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On Z_p-extensions of real quadratic fields

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§ 0. Introduction.

Let k be a finite totally real extension of $\boldsymbol{Q}$, and p an odd prime number. Concerning the Greenberg's conjecture (cf. [2]) which states that Iwasawa invariants $\mu_p(k)$ and $\lambda_p(k)$ both vanish, we have obtained some results in the previous paper [1]. The purpose of this paper is to extend the results in our previous work.

For a finite algebraic number field K, we denote by h_K, C_K, and E_K the class number of K, the ideal class group of K, and the unit group of K, respectively. We denote also by $|X|$ the cardinality of a finite set X.

In the following, we assume that k is a real quadratic field and ε denotes the fundamental unit of k. Let p be an odd prime number which splits in $k/\boldsymbol{Q}$, and $\mathfrak{P}$ a prime of k lying above p. Take $\alpha \in k$ such that $\mathfrak{P}^{h_k}=(\alpha)$. We define n_1 (resp. n_2) to be the maximal integer such that $\alpha^{p-1}\equiv 1 \pmod{p^{n_1}\boldsymbol{Z}_p}$ (resp. $\varepsilon^{p-1}\equiv 1 \pmod{p^{n_2}\boldsymbol{Z}_p}$). Note that n_1 is uniquely determined under the condition $n_1 \leqq n_2$. For the cyclotomic $\boldsymbol{Z}_p$-extension

$$k=k_0\subset k_1\subset k_2\subset \cdots \subset k_n\subset \cdots \subset k_\infty,$$

let A_n be the p-primary part of the ideal class group of k_n, B_n the subgroup of A_n consisting of ideal classes which are invariant under the action of $\mathrm{Gal}(k_n/k)$, and D_n the subgroup of A_n consisting of ideal classes which contain a product of ideals lying over p. Let E_n be the unit group of k_n. For $m\geqq n\geqq 0$, $N_{m,n}$ denote the norm maps. We fix a topological generator σ of $G(k_\infty/k)$. Let ζ_p be a primitive p-th root of unity, and A_0^* the p-primary part of the ideal class group of $k(\zeta_p)$. Our main theorems are

THEOREM 1. *Let k be a real quadratic field and p an odd prime number which splits in $k/\boldsymbol{Q}$. Assume that*

(1) $n_1=1$, *and*

(2) $A_0=D_0$.

Then, for $n\geqq n_2-1$, we have $|A_n|=|D_n|=|D_0|\cdot p^{n_2-1}$.

Concerning the Iwasawa invariants $\mu_p(k)$, $\lambda_p(k)$ and $\nu_p(k)$, we obtain the next corollary.

COROLLARY. *Under the same assumption as Theorem* 1, *we have* $\mu_p(k)=\lambda_p(k)=0$ *and* $\nu_p(k)=\nu+n_2-1$, *where* ν *is the integer such that* $|A_0|=p^{\nu}$.

THEOREM 2. *Let* k *be a real quadratic field and* p *an odd prime number which splits in* $k/\boldsymbol{Q}$. *Assume that*

(1) $2\leqq n_1<n_2$,

(2) $|A_0|=1$, *and*

(3) A_0^* *is an elementary* p*-abelian group.*

Then, for $n\geqq n_1+n_2-2$, *we have* $(A_n:D_n)=p^{n_1-1}$, $|D_n|=p^{n_2-1}$, *and* $|A_n|=p^{n_1+n_2-2}$.

COROLLARY. *Under the same assumption as Theorem* 2, *we have* $\mu_p(k)=\lambda_p(k)=0$ *and* $\nu_p(k)=n_1+n_2-2$.

REMARK. Note that $1=n_1<n_2$ implies $A_0=D_0$. Hence, the assumption (2) of Theorem 1 is essential only when $n_1=n_2=1$ and $|A_0|\neq 1$.

§ 1. Proof of theorems.

We first refer to the following proposition.

PROPOSITION 1 (cf. Proposition 1 [**1**]). *Let* k *be a real quadratic field and* p *an odd prime number which splits in* $k/\boldsymbol{Q}$. *Then, for* $n\geqq n_2-1$, *we have* $|B_n|=|A_0|\cdot p^{n_2-1}$.

LEMMA 1. *Let* k *and* p *be as in Proposition* 1. *Then,*

(1) $D_n^{hkp^{n_2-n_1-1}}\not\subset A_n^{\sigma-1}$ *for* $n\geqq n_2-1$, *and*

(2) $D_n^{hkp^{n_2-n_1}}\subset A_n^{\sigma-1}$ *for* $n\geqq n_2-1$.

