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Limits on $P(\omega)$/finite

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§ 1. Introduction.

Define the quasi-order $\leqq^*$ on $P(\omega)$ by $x \leqq^* y$, if $x \setminus y$ is finite. $x <^* y$ means that $x \leqq^* y$ and not $y \leqq^* x$. $x \sim y$ means that $x \leqq^* y$ and $y \leqq^* x$. $x \not\sim y$ means that not $x \sim y$. For any cardinal κ, a κ-sequence $X = \langle a_\alpha \mid \alpha < \kappa \rangle$ is said to be a κ-limit, if X is a $<^*$-descending sequence and, whenever $y \subset \omega$ and $\forall \alpha < \kappa$ $(y <^* a_\alpha)$, $y \sim \emptyset$. We abbreviate the statement "There is a κ-limit" by $\exists\kappa$-limit. Since $\exists\kappa$-limit holds for some cardinal κ, under the continuum hypothesis (CH), ω_1 is the unique cardinal κ such that $\exists\kappa$-limit. And, if $2^\omega = \omega_2$ holds, then the following (A), (B) and (C) are the only possible cases. (A) $\exists\omega_1$-limit$+\neg\exists\omega_2$-limit. (B) $\neg\exists\omega_1$-limit$+\exists\omega_2$-limit. (C) $\exists\omega_1$-limit$+\exists\omega_2$-limit. In fact, each of them is known to be compatible with $2^\omega = \omega_2$. If we start with a ground model of CH and add ω_2 Cohen reals, then we get a model of (A) (see [3]). The Martin's Axiom (MA)$+2^\omega = \omega_2$ implies (B). And, if we start with a ground model of (B) and add ω_1 Cohen reals, then we get a model of (C). The existence of κ-limits provides still a few problems when 2^ω is much more large. In this paper, we would like to make a contribution to this subject. Since $\exists\kappa$-limit implies $\exists \mathrm{cf}\kappa$-limit, we may restrict our interest to regular cardinals. Our result is the following.

THEOREM 1 (GCH). *Let n be a natural number. Let $\kappa_0, \cdots, \kappa_n$ and λ be regular cardinals such that $\omega_1 \leqq \kappa_0 < \cdots < \kappa_n \leqq \lambda$. Then, there exists a poset P which satisfies the following* (i)~(iv).

(i) *P satisfies the countable chain condition* (*the c.c.c.*).

(ii) $\Vdash_P$ "$2^\omega = \check{\lambda}$".

(iii) $\forall m \leqq n$ ($\Vdash_P$ "$\exists\check{\kappa}_m$-limit").

(iv) $\forall \theta$: *regular* $(\forall m \leqq n\, (\theta \neq \kappa_m) \Rightarrow \Vdash_P$ "$\neg\exists\check{\theta}$-limit").

The rest of the paper consists of three sections. Section 2 is for preliminaries. Sections 3 and 4 are entirely devoted to the proof of the theorem.

§ 2. Notions and notations.

We shall use current set theoretical notions and notations (see [1] or [2]). We assume that the reader is familiar with notions of finite support (FS)-iterated forcing. k, m and n denote natural numbers. α, β, η, ξ, δ, τ and σ denote ordinals. κ, λ and θ denote regular cardinals. For any set X, $P_{<\lambda}(X)$ denotes $\{x \subset X;\ |x|<\lambda\}$ and $P_{\leq\lambda}(X)$ denotes $\{x \subset X;\ |x| \leqq \lambda\}$. Let X be a subset of $P(\omega)$. X has *the strong finite intersection property* (*the sfip*), if $\forall x \subset X\ (|x|<\omega \Rightarrow \bigcap x \neq \emptyset)$. Let P and Q be posets. For any p, p' in P, $p \uparrow p'$ in P means that p and p' are compatible in P. P satisfies *the strong countable chain condition* (*the strong c.c.c.*), if $\forall W \subset P\ (|W|=\omega_1 \Rightarrow \exists W' \subset W (|W'|=\omega_1\ \&\ W'$ is pairwise compatible)). The complete Boolean algebra consisting of all regular open subsets of P is denoted by r.o.(P). The Boolean valued class associated with r.o.(P) is denoted by $V^{\mathrm{r.o.}(P)}$. We call elements in $V^{\mathrm{r.o.}(P)}$ *P-names.* If φ is an isomorphism from P to Q, then $\tilde{\varphi}$ denotes the isomorphism from $V^{\mathrm{r.o.}(P)}$ to $V^{\mathrm{r.o.}(Q)}$ induced by φ which is defined by the following:

For any x in $V^{\mathrm{r.o.}(P)}$,

$$\mathrm{dom}(\tilde{\varphi}(x))=\{\tilde{\varphi}(t);\ t \in \mathrm{dom}(x)\},$$

$$\tilde{\varphi}(x)(\tilde{\varphi}(t))=\varphi'' x(t) \qquad \text{for any} \quad t \in \mathrm{dom}(x).$$

P is *a complete subposet of* Q (denoted by $P \subset_{\mathrm{c}} Q$), if the following (i)~(iii) are satisfied.

