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The existence of nonexpansive retractions in Banach spaces

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1. Introduction.

In this paper, we consider the existence of nonexpansive retractions concerning nonexpansive mappings. The study of nonexpansive retraction is closely connected to differential equations and geometry of Banach spaces. Especially, it plays an important role in the theory of nonexpansive mappings (cf. Goebel and Reich [5]). In recent years, the existence of nonexpansive retractions has been shown from the mean ergodic theorems for nonexpansive mappings. In [1] Baillon proved the first mean ergodic theorem for nonexpansive mappings: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x\in C$, the Cesàro means

$$S_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}T^k x$$

converge weakly to some $y\in F(T)$. This theorem was extended to a uniformly convex Banach space with a Fréchet differentiable norm by Bruck [3], Hirano [6] and Reich [10]. In this case, putting $y=Px$ for each $x\in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT=TP=P$ and $Px\in\overline{\text{co}}\{T^n x: n=0, 1, \cdots\}$ for each $x\in C$. In [13], Takahashi proved the existence of such a retraction for an amenable semigroup of nonexpansive mappings in a Hilbert space. Recently Hirano-Takahashi [7] studied this result in a uniformly convex and uniformly smooth Banach space.

In this paper, we prove the existence of such a retraction in a uniformly convex Banach space without additional assumption. Further we find a sequence of means on $N=\{0, 1, \cdots\}$, generalizing the Cesàro means on N, such that the corresponding sequence of mappings converges to such a retraction.

2. Preliminaries.

Let S be a commutative semigroup and let $m(S)$ be the Banach space of all bounded real valued functions on S with supremum norm. An element $\mu \in m(S)^*$ (the conjugate space of $m(S)$) is called a mean on S if $\|\mu\|=\mu(1)=1$. Let μ be a mean on S and $f \in m(S)$. Then we denote by $\mu(f)$ the value of μ at the function f. According to the time and circumstances, we write by $\mu_t(f(t))$ or $\int f(t)d\mu(t)$ the value $\mu(f)$. For each $s \in S$, we also define an operator r_s of $m(S)$ into itself by $(r_s f)(t)=f(ts)$ for every $t \in S$ and $f \in m(S)$. We denote by $r_s{}^*$ the conjugate operator of r_s. A mean μ is said to be invariant if $\mu(r_s f)=\mu(f)$ for all $s \in S$ and $f \in m(S)$. For $s \in S$, we also define a mean δ_s on S by $\delta_s(f)=f(s)$ for every $f \in m(S)$. A mean μ is said to be finite if $\mu=\sum_{i=1}^n a_i \delta_{s_i}$, where $a_1, \cdots, a_n \geqq 0$ with $\sum_{i=1}^n a_i=1$ and $s_1, \cdots, s_n \in S$.

Let C be a closed convex subset of a uniformly convex Banach space E, let T be a nonexpansive mapping of C into itself with $F(T)\neq\emptyset$ and let μ be an element of $m(S)^*$. Fix $x \in C$ and consider a functional F on the conjugate space E^* of E given by $F(x^*)=\int\langle T^n x, x^*\rangle d\mu(n)$ for every $x^* \in E^*$, where $\langle y, x^*\rangle$ is the value of $x^* \in E^*$ at $y \in E$. Then F is bounded and linear. Since E is reflexive, there is an $x_0 \in E$ such that $\int\langle T^n x, x^*\rangle d\mu(n)=\langle x_0, x^*\rangle$ for every $x^* \in E^*$. Such an $x_0 \in E$ is denoted by $\mathcal{T}_\mu x$ or $\int T^n x\, d\mu(n)$. If μ is a mean on S, $\mathcal{T}_\mu x$ is an element in the closure of the convex hull of $\{x, Tx, T^2x, \cdots\}$. Let E be a Banach space. Then, with each $x \in E$, we associate the set $J(x)=\{x^* \in E^* : \langle x, x^*\rangle=\|x\|^2=\|x^*\|^2\}$. Using the Hahn-Banach theorem it is immediately clear that $J(x)\neq\emptyset$ for each $x \in E$. Then the multivalued operator $J: E \to E^*$ is called the duality mapping of E. Let $B=\{x \in E : \|x\|=1\}$. Then, the norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x+ty\|-\|x\|}{t}$$

exists for each x and y in B. It is said to be Fréchet differentiable if for each x in B, the limit is attained uniformly for y in B. It is well known that if E is smooth, then the duality mapping J is single valued. We also know the following result [3]: Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm and let T be a nonexpansive mapping of C into itself with $F(T)\neq\emptyset$. Then for each $x \in C$, the set

$$F(T) \cap \bigcap_{n \in N} \overline{\mathrm{co}}\{T^k x : k \geqq n\}$$

consists of at most one point.

3. Asymptotic behavior of nonexpansive mappings.

We begin with the following lemma which is obtained by using Bruck's result [**4**].

LEMMA 1. *Let C be a bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Let $\{\mu_\alpha\}$ be a net of finite means on $N=\{0, 1, \cdots\}$ such that $\|\mu_\alpha - r_1{}^*\mu_\alpha\| \to 0$ and let λ be a finite mean on N. Then, for any $x \in C$,*

$$\lim_\alpha \int \|T\mathcal{T}_\lambda T^{s+k}x - \mathcal{T}_\lambda T^{s+k+1}x\| \, d\mu_\alpha(s) = 0$$

uniformly in $k=0, 1, 2, \cdots$.