PROOF. Let $p=\mathfrak{P}\mathfrak{P}'$ be the prime factorization of p in k, and $\mathfrak{P}^{hk}=(\alpha)$ for $\alpha\in k$. Let $\mathfrak{P}_n$ be the prime of k_n lying above $\mathfrak{P}$. (1) Assume that $D_n^{hkp^{n_2-n_1-1}}\subset A_n^{\sigma-1}$. Then, $\mathfrak{P}_n^{hkp^{n_2-n_1-1}}=I_n^{\sigma-1}(\alpha_n)$ for some ideal I_n of k_n and $\alpha_n\in k_n$. Hence, $\alpha^{p^{n_2-n_1-1}}=\pm N_{n,0}(\alpha_n)\varepsilon^m$ for some integer m, where ε is the fundamental unit of k. Now, $N_{n,0}(\alpha_n)$ is $\mathfrak{P}'$-adic p^{n_2-1}-th power for $n\geqq n_2-1$ by local class field theory, and so is ε^m by definition of n_2. But $\alpha^{p^{n_2-n_1-1}}$ is just $\mathfrak{P}'$-adic p^{n_2-2}-th power. It is a contradiction. (2) By definition of n_1 and n_2, $\alpha^{(p-1)p^{n_2-n_1}}$ and ε^{p-1} are both generators of $1+p^{n_2}\boldsymbol{Z}_p$. Hence, if we put $x=\alpha^{p^{n_2-n_1}}\varepsilon^m$, then $x^{p-1}\in 1+p^{n+1}\boldsymbol{Z}_p$ for some integer m. Then, by local class field theory, x is a $\mathfrak{P}'$-adic norm for k_n/k and also $\mathfrak{L}$-adic norm if $\mathfrak{L}$ is a prime of k_n prime to p. Hence, by the product formula of the norm residue symbol and Hasse's norm theorem, x is a global norm. Let $x=N_{n,0}(\alpha_n)$ for some $\alpha_n\in k_n$ and put $I_n=\mathfrak{P}_n^{hkp^{n_2-n_1}}(\alpha_n)^{-1}$. Then,

$$N_{n,0}(I_n)=(\alpha^{p^{n_2-n_1}}N_{n,0}(\alpha_n)^{-1})=(\varepsilon^{-m})=(1)\,.$$

Hence, $I_n=J_n^{\sigma-1}$ for some ideal J_n of k_n.

We consider A_n as a $\boldsymbol{Z}_p[G(k_\infty/k)]$-module.

LEMMA 2. *Let k and p be as in Proposition 1. If $|A_0|=1$ and $\rho\in 1+p\boldsymbol{Z}_p$, then $A_n/A_n^{\sigma-\rho}$ is a cyclic group for all $n\geqq 0$.*

PROOF. We denote also by σ the restriction of σ to k_n. Let L be the intermediate field of the Hilbert p-class field of k_n which corresponds to $A_n^{\sigma-\rho}$ and put $X=G(L/k_n)$. By class field theory, $X\cong A_n/A_n^{\sigma-\rho}$. Hence, if $\tilde{\sigma}$ denotes an extension of σ to L, then $\tilde{\sigma}^{-1}\tau\tilde{\sigma}=\tau^{\rho}$ for $\tau\in X$. If M denotes the intermediate field of L/k_n corresponding to X^p, we can easily see that M/k is an abelian extension. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be the primes of k lying above p and N the intermedite field of M/k corresponding to the inertia group of $\mathfrak{P}'$ for M/k. By the assumption $|A_0|=1$, $\mathfrak{P}$ is totally ramified in N/k. Therefore, $X/X^p\cong G(M/k_n)\cong G(N/k)$ is isomorphic to the Galois group of a finite abelian totally ramified extension of $k_{\mathfrak{P}}=\boldsymbol{Q}_p$ which is cyclic by local class field theory. Hence, X is cyclic.

Let $k^*=k(\zeta_p)$ where ζ_p is a primitive p-th root of unity and

$$k^*=k_0^*\subset k_1^*\subset k_2^*\subset\cdots\subset k_n^*\subset\cdots\subset k_\infty^*$$

the cyclotomic $\boldsymbol{Z}_p$-extension. We identify $G(k_\infty/k)$ and $G(k_\infty^*/k^*)$, and use the same topological generator σ. There exists a p-adic unit κ such that $\zeta^\sigma=\zeta^\kappa$ for all p-power-th roots of unity ζ.