(i) $P \subset Q\ \&\ \forall p, p' \in P\ (p \leqq p'$ in $P \Leftrightarrow p \leqq p'$ in $Q)$.

(ii) $\forall p, p' \in P\ (p \uparrow p'$ in $Q \Rightarrow p \uparrow p'$ in $P)$.

(iii) $\forall q \in Q\ \exists p \in P\ \forall p' \in P\ (p' \leqq p \Rightarrow p' \uparrow q)$.

Let $P \subset_{\mathrm{c}} Q$. Then, we regard r.o.(P) as a complete subalgebra of r.o.(Q). So, $V^{\mathrm{r.o.}(P)}$ is a subclass of $V^{\mathrm{r.o.}(Q)}$. The Boolean subclass associated with $V^{\mathrm{r.o.}(P)}$ in $V^{\mathrm{r.o.}(Q)}$ is the Boolean subclass $\boldsymbol{U}$ of $V^{\mathrm{r.o.}(Q)}$ which is defined by

$$\|x \in \boldsymbol{U}\| = \sum_{y \in V^{\mathrm{r.o.}(P)}} \|x=y\|, \qquad \text{for any} \quad x \in V^{\mathrm{r.o.}(Q)}.$$

For any posets $\langle P_i | i \in I \rangle$, *the finite-product of* $\langle P_i | i \in I \rangle$ is the poset $\{f;\ \exists J \subset I\ (|J|<\omega\ \&\ f \in \prod_{j \in J} P_j)\}$. Let P be a κ-stage FS-iteration. For any p in P, *the support of* p $(=\{\alpha<\kappa;\ p(\alpha) \neq \mathbf{1}\})$ is denoted by supp(p).

§ 3. Definition of the poset P.

Henceforth, in order to prove Theorem 1, we assume the generalized continuum hypothesis (GCH). Let n be any natural number. Let $\kappa_0, \cdots, \kappa_n$ and λ be any regular cardinals such that $\omega_1 \leqq \kappa_0 < \cdots < \kappa_n \leqq \lambda$. Set $\kappa=\kappa_n$ and $\bar{\kappa}=\kappa_0$.

Define the κ-stage FS-iteration S_ξ $(\xi \leqq \kappa)$ associated with T_ξ $(\xi<\kappa)$ and the

$S_{\xi+1}$-name a_ξ ($\xi<\kappa$) by the following induction on ξ.

Case 1. $\xi=0$. Define T_0 and a_0 by

$$T_0=2^{<\omega}\ (=\{t;\ \exists k<\omega\ (t: k\to 2)\}),$$
$$\Vdash_1 \text{“}a_0\subset\omega\text{”},$$
$$\|\check{k}\in a_0\|=\{s\in S_1;\ s(0)(\check{k})=1\},\qquad \text{for any}\quad k<\omega.$$

Case 2. $\xi=\eta+1$ for some η. Define T_ξ and a_ξ by

$$\Vdash_\xi \text{“}T_\xi=2^{<\omega}\text{”},$$
$$\Vdash_{\xi+1} \text{“}a_\xi\subset\omega\text{”},$$
$$\|\check{k}\in a_\xi\|=\{s\in S_{\xi+1};\ s\restriction\xi\ \Vdash_\xi \text{“}\check{k}\in a_\eta\ \&\ s(\xi)(\check{k})=1\text{”}\},$$

for any $k<\omega$.

Case 3. ξ is a limit ordinal. Define T_ξ and a_ξ by

$$\Vdash_\xi \text{“}T_\xi=P_{<\omega}(\omega)\times P_{<\omega}(\check{\xi})\text{”},$$
$$\Vdash_\xi \text{“}(u,x)\leqq(v,y) \text{ in } T_\xi \Leftrightarrow u\supset v\ \&\ x\supset y\ \&\ u\setminus v\subset \bigcap_{\eta\in y} a_\eta\text{”},$$
$$\Vdash_{\xi+1} \text{“}a_\xi\subset\omega\text{”},$$
$$\|\check{k}\in a_\xi\|=\{s\in S_{\xi+1};\ s\restriction\xi\ \Vdash_\xi \text{“}\check{k}\in \mathrm{dom}(s(\xi))\text{”}\},\qquad \text{for any}\quad k<\omega.$$

For any $\xi\leqq\kappa$, set

$$\bar{S}_\xi=\{s\in S_\xi;\ \forall\eta\in\mathrm{supp}(s)\setminus\{0\}\ \exists x(s(\eta)=\check{x})\}.$$

The following Lemmas 1 and 2 are easy. We omit proofs.

