k_\infty$. Then H and K are given by $H=h_0+(n+l)(h_0n+h_\infty)$ and $K=k_0+(n+l)(k_0n+k_\infty)$, respectively, where $l=(D(h_0, k_0)-h_0+k_\infty)/k_0$. The lower bound of n is automatically determined by the conditions $D(h, k)=K-3k+h>0$ and $D(H, K)=H-k>0$.

Example. Let us take $h_0/k_0=2/9$, and so $h_\infty/k_\infty=1/4$ and $l=(D(2, 9)-2+4)/9=2$. Hence we have a series of Rademacher's pairs: $H/K=(2n^2+5n+4)/(9n^2+22n+17)<h/k=(2n+1)/(9n+4)$, whose Dedekind sums are $D(h, k)=9n^2-3n+6$, $D(H, K)=2n(n-2)$ and $D(H+h, K+k)=-(n+4)(7n+5)$, so that $n\geqq 3$, necessarily. It should be noticed that a Rademacher's pair $8/35<3/13$ is not included in the above series, though the left parent of $3/13$ is $2/9$. These exceptional cases must be handled by Theorem 1.

4. Exceptional values.

As it was shown by H. Salié, it is known that

$$D(h, k)\equiv 0,\quad \pm 2 \quad \text{or} \quad \pm 6 \pmod{18}$$

for every reduced fraction h/k ([5], p. 75). Salié considered the converse problem whether W is a value of Dedekind sums for a given integer $W\equiv 0$, ± 2 or ± 6 (mod 18), and he observed as a computational phenomenon that ± 24, ± 34, ± 88 and ± 214 would be exceptional values, that is, the function $D(h, k)$ never takes these values. Now we can prove this as an application of Lemma 9, though the intrinsical nature of these exceptional values is still unclear.

Lemma 10. *If $2<|D(h, k)|\leqq k$ for a reduced fraction h/k, then there exists a reduced fraction h_1/k_1 such that $0<k_1<k$ and $|D(h_1, k_1)|=|D(h, k)|$.*

Proof. We may assume $0<h<k$, and by replacing $k-h$ by h if necessary, we can suppose that $D(h, k)=-W$ and $W>2$. By replacing again h by $\bar{h}$ if necessary, we may assume that $0<h\leqq \bar{h}<k$, where $\bar{h}$ is the minimal positive integer such that $h\bar{h}\equiv 1 \pmod k$. Now let us put $h_1=h-D(h, k)=h+W$ and $k_1=3h-k+D(k, h)$, so that h_1 and k_1 are positive. Hence we have $D(h_1, k_1)=-D(h, k)=W$ by means of Lemma 9, and $kk_1=hh_1+1=h^2+hW+1$. On the other hand it holds that $h+\bar{h}\equiv -W \pmod k$, since $h(D(h, k)-h)\equiv 1 \pmod k$ by the reciprocity. From this and our assumption $2<W\leqq k$ it follows that $h+\bar{h}=k-W$ or $2k-W$. Hence $2h+W\leqq h+\bar{h}+W\leqq 2k$. Therefore $4kk_1=4h^2+4hW+4<(2h+W)^2\leqq 4k^2$, and so we have $k_1<k$. This completes the proof.

The descending procedure of Lemma 10 can be repeated as far as the denominator k is not less than $|D(h, k)|$. Hence we have

Theorem 3. *If W ($|W|>2$) is a value of Dedekind sums, then it can be attained by a fraction with the denominator less than $|W|$, that is, there exist* *relatively prime integers h and k such that* $D(h, k)=W$ *and* $0<h<k<|W|$.

EXAMPLE. Let us start with the fact $D(31, 44)=-30$. Since $31\cdot 27\equiv 1 \pmod{44}$, we put $h=27$ and $k=44$. Then $h_1=h-D(h, k)=57$ and $k_1=3h-k+D(k, h)=35$, so that we have $D(57, 35)=D(22, 35)=30$. Thus $D(13, 35)=-30$. Since $13\cdot 27\equiv 1 \pmod{35}$, we put anew $h=13$ and $k=35$. In this case we have $h_1=43$ and $k_1=16$. In this way we obtain $D(43, 16)=D(11, 16)=30$ or $D(5, 16)=-30$. The denominator 16 is now less than 30. If we apply the descending once more, we can obtain $D(2, 11)=30$. On the other hand it can be known that the least denominator is 7, namely, $D(1, 7)=30$. In general, it seems difficult to find the least denominator for a given value of Dedekind sums.

In view of Theorem 3 it is now possible to know whether or not a given integer W is a value of Dedekind sums by evaluating the sums $D(h, k)$ for only a finite number of fractions h/k, namely, those which satisfy the conditions $0\leqq h/k<1$ and $1\leqq k\leqq |W|$. In practice even a half of the number of fractions suffices because $D(k-h, k)=-D(h, k)$. For instance the number of reduced fractions h/k satisfying the conditions $0\leqq h/k\leqq 1/2$ and $1\leqq k\leqq 23$ is 87 and we can easily make certain that the set of these 87 values of $D(h, k)$ does not contain the value 24. It is thus verified that the number 24 is certainly an exceptional value. By a calculation of 170,616 values of $D(h, k)$, that is, for reduced fractions h/k with the conditions $0\leqq h/k\leqq 1/2$ and $1\leqq k\leqq 1059$, we have proved the following

THEOREM 4. *The numbers* ± 24, ± 34, ± 88, ± 214, ± 304, ± 344, ± 394 *and* ± 1060 *are exceptional values of Dedekind sums.*

REMARK. After the preparation of this paper, some larger exceptional values have been determined: ± 1924, ± 2050, ± 3364, ± 4804, ± 9250, ± 17674, ± 21220, ± 25090, ± 25540 by Mr. Norimune Saito; ± 49930, ± 55780, ± 67714, ± 74500, ± 75274 by Dr. Chiaki Nagasaka. Furthermore Dr. Nagasaka has proposed a very exciting conjecture: for $W>344$, $\pm W$ is exceptional if and only if $W=2(n^2+1)$; $n\equiv\pm 4 \pmod 9$ and each odd prime factor of n is congruent to ± 1 modulo 8 ([7]).

Added after submission. Very recently a big progress has been made in the exceptional value problem. The author has succeeded in reforming Nagasaka's to a more general conjecture ([8]), and now it has been proved by Prof. Hiroshi Saito ([9]).

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