LEMMA 3. *Let F be a finite extension of $\boldsymbol{Q}$, K a cyclic extension of F and ζ_m a primitive m-th root of unity. We assume that K contains ζ_m. Let L be a cyclic extension of K of degree m such that L is a Galois extension of F. We assume that there exists an element σ of $G(L/F)$ of order m such that the restriction of σ to K is a generator of $G(K/F)$. Let κ be an integer such that $\zeta_m^\sigma=\zeta_m^\kappa$. If $\sigma^{-1}\rho\sigma=\rho^\kappa$ for any element ρ of $G(L/K)$, then there exists an element a of F such that $L=K(a^{1/m})$.*

PROOF. Since the extension L/K is a Kummer extension, there exists an element α of K such that $L=K(\alpha^{1/m})$. Hence, there exists a generator τ of $G(L/K)$ such that $(\alpha^{1/m})^{\tau-1}=\zeta_m$. Now, we have

$$\left(\frac{(\alpha^{1/m})^{\sigma-1}}{\alpha^{1/m}}\right)^{\tau}=\frac{((\alpha^{1/m})^{\tau^\kappa})^{\sigma-1}}{\alpha^{1/m}\zeta_m}=\frac{(\alpha^{1/m}\zeta_m^\kappa)^{\sigma-1}}{\alpha^{1/m}\zeta_m}=\frac{(\alpha^{1/m})^{\sigma-1}}{\alpha^{1/m}}\,.$$

Hence, we have $(\alpha^{1/m})^{\sigma-1}\in K$. Since $N_{K/F}((\alpha^{1/m})^{\sigma-1})=1$, there exists an element d of K such that $(\alpha^{1/m})^{\sigma-1}=d^{1-\sigma}$. Put $a=\alpha d^m$. Then we have $a\in F$ and $L=K(a^{1/m})$.

LEMMA 4. *Let k and p be as in Proposition* 1. *Assume that* $|A_0|=1$ *and* A_0^* *is an elementary p-abelian group. Then, for* $n\geqq 1$, $|A_n/A_n^{\sigma-\kappa}|=p$ *if* $|A_n|\neq 1$.

PROOF. It is easy to see that $|A_n|\neq 1$ implies $|A_n/A_n^{\sigma-\kappa}|\neq 1$. By Lemma 2, $A_n/A_n^{\sigma-\kappa}$ is cyclic. Assume that $p^2 \mid |A_n/A_n^{\sigma-\kappa}|$. Let L_n be the intermediate field of the Hilbert p-class field of k_n which corresponds to $A_n^{p^2}A_n^{\sigma-\kappa}$. Then L_n/k_n is a cyclic unramified extension of degree p^2 and L_n/k is a Galois extension. By Lemma 3, there exists $a\in k^*$ such that $k_n^*L_n=k_n^*(a^{1/p^2})$. Since $k_n^*(a^{1/p^2})/k_n^*$ is unramified, $(a)=I_n^{p^2}$ for some ideal I_n of k_n^*. Let $p=(\mathfrak{P}\mathfrak{P}')^{p-1}$ be the prime factorization of p in k^*. Since an ideal of k^* prime to p is unramified in k_n^*/k^*, $(a)=I^{p^2}\mathfrak{P}^m\mathfrak{P}'^{m'}$ for some ideal I of k^* and $m, m'\in \boldsymbol{Z}$. Put $b=a^{(p-1)h_k}$. Since $\mathfrak{P}^{(p-1)h_k}$ and $\mathfrak{P}'^{(p-1)h_k}$ are both principal ideals of k, $(b)=I^{(p-1)h_kp^2}(\beta)$ for some $\beta\in k$. By the assumption on A_0^*, $I^{(p-1)h_kp}=(\gamma)$ for some $\gamma\in k^*$. Hence, $(b)=(\gamma^p\beta)$ in k^* and $b=\gamma^p\beta\eta$ for some unit η of k^*. Since k^* is a CM-field, $b^2=\pm\gamma^{2p}\beta^2\eta_+\zeta_p^r$ for some real unit η_+ of k^* and an integer r. Then, $k_n^*(a^{1/p})=k_n^*((b^2)^{1/p})=k_n^*((\beta^2\eta_+)^{1/p})$ is a cyclic extension of k_n^* of degree p. Since $k_n^*(a^{1/p^2})/k_n$ is an abelian extension, if we denote by $k_{n,+}^*$ the maximal real subfield of k_n^*, then the subextension $k_{n,+}^*((\beta^2\eta_+)^{1/p})/k_{n,+}^*$ is also a Galois extension of degree p. Now, $\beta^2\eta_+\in k_{n,+}^*$. Hence, it follows that $k_{n,+}^*((\beta^2\eta_+)^{1/p})$ contains ζ_p which is a contradiction.