LEMMA 1. *Let ξ and η be ordinals such that $\xi<\eta<\kappa$. Then,*

(i) $\Vdash_{\eta+1}$ “$a_\eta\not\sim\emptyset$ & $a_\eta<^* a_\xi$”,

(ii) $\bar{S}_\eta$ *is dense in* S_η & $|\bar{S}_\eta|\leqq|\eta|+\omega$.

LEMMA 2. *Let $\xi<\kappa$ and W be a set. Suppose that $|W|=\omega_1$ and $\forall w\in W$ ($\Vdash_\xi$ “$\check{w}\in T_\xi$”). Then, there is $W'\subset W$ such that*

(i) $|W'|=\omega_1$,

(ii) $\forall w, z\in W'$ ($\Vdash_\xi$ “$\check{w}\uparrow\check{z}$ in T_ξ”).

LEMMA 3. *S_κ satisfies the strong c.c.c.*

PROOF. By Lemma 1 (ii), it suffices to show that $\bar{S}_\kappa$ satisfies the strong c.c.c. To show this, let W be any subset of $\bar{S}_\kappa$ with $|W|=\omega_1$. By the Δ-system lemma, there exist $W_1\subset W$ and $u\subset\kappa$ such that

$$|W_1|=\omega_1\quad\&\quad\forall s, s'\in W_1\ (s\neq s'\Rightarrow \mathrm{supp}(s)\cap\mathrm{supp}(s')=u).$$

Since u is finite, by using Lemma 2 $|u|$ times, we can obtain $W'\subset W_1$ such that

$$|W'|=\omega_1 \quad \& \quad \forall s, s'\in W' \,\forall \xi \in u\, (\Vdash_\xi \text{“}s(\xi)\uparrow s'(\xi) \text{ in } T_\xi\text{”}).$$

Then, W' is pairwise compatible in $\bar{S}_\kappa$. □

Define the poset $I=(I, \leqq)$ by

$$I=\kappa_0\times \cdots \times\kappa_n\times P_{<\kappa}(\lambda),$$

$$(\xi_0, \cdots, \xi_n, A)\leqq(\eta_0, \cdots, \eta_n, B) \iff \forall m\leqq n\,(\xi_m\leqq\eta_m) \;\&\; A\subset B.$$

It holds that $\forall i, i'\in I\ \exists j\in I\ (i\leqq j \;\&\; i'\leqq j)$.

DEFINITION. For any subset X of $P(\omega)$ with the sfip, define the poset $R_X=(P_{<\omega}(\omega)\times P_{<\omega}(X), \leqq)$ by

$$(u, x)\leqq(v, y) \iff u\supset v \;\&\; x\supset y \;\&\; u\setminus v\subset \bigcap y.$$

LEMMA 4. *Let X be a subset of $P(\omega)$ with the sfip. Then,*
(i) *R_X satisfies the strong c.c.c.,*
(ii) *there exists an R_X-name b such that*

$$\Vdash \text{“}b\subset\omega \;\&\; b\not\approx\emptyset\text{”} \quad \text{and} \quad \forall x\in X\ (\Vdash \text{“}b\leqq^*\check{x}\text{”}).$$

PROOF. Let $X\subset P(\omega)$ with the sfip.
(i) This follows from the fact that

$$\forall (u, x), (u, y)\in R_X \quad ((u, x\cup y)\leqq(u, x) \;\&\; (u, x\cup y)\leqq(u, y)).$$

(ii) Define the R_X-name b by

$$\Vdash \text{“}b\subset\omega\text{”},$$

$$\|\check{k}\in b\|=\{r\in R_X\,;\ k\in\mathrm{dom}(r)\} \qquad \text{for any } k<\omega.$$

Then, since X has the sfip, it is easy to see that b is as required. □

For each $i=(\xi_0, \cdots, \xi_n, A)\in I$, define the $\bar{\kappa}$-stage FS-iteration $P_\alpha(i)$ $(\alpha\leqq\bar{\kappa})$ associated with $Q_\alpha(i)$ $(\alpha<\bar{\kappa})$ by

$$Q_0(i)=\bar{S}_{\xi_0}\times \cdots \times\bar{S}_{\xi_n}\times\{f\,;\ \exists x\subset A\,(|x|<\omega \;\&\; f: x\to 2)\}$$

and, for $0<\alpha<\bar{\kappa}$,

$$\Vdash_\alpha \text{“}\Gamma_\alpha(i)=\{X\subset P(\omega)\,;\ |X|<\check{\kappa} \;\&\; X \text{ has the sfip}\}\text{”},$$

$$\Vdash_\alpha \text{“}Q_\alpha(i)=\text{the finite-product of } \langle R_X\,|\,X\in\Gamma_\alpha(i)\rangle\text{”}.$$

Set $P(i)=P_{\bar{\kappa}}(i)$.