PROOF OF THEOREM 1. Note that $A_n=D_nA_n^{\sigma-1}$ implies $A_n=D_n$. By using Proposition 1, it suffices to prove that $A_n=D_nA_n^{\sigma-1}$. (1) If $n_2=1$, then $|A_n/A_n^{\sigma-1}|=|B_n|=|A_0|=|D_0|$ for $n\geqq 0$ by Proposition 1. If follows that $A_n=D_nA_n^{\sigma-1}$. (2) If $n_2>1$, then

$$A_n\supset D_nA_n^{\sigma-1}\supset D_n^{h_kp^{n_2-2}}A_n^{\sigma-1}\supsetneqq A_n^{\sigma-1} \qquad \text{for} \quad n\geqq n_2-1$$

by Lemma 1. We see that $(A_n : A_n^{\sigma-1})=|B_n|=|A_0|\cdot p^{n_2-1}$ by Proposition 1. It is easy to see that

$$(D_nA_n^{\sigma-1} : D_n^{h_kp^{n_2-2}}A_n^{\sigma-1})=|A_0|\cdot p^{n_2-2}$$

by a group theoretical argument. Hence, we have $A_n=D_nA_n^{\sigma-1}$.

PROOF OF THEOREM 2. We see that $|A_n/A_n^{\sigma-\kappa}|=p$ and $|B_n|\geqq p^2$ for $n\geqq n_2-1$ by Proposition 1 and Lemma 4. From this together with the Greenberg's argument [**1**, p. 281-282], it can be shown that A_n is cyclic for all $n\geqq 0$. Thus Lemma 1 shows that $A_n^{\sigma-1}=D_n^{p^{n_2-n_1}}$ for $n\geqq n_2-1$. Hence, for $n\geqq n_2-1$,

$$|A_n|=(A_n : A_n^{\sigma-1})|A_n^{\sigma-1}|=|B_n||D_n^{p^{n_2-n_1}}|=\frac{|B_n||B_n^{p^{n_2-n_1}}|}{(B_n^{p^{n_2-n_1}} : D_n^{p^{n_2-n_1}})}=\frac{p^{n_1+n_2-2}}{(B_n : D_n)}.$$

Now, $(B_n : D_n) \leqq p^{n_1-1}$ since

$$B_n \supset D_n \supset D_n^{p^{n_2-n_1-1}} \supsetneqq 1.$$

We will show that if $n \geqq n_2-1$ and $(B_n : D_n)>1$, then $(B_n : D_n)>(B_{n+1} : D_{n+1})$, from which Theorem 2 follows. Assume that $(B_n : D_n)=(B_{n+1} : D_{n+1})=p^s$ for some $n \geqq n_2-1$ and $s \geqq 1$. We first observe that

$$(E_0 \cap N_{n,0}(k_n) : N_{n,0}(E_n))=(E_0 \cap N_{n+1,0}(k_{n+1}) : N_{n+1,0}(E_n))=p^s$$

by the assumption $|A_0|=1$ and Lemma 1 of [**1**] and $(E_0 \cap N_{n+1,0}(k_{n+1}) : N_{n,0}(E_n)) =p^{s-1}$ by Lemma 2 of [**1**]. Now, $|A_n|=|A_{n+1}|$ implies that $N_{n+1,n} : A_{n+1} \rightarrow A_n$ is an isomorphism and hence $|B_n|=|B_{n+1}|$ implies that $N_{n+1,n} : D_{n+1} \rightarrow D_n$ is also an isomorphism. Let I_n be an ideal of k_n such that $\mathrm{cl}(I_n) \in B_n$ and I_{n+1} an ideal of k_{n+1} such that $\mathrm{cl}(I_{n+1}) \in B_{n+1}$ and $N_{n+1,n}(\mathrm{cl}(I_{n+1}))=\mathrm{cl}(I_n)$. Then, $I_n^{\sigma-1}=(\alpha_n)$ for some $\alpha_n \in k_n$ and $I_{n+1}^{\sigma-1}=(\alpha_{n+1})$ for some $\alpha_{n+1} \in k_{n+1}$. We can choose α_n and α_{n+1} so that $N_{n+1,n}(\alpha_{n+1})=\alpha_n$. Since