LEMMA 5. $\forall i\in I$ *($P(i)$ satisfies the c.c.c. & $|P(i)|\leqq\kappa$).*
PROOF. This is easy. □

LEMMA 6. $\forall i, j\in I$ $(i\leqq j \Rightarrow P(i)\subset_c P(j))$.
PROOF. Let i and j be in I such that $i\leqq j$. We shall show by induction

on α ($\leqq\bar{\kappa}$) that

$$(*)\qquad P_\alpha(i)\subset_c P_\alpha(j).$$

The case which $\alpha\leqq 1$ or α is limit is easily checked. So, suppose that $\alpha=\beta+1$ ($\geqq 2$). By the induction hypothesis, $V^{\mathrm{r.o.}(P_\beta(i))}$ is a subclass of $V^{\mathrm{r.o.}(P_\beta(j))}$. Set $\boldsymbol{U}$ to be the Boolean subclass associated with $V^{\mathrm{r.o.}(P_\beta(i))}$ in $V^{\mathrm{r.o.}(P_\beta(j))}$. Since $\Vdash_\beta$ "$\Gamma_\beta(j)\cap\boldsymbol{U}=\Gamma_\beta(i)$", it holds that

$$\Vdash_\beta \text{“}Q_\beta(j)\cap\boldsymbol{U}=Q_\beta(i)\ \&\ Q_\beta(i)\subset_c Q_\beta(j)\text{”}.$$

We show first that $\forall p,\ p'\in P_\alpha(i)$ ($p\Uparrow p'$ in $P_\alpha(j)\Rightarrow p\Uparrow p'$ in $P_\alpha(i)$). Let p and p' be any elements of $P_\alpha(i)$ such that $p\Uparrow p'$ in $P_\alpha(j)$. Take $r\in P_\alpha(j)$ such that $r\leqq p$ and $r\leqq p'$. Then, it holds that

$$r\restriction\beta\leqq p\restriction\beta\ \ \&\ \ r\restriction\beta\leqq p'\restriction\beta\ \ \&\ \ r\restriction\beta\Vdash \text{“}p(\beta)\Uparrow p'(\beta)\text{ in }Q_\beta(j)\text{”}.$$

Since $\Vdash_\beta$ "$Q_\beta(i)\subset_c Q_\beta(j)$", we have that

$$r\restriction\beta\Vdash_\beta \text{“}p(\beta)\Uparrow p'(\beta)\text{ in }Q_\beta(i)\text{”}.$$

So, there are $\bar{r}\in P_\beta(j)$ and a $P_\beta(i)$-name q such that

$$\Vdash \text{“}q\in Q_\beta(i)\text{”}\ \ \&\ \ \bar{r}\leqq r\restriction\beta\ \ \&\ \ \bar{r}\Vdash\text{“}q\leqq p(\beta)\ \&\ q\leqq p'(\beta)\text{”}.$$

By the induction hypothesis, take $\bar{p}\in P_\beta(i)$ such that

$$\forall p''\in P_\beta(i)\ (p''\leqq\bar{p}\Rightarrow p''\Uparrow\bar{r}).$$

Since $\bar{r}\leqq r\restriction\beta\leqq p\restriction\beta,\ p'\restriction\beta$, exchanging $\bar{p}$ if necessary, we may assume that $\bar{p}\leqq p\restriction\beta$ and $\bar{p}\leqq p'\restriction\beta$. Set $p_1=\bar{p}^\frown\langle q\rangle$. Then, it is easy to see that $\bar{p}\Vdash$"$q\leqq p(\beta)$ & $q\leqq p'(\beta)$". Thus, p_1 is as required.