$$(N_{n+1,0}(\alpha_{n+1}))=(N_{n,0}(\alpha_n))=N_{n,0}(I_n^{\sigma-1})=(1)$$

we see that $N_{n,0}(\alpha_n) \in E_0 \cap N_{n+1,0}(k_{n+1})$. Thus, $N_{n,0}(\alpha_n^{p^{s-1}}) \in N_{n,0}(E_n)$ and $N_{n,0}(\alpha_n^{p^{s-1}})=N_{n,0}(\varepsilon_n)$ for some $\varepsilon_n \in E_n$. The Hilbert Theorem 90 implies that $\alpha_n^{p^{s-1}} \varepsilon_n^{-1}=\beta_n^{\sigma-1}$ for some $\beta_n \in k_n$. Then, $I_n^{p^{s-1}}(\beta_n^{-1})$ is σ-invariant and $\mathrm{cl}(I_n^{p^{s-1}}) \in D_n$ by the assumption $|A_0|=1$. Hence, $B_n^{p^{s-1}} \subset D_n$. It is a contradiction.

§ 2. Examples.

Let $k=\boldsymbol{Q}(\sqrt{m})$. We will give all m less than 2000 such that $n_2 \geqq 2$ for $p=3$, 5 and 7. A star in the table indicates that we can not apply Theorem 1 or Theorem 2 to such m. We have been able to determine A_0^* only when $p=3$ or 5.

p=3

m	n_1	n_2	h_k	A_0^*	λ	m	n_1	n_2	h_k	A_0^*	λ
43	1	2	1			181	1	2	1		
58	1	2	2			199	1	2	1		
67	2	3	1	(3)		202	1	2	2		
79	1	2	3			238	2	3	2	(3)	
82	1	2	4			247	1	4	2		
85	1	2	2			253	2	2	1		*
103	2	2	1		*	271	1	3	1		
106	2	2	2		*	295	2	2	2		*
109	1	2	1			310	1	2	2		
139	2	2	1		*	322	1	2	4		
151	1	2	1			331	1	2	1		

p=3 (continued)

m	n_1	n_2	h_k	A_0^*	λ	m	n_1	n_2	h_k	A_0^*	λ
337	1	2	1			1249	1	2	1		
391	1	2	2			1258	1	3	4		
397	2	2	1		*	1261	2	2	2		*
406	1	2	2			1294	2	2	7		*
418	2	2	2		*	1297	1	2	11		
454	2	2	1		*	1318	2	2	1		*
457	1	2	1			1330	1	4	4		
502	1	2	1			1333	2	2	1		*
505	2	2	4		*	1339	1	3	6		
511	1	4	2			1390	3	4	2	(3)	
571	1	2	1			1399	1	2	1		
607	2	2	1		*	1429	1	3	5		
610	2	4	4	(3)		1453	1	4	1		
634	1	3	2			1462	2	2	4		*
667	1	2	2			1477	1	3	1		
679	2	2	2		*	1486	1	2	5		
694	1	2	1			1498	1	2	2		
727	2	3	5	(9)	*	1501	1	2	1		
730	1	3	12			1507	1	3	4		
733	1	3	3			1546	1	3	6		
745	2	2	2		*	1555	1	2	2		
751	1	2	1			1579	1	2	1		
754	1	2	4			1585	1	2	2		
787	2	2	1		*	1603	1	2	2		
790	2	2	2		*	1609	2	2	1		*
802	1	2	2			1642	2	2	2		*
865	1	2	2			1645	1	2	2		
871	1	2	2			1663	1	2	1		
874	1	2	6			1669	2	2	1		*
877	1	3	1			1678	1	3	1		
886	2	2	1		*	1699	1	2	1		
979	1	2	4			1714	2	2	12		*
994	2	2	8		*	1726	2	2	1		*
997	1	2	1			1738	2	2	2		*
1006	1	3	1			1753	2	2	1		*
1009	1	3	7			1795	1	2	2		
1027	1	3	2			1810	2	2	4		*
1051	1	3	1			1834	1	2	2		
1102	2	2	2		*	1855	1	3	4		
1114	1	2	2			1858	1	2	2		
1117	1	3	1			1867	2	6	1	(3)	
1126	1	2	5			1870	1	2	4		
1135	1	3	2			1882	1	3	6		
1153	2	2	1		*	1894	2	3	1	(3)	
1162	1	2	2			1897	1	2	5		
1165	1	3	2			1903	1	2	2		
1195	1	2	2			1921	1	2	2		
1213	1	7	1			1945	1	2	2		
1222	1	2	2			1951	1	3	1		
1237	1	2	1			1966	1	4	1		
1246	1	3	2			1993	1	2	1		