Now, we show that $\forall p\in P_\alpha(j)\ \exists p_1\in P_\alpha(i)\ \forall p'\in P_\alpha(i)\ (p'\leqq p_1\Rightarrow p'\Uparrow p)$. Let p be in $P_\alpha(j)$. Since

$$\Vdash_\beta \text{“}p(\beta)\cap\boldsymbol{U}\in Q_\beta(j)\ \&\ p(\beta)\text{ is finite”},$$

we have that

$$\Vdash_\beta \text{“}p(\beta)\cap\boldsymbol{U}\in Q_\beta(i)\text{”}.$$

Take $\bar{p}\in P_\beta(j)$ and a $P_\beta(i)$-name q_1 such that

$$\bar{p}\leqq p\restriction\beta\ \ \&\ \ \Vdash_\beta\text{“}q_1\in Q_\beta(i)\text{”}\ \ \&\ \ \bar{p}\Vdash\text{“}p(\beta)\cap\boldsymbol{U}=q_1\text{”}.$$

Then, by the induction hypothesis, there is $\bar{p}_1\in P_\beta(i)$ such that

$$\forall p'\in P_\beta(i)\ (p'\leqq\bar{p}_1\Rightarrow p'\Uparrow\bar{p}).$$

Set $p_1=\bar{p}_1{}^\frown\langle q_1\rangle$. Then, p_1 is as required. □

Set $P=\operatorname{dir\,lim}\langle P(i)\,|\,i\in I\rangle$, i.e.,

$$P=\bigcup_{i\in I} P(i),$$

$$p\leqq p' \text{ in } P \iff \exists i\in I\ (p,\ p'\in P(i)\ \&\ p\leqq p' \text{ in } P(i)).$$

CONVENTION. For each $p\in P$, let $p(0)=(s_0^p, \cdots, s_n^p, f^p)$.

By Lemmas 5 and 6 and by the fact that $\forall J\subset I\ (|J|\leqq\omega \Rightarrow \exists i\in I\ \forall j\in J\ (j\leqq i))$, it holds that P satisfies the c.c.c. and $\forall i\in I\ (P(i)\subset_c P)$.

In the rest of this section, we shall show that P satisfies Theorem 1 (ii). First, since $|P|\leqq\sum_{i\in I}|P(i)|\leqq\kappa|I|=\lambda$, it holds that

$$\Vdash_P \text{"}2^\omega\leqq\check{\lambda}\text{"}.$$

The following Lemma shows that $\Vdash_P$ "$2^\omega\geqq\check{\lambda}$".

LEMMA 7. *$\Vdash_P$ "There are $\check{\lambda}$ Cohen generic reals over $\check{V}$".*

PROOF. Set $Q=\{p\in P;\ \mathrm{supp}(p)=\{0\}\ \&\ \forall m\leqq n\,(s_m^p=\mathbf{1})\}$. Then, Q is order isomorphic to the poset adding λ Cohen generic reals. And it is easy to see that $Q\subset_c P$. This lemma follows immediately from these facts. □

§ 4. Proofs of Theorem 1 (iii) and (iv).

LEMMA 8. *Let x be a P-name such that $\Vdash_P$ "$x\subset\omega$". "Then, there are $i\in I$ and a $P(i)$-name $\bar{x}$ such that $\Vdash_P$ "$x=\bar{x}$".*

PROOF. This lemma follows from the facts that P satisfies the c.c.c. and that $\forall J\subset I\ (|J|\leqq\omega \Rightarrow \exists i\in I\ \forall j\in J\ (j\leqq i))$. □

For each $\alpha\leqq\bar{\kappa}$ and $i\in I$, since $P_\alpha(i)\subset_c P$, we denote by $\boldsymbol{U}_\alpha(i)$ the Boolean subclass in $V^{\mathrm{r.o.}(P)}$ associated with $V^{\mathrm{r.o.}(P_\alpha(i))}$. For each $i\in I$, define the subset E_i of P by

$$E_i=\{p\in P;\ \forall\alpha\in\mathrm{supp}(p)\setminus\{0\}\ \exists q: P_\alpha(i)\text{-name}\ (p\restriction\alpha \Vdash \text{"}p(\alpha)\cap\boldsymbol{U}_\alpha(i)=q\text{"})\}.$$

LEMMA 9. *$\forall i,\ j\in I\ (i\leqq j \Rightarrow \forall\beta\leqq\bar{\kappa}\ (E_i\cap P_\beta(j)$ is dense in $P_\beta(j)))$.*

PROOF. Let $i,\ j\in I$ such that $i\leqq j$. We shall show by induction on $\beta\leqq\bar{\kappa}$ that

$$(*)' \qquad E_i\cap P_\beta(j) \text{ is dense in } P_\beta(j).$$

Since $(*)'$ is clear in cases that $\beta\leqq 1$ and that β is limit, we suppose that $\beta=\alpha+1$. Let $p\in P_\beta(j)$. By a similar argument to the proof of Lemma 6, there are $\bar{p}\in P_\alpha(j)$ and a $P_\alpha(i)$-name q such that

$$\bar{p}\leqq p\restriction\alpha \quad \& \quad \bar{p} \Vdash \text{"}p(\alpha)\cap\boldsymbol{U}_\alpha(i)=q\text{"}.$$

By the induction hypothesis, take $\bar{p}_1$ in $P_\alpha(j)$ such that $\bar{p}_1\leqq\bar{p}$ and $\bar{p}_1\in E_i$. Put $\tilde{p}=\bar{p}_1{}^\frown\langle p(\alpha)\rangle$. Then, $\tilde{p}$ is as required. □

LEMMA 10. *Let $i=(\xi_0, \cdots, \xi_n, A)\in I$ and $p\in E_i$. Then, there is $p_1\in P(i)$ which satisfies the following* $(\mathrm{R})_p$.

$$(\mathrm{R})_p \quad \begin{array}{l} \text{“For any } p'\in P(i) \text{ such that } p'\leqq p_1, \text{ there is } q\in P \text{ such that} \\ \text{(i)} \quad q\leqq p \ \& \ q\leqq p', \\ \text{(ii)} \quad \forall m\leqq n \ (s_m^q\restriction[\xi_m, \kappa_m)=s_m^p\restriction[\xi_m, \kappa_m)), \\ \text{(iii)} \quad f^q\restriction(\lambda\setminus A)=f^p\restriction(\lambda\setminus A).\text{”} \end{array}$$

PROOF. Let $i=(\xi_0, \cdots, \xi_n, A)\in I$ and $p\in E_i$. For each $\alpha\in\mathrm{supp}(p)\setminus\{0\}$, take a $P_\alpha(i)$-name r_α such that

$$\Vdash \text{“}r_\alpha\in Q_\alpha(i)\text{”} \qquad \text{and} \qquad p\restriction\alpha \Vdash \text{“}p(\alpha)\cap U_\alpha(i)=r_\alpha\text{”}.$$

Define $p_1\in P(i)$ by

$$\mathrm{supp}(p_1)=\mathrm{supp}(p),$$

$$p_1(\alpha)=\begin{cases} r_\alpha, & \text{if } \alpha\in\mathrm{supp}(p)\setminus\{0\}, \\ (s_0^p\restriction\xi_0, \cdots, s_n^p\restriction\xi_n, f^p\restriction A), & \text{if } \alpha=0. \end{cases}$$

Then, p_1 is as required. □

PROOF OF THEOREM 1 (iii). Let $m\leqq n$. For each $\delta<\kappa_m$, define the P-name b_δ by

$$\Vdash \text{“}b_\delta\subset\omega\text{”},$$

$$\|\check{k}\in b_\delta\|=\{p\in P;\ s_m^p \Vdash_{S_\kappa} \text{“}\check{k}\in a_\delta\text{”}\}, \qquad \text{for any } k<\omega.$$

LEMMA 11. $\forall\delta<\forall\tau<\kappa_m$ $(\Vdash_P \text{“}b_\tau\not\sim\emptyset \ \& \ b_\tau<^*b_\delta\text{”})$.

PROOF. Let δ and τ be ordinals such that $\delta<\tau<\kappa_m$. Set $i=(0, \cdots, 0, \underset{\widehat{m}}{\tau+1}, 0, \cdots, 0)\in I$. Define $\varphi: \bar{S}_{\tau+1}\to P_1(i)$ by

$$\varphi(s)=(0, \cdots, 0, \underset{\widehat{m}}{s}, 0, \cdots, 0) \qquad \text{for any } s\in\bar{S}_{\tau+1}.$$

Then, φ is an order isomorphism from $\bar{S}_{\tau+1}$ to $P_1(i)$ and $\tilde{\varphi}(a_\xi)=b_\xi$ for any $\xi\leqq\tau$. So, by Lemma 1, we have that

$$\Vdash \text{“}b_\tau\not\sim\emptyset \ \& \ b_\tau<^*b_\delta\text{”}. \qquad □$$

We claim that $\Vdash_P$ “X is a $\check{\kappa}_m$-limit”, where X is the P-name $\{(\check{\delta}, b_\delta)^{\mathrm{r.o.}(P)};\ \delta<\kappa_m\}\times\{1\}$. In order to show this claim, let x be any P-name such that

$$\Vdash \text{“}x\subset\omega \ \& \ x\not\sim\emptyset\text{”}.$$

We need to show that there is $\delta<\kappa_m$ such that $\Vdash$ “$x\setminus b_\delta\not\sim\emptyset$”. Using Lemma 8, if necessary, take $i=(\xi_0, \cdots, \xi_n, A)\in I$ such that x is a $P(i)$-name. Set $\delta=\xi_m+1$. We shall show that $\Vdash$ “$x\setminus b_\delta\not\sim\emptyset$”. To see this, let p be any element in P and k be any element in ω. Take $j=(\eta_0, \cdots, \eta_n, B)\in I$ such that