p=5

m	n_1	n_2	h_k	A_0^*	λ
39	1	2	2		
51	1	2	2		
69	1	2	1		
89	1	3	1		
109	1	2	1		
114	1	2	2		
134	1	2	1		
139	1	3	1		
161	1	2	1		
186	2	2	2		*
191	2	3	1	(5,5)	
211	1	2	1		
214	1	2	1		
241	1	2	1		
259	2	2	2		*
271	1	2	1		
314	1	2	2		
326	1	2	3		
366	1	2	2		
426	1	2	2		
434	2	3	4	(25)	*
466	1	2	2		
489	1	2	1		
501	2	2	1		*
509	1	2	1		
514	1	3	4		
519	1	2	2		
526	1	2	1		
534	2	2	2		*
541	1	2	1		
574	1	2	6		
581	1	2	1		
589	1	3	1		
606	1	4	2		
626	1	2	4		
629	1	2	2		
634	1	4	2		
674	1	2	4		
699	2	2	2		*
719	1	2	1		
734	2	2	1		*
761	1	2	3		
789	1	2	1		
791	2	2	4		*
869	1	2	1		
874	1	2	6		
881	2	2	1		*
966	1	2	4		
1031	2	2	1		*
1041	1	2	1		
1051	2	3	1	(5)	

m	n_1	n_2	h_k	A_0^*	λ
1074	1	2	2		
1079	2	2	2		*
1086	1	2	6		
1111	2	2	10		*
1191	1	2	6		
1194	1	3	2		
1214	1	2	1		
1231	1	2	1		
1261	2	2	2		*
1279	1	3	1		
1281	2	2	2		*
1289	1	2	1		
1301	1	2	1		
1321	1	2	1		
1339	1	2	6		
1351	1	2	8		
1366	1	2	1		
1389	1	2	1		
1406	1	2	6		
1426	1	2	4		
1434	1	2	2		
1441	1	2	1		
1461	1	2	1		
1479	1	2	4		
1529	1	2	1		
1531	1	2	1		
1586	1	2	4		
1621	1	3	1		
1631	1	2	4		
1641	1	4	5		
1686	1	2	2		
1699	1	2	1		
1731	1	3	4		
1741	1	2	1		
1754	2	2	2		*
1761	1	2	7		
1786	1	2	2		
1829	1	2	1		
1834	1	2	2		
1851	1	2	6		
1861	2	2	1		*
1874	1	2	2		
1891	1	2	2		
1914	2	2	8		*
1921	1	3	2		
1959	2	2	2		*
1966	1	2	1		
1969	2	2	1		*
1986	1	3	2		
1999	1	2	1		

p=7

m	n_1	n_2	h_k	λ	m	n_1	n_2	h_k	λ
23	1	2	1		989	1	2	1	
37	1	2	1		995	2	2	2	*
39	1	2	2		1045	1	2	4	
74	1	2	2		1073	1	2	2	
123	2	2	2	*	1103	1	2	1	
149	1	2	1		1121	1	2	1	
179	1	2	1		1171	1	2	3	
214	1	2	1		1194	1	2	2	
218	1	2	2		1247	1	2	2	
219	1	2	4		1306	1	4	2	
253	1	2	1		1311	1	2	4	
267	1	2	2		1353	1	2	2	
295	1	2	2		1373	3	3	3	*
303	1	2	2		1383	1	2	2	
337	1	2	1		1415	1	3	2	
403	1	2	2		1446	1	2	8	
415	1	2	2		1451	1	2	1	
417	2	2	1	*	1486	2	2	5	*
449	1	3	1		1590	2	2	8	*
470	1	2	2		1635	1	2	4	
478	1	2	1		1702	1	2	2	
494	1	2	2		1709	1	2	1	
501	1	2	1		1751	1	4	4	
505	1	2	4		1754	1	2	2	
519	1	2	2		1759	1	2	1	
554	2	2	2	*	1789	1	2	1	
583	1	2	2		1887	1	2	4	
695	2	2	2	*	1898	1	2	4	
751	1	2	1		1934	1	2	7	
758	1	2	1		1954	1	2	6	
767	1	2	2		1978	1	2	2	
771	1	2	2		1985	1	2	2	
834	2	3	2	*					

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