$$i \leqq j \quad \& \quad \delta < \eta_m \quad \& \quad p \in P(j).$$

By Lemma 9, there is $q \in P(j) \cap E_i$ such that $q \leqq p$. Take $k_1 < \omega$ such that $k \leqq k_1$ & $s_m^q \restriction \delta \Vdash$ "$\mathrm{dom}(s_m^q(\delta)) \subset \check{k}_1$". Applying Lemma 10 to q, take $p_1 \in P(i)$ satisfying $(\mathrm{R})_q$. Since $p_1 \in P(i)$, x is a $P(i)$-name and $\Vdash$ "$x \not\sim \emptyset$", there exist $p' \in P(i)$ and $k' < \omega$ such that

$$k_1 \leqq k' \quad \& \quad p' \leqq p_1 \quad \& \quad p' \Vdash \text{“}\check{k}' \in x\text{”}.$$

Since $p' \leqq p_1$ and $p' \in P(i)$, by virtue of the fact that p_1 satisfies $(\mathrm{R})_q$, there is $q_1 \in P$ such that

$$q_1 \leqq p' \quad \& \quad q_1 \leqq q \quad \& \quad s_m^{q_1} \restriction [\xi_m, \kappa_m) = s_m^q \restriction [\xi_m, \kappa_m).$$

Especially, $s_m^{q_1}(\delta) = s_m^q(\delta)$. So, take $s \in S_{\kappa_m}$ such that $s \leqq s_m^{q_1}$ and $s \restriction \delta \Vdash$ "$s(\delta)(\check{k}') = 0$". Take $q_2 \in P$ such that $q_2 \leqq q_1$ and $s_m^{q_2} = s$. Since $s \Vdash$ "$\check{k}' \notin a_\delta$", it holds that

$$q_2 \Vdash \text{“}\check{k}' \notin b_\delta\text{”}.$$

By this and by the fact that $q_2 \leqq p'$, we have that

$$q_2 \Vdash \text{“}\check{k}' \in x \setminus b_\delta\text{”} \quad \& \quad q_2 \leqq p.$$

The proof of our claim completes. □□

PROOF OF THEOREM 1 (iv). Suppose that
(1) θ is a regular cardinal and $\forall m \leqq n\, (\theta \neq \kappa_m)$,
(2) $p \in P$ and Y is a P-name,
(3) $\Vdash$ "$Y : \check{\theta} \to P(\omega)$" and $p \Vdash$ "Y is a $\check{\theta}$-limit".
We shall derive a contradiction. Take $i \in I$ such that $p \in P(i)$. By Lemma 8, for each $\delta < \theta$, take y_δ and i_δ such that

$$i_\delta \in I \quad \& \quad i \leqq i_\delta \quad \& \quad y_\delta \text{ is a } P(i_\delta)\text{-name} \quad \& \quad \Vdash \text{“}Y(\check{\delta}) = y_\delta\text{”}.$$

Let $i_\delta = (\xi_0^\delta, \cdots, \xi_n^\delta, A^\delta)$ for each $\delta < \theta$.

LEMMA 12. *There are $D \subset \theta$, $(\xi_0, \cdots, \xi_n) \in \kappa_0 \times \cdots \times \kappa_n$ and $\alpha < \bar{\kappa}$ such that*
(i) $|D| = \theta$,
(ii) $\forall \delta \in D\, (\xi_0^\delta \leqq \xi_0$ & $\cdots$ & $\xi_n^\delta \leqq \xi_n$ & y_δ *is a* $P_\alpha(i_\delta)$*-name*).
PROOF. This lemma follows easily from (1). □

Take $D \subset \theta$, $(\xi_0, \cdots, \xi_n) \in \kappa_0 \times \cdots \times \kappa_n$ and $\alpha < \bar{\kappa}$ which satisfy Lemma 12 (i) and (ii). Extending i_δ $(\delta \in D)$, we may assume that

$$i_\delta = (\xi_0, \cdots, \xi_n, A^\delta) \qquad \text{for each} \quad \delta \in D.$$

Case 1. $\theta < \kappa$. Set $A = \bigcup_{\delta \in D} A^\delta$ and $\bar{i} = (\xi_0, \cdots, \xi_n, A)$. Since $|A| < \theta \cdot \kappa = \kappa$, it holds that $\bar{i} \in I$ and $i \leqq \bar{i}$. Define the $P_\alpha(\bar{i})$-name X by

$$\mathrm{dom}(X) = \{y_\delta ; \delta \in D\},$$

$$X(y_\delta)=\mathbf{1} \quad \text{for any} \quad \delta\in D.$$

Since $p \Vdash$ "$X\in\Gamma_\alpha(\bar{i})$" in $P_\alpha(\bar{i})$, there exists a $P_{\alpha+1}(\bar{i})$-name b such that

$$p \Vdash \text{“}b\subset\omega \ \& \ b\not\sim\emptyset \ \& \ \forall x\in X\,(b\leqq^* x)\text{”}.$$

Since D is cofinal in θ, we have that

$$p \Vdash \text{“}\forall\delta<\check{\theta}\,(b\leqq^* Y(\delta))\text{”}.$$

This is a contradiction.

Case 2. $\kappa<\theta$. Since $\forall\delta\in D$ $(|A^\delta|<\kappa<\theta)$, by the Δ-system lemma, there exist $\bar{D}\subset D$ and $\bar{A}\subset\lambda$ such that

$$|\bar{D}|=\theta \quad \text{and} \quad \forall\delta,\tau\in\bar{D}\ (\delta\neq\tau\Rightarrow A^\delta\cap A^\tau=\bar{A}).$$

Thinning out $\bar{D}$, we may assume that

$$\forall\delta,\tau\in\bar{D}\ (|A^\delta\setminus\bar{A}|=|A^\tau\setminus\bar{A}|).$$

Set $j=(\xi_0,\cdots,\xi_n,\bar{A})$ and $\sigma=\min(\bar{D})$. Then, p is in $P(j)$.

In order to define the $P(i_\sigma)$-name x_δ $(\delta\in\bar{D})$, let δ be any element in $\bar{D}$. Take a bijection g_δ from $A^\delta\setminus\bar{A}$ to $A^\sigma\setminus\bar{A}$, and set $h_\delta=g_\delta\cup(\mathrm{id}\upharpoonright\bar{A})$. Define the order isomorphism ψ_δ from $P_1(i_\delta)$ to $P_1(i_\sigma)$ by

$$\psi_\delta(p)(0)=(s_0^p,\cdots,s_n^p,f^p\circ h_\delta^{-1}) \quad \text{for any} \quad p\in P_1(i_\delta).$$

ψ_δ can be extended canonically to the order isomorphism from $P(i_\delta)$ to $P(i_\sigma)$. We denote this isomorphism by φ_δ. Set $x_\delta=\tilde{\varphi}_\delta(y_\delta)$.

Since $\forall\delta\in\bar{D}$ ($\Vdash$ "$x_\delta\subset\omega$") and $|P(i_\sigma)|\leqq\kappa<\theta$, there are δ and τ in $\bar{D}$ such that $\delta<\tau$ and $\Vdash$ "$x_\delta=x_\tau$". Set $\tilde{j}=(\xi_0,\cdots,\xi_n,A^\delta\cup A^\tau)$. Define the permutation H on $A^\delta\cup A^\tau$ by $H=((g_\tau)^{-1}\circ g_\delta)\cup((g_\delta)^{-1}\circ g_\tau)\cup(\mathrm{id}\upharpoonright\bar{A})$. Define the automorphism $\Psi: P_1(\tilde{j})\to P_1(\tilde{j})$ by

$$\Psi(p)(0)=(s_0^p,\cdots,s_n^p,f^p\circ H^{-1}) \quad \text{for any} \quad p\in P_1(\tilde{j}).$$

Let $\Phi: P(\tilde{j})\to P(\tilde{j})$ be the canonical extension of Ψ. Since $H\upharpoonright\bar{A}=\mathrm{id}\upharpoonright\bar{A}$, it holds that $\Phi\upharpoonright P(j)=\mathrm{id}\upharpoonright P(j)$. So, especially, $\Phi(p)=p$. Moreover, since $\Phi\upharpoonright P(i_\tau)=(\varphi_\delta)^{-1}\circ\varphi_\tau$, it holds that

$$\tilde{\Phi}(y_\tau)=(\tilde{\varphi}_\delta)^{-1}\circ\tilde{\varphi}_\tau(y_\tau)=(\tilde{\varphi}_\delta)^{-1}(x_\tau).$$

Since $\Vdash$ "$x_\tau=x_\delta$", we have that

$$\Vdash \text{“}(\tilde{\varphi}_\delta)^{-1}(x_\tau)=y_\delta\text{”}.$$

Hence,

$$\Vdash \text{“}\tilde{\Phi}(y_\tau)=y_\delta\text{”}.$$

Similarly, $\Vdash$ "$\tilde{\Phi}(y_\delta)=y_\tau$". Since $p \Vdash$ "$y_\tau<^* y_\delta$", it holds that $\Phi(p) \Vdash$ "$\tilde{\Phi}(y_\tau)$

$<^*\tilde{\Phi}(y_\delta)$". So,

$$p \Vdash \text{“} y_\delta <^* y_\tau \text{”}.$$

This is a desired contradiction. □□

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