PROOF. By [**4**], we know that there exists a continuous, strictly increasing and convex function γ of $[0, \infty)$ into itself with $\gamma(0)=0$ such that for any $u_1, \cdots, u_m \in C$ and $\lambda_1, \cdots, \lambda_m \geqq 0$ with $\sum_{i=1}^m \lambda_i = 1$,

$$\gamma\Big(\Big\|T\Big(\sum_{i=1}^m \lambda_i u_i\Big) - \sum_{i=1}^m \lambda_i T u_i\Big\|\Big) \leqq \max_{1 \leq i, j \leq m} (\|u_i - u_j\| - \|Tu_i - Tu_j\|).$$

Let $\lambda = \sum_{i=1}^n a_i \delta_{t_i}$, where $a_1, \cdots, a_n \geqq 0$ with $\sum_{i=1}^n a_i = 1$ and $t_1, \cdots, t_n \in N$. Let $\varepsilon > 0$ and choose α_0 so large that $\|\mu_\alpha - r_1{}^*\mu_\alpha\| < \varepsilon / dn^2$ for every $\alpha \geqq \alpha_0$, where d is the diameter of C. Then, we have

$$\begin{aligned}
&\gamma\Big(\int \|T\mathcal{T}_\lambda T^{s+k}x - \mathcal{T}_\lambda T^{s+k+1}x\| \, d\mu_\alpha(s)\Big) \\
&\leqq \int \gamma(\|T\mathcal{T}_\lambda T^{s+k}x - \mathcal{T}_\lambda T^{s+k+1}x\|) \, d\mu_\alpha(s) \\
&\leqq \int \max_{1 \leq i, j \leq n} (\|T^{t_i+s+k}x - T^{t_j+s+k}x\| - \|T^{t_i+s+k+1}x - T^{t_j+s+k+1}x\|) \, d\mu_\alpha(s) \\
&\leqq \int \sum_{1 \leq i, j \leq n} (\|T^{t_i+s+k}x - T^{t_j+s+k}x\| - \|T^{t_i+s+k+1}x - T^{t_j+s+k+1}x\|) \, d\mu_\alpha(s) \\
&= \sum_{1 \leq i, j \leq n} \int \|T^{t_i+s+k}x - T^{t_j+s+k}x\| \, d(\mu_\alpha - r_1{}^*\mu_\alpha)(s) \\
&\leqq n^2 \cdot d \cdot \|\mu_\alpha - r_1{}^*\mu_\alpha\| < \varepsilon
\end{aligned}$$

for every $\alpha \geqq \alpha_0$ and $k \in N$. (Indeed, the first inequality follows from that μ_α is a finite mean and γ is a convex function. The third inequality follows from nonexpansiveness of T.) Since γ^{-1} is continuous and $\gamma^{-1}(0)=0$, we have the desired result.

By using Lemma 1, we prove the following:

THEOREM 1. *Let C be a bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Let $\{\mu_\alpha\}$ be a net of finite means on N such that $\|\mu_\alpha - r_1{}^*\mu_\alpha\| \to 0$ and let $x \in C$. Then for any weak neighborhood W of $F(T)$, there exists α_0 such that*

$$\mathcal{T}_{\mu_\alpha} T^k x \in W \qquad \text{for all } \alpha \geqq \alpha_0 \text{ and } k \in N.$$

PROOF. We follow an idea of Bruck in [3]. Since $F(T)$ is weakly compact, it is easy to see that there exist a convex weak neighborhood W' of $F(T)$ and $\delta>0$ such that $W'+B_\delta\subset W$, where $B_\delta=\{x\in E:\|x\|\leqq\delta\}$. By Lemma 1.4 in [3], choose $\varepsilon>0$ so small that $\|x-Tx\|\leqq\varepsilon\Rightarrow x\in W'$ and $0<\varepsilon<\delta/3d$, where $d=\sup\{\|x\|:x\in C\}$. Then choose α_1 so large that $\|\mu_\alpha-r_1{}^*\mu_\alpha\|<\varepsilon^2/2d$ for all $\alpha\geqq\alpha_1$. Let $\lambda=\mu_{\alpha_1}=\sum_{i=1}^n a_i\delta_{t_i}$, where $a_1,\cdots,a_n\geqq 0$ with $\sum_{i=1}^n a_i=1$ and $t_1,\cdots,t_n\in N$. By Lemma 1, there exists α_0 with $\alpha_0\geqq\alpha_1$ such that

$$\int\|T\mathcal{T}_\lambda T^{s+k}x-\mathcal{T}_\lambda T^{s+k+1}x\|\,d\mu_\alpha(s)<\varepsilon^2/2$$

and $\|\mu_\alpha-r_1{}^*\mu_\alpha\|<\varepsilon/\tau$ for every $\alpha\geqq\alpha_0$, where $\tau=\max\{t_i:1\leqq i\leqq n\}$. Fix k and let $y=T^k x$. Then for any $\alpha\geqq\alpha_0$ we have

$$\begin{aligned}&\int\|\mathcal{T}_\lambda T^s y-T\mathcal{T}_\lambda T^s y\|\,d\mu_\alpha(s)\\ &\leqq\int\|\mathcal{T}_\lambda T^s y-\mathcal{T}_\lambda T^{s+1}y\|\,d\mu_\alpha(s)+\int\|\mathcal{T}_\lambda T^{s+1}y-T\mathcal{T}_\lambda T^s y\|\,d\mu_\alpha(s)\\ &<\varepsilon^2/2+\varepsilon^2/2=\varepsilon^2.\end{aligned}$$

In fact, we have

$$\begin{aligned}\|\mathcal{T}_\lambda T^s y-\mathcal{T}_\lambda T^{s+1}y\|&=\sup_{\|x^*\|\leqq 1}\langle\mathcal{T}_\lambda T^s y-\mathcal{T}_\lambda T^{s+1}y,\ x^*\rangle\\ &=\sup_{\|x^*\|\leqq 1}\int\langle T^{t+s}y-T^{t+1+s}y,\ x^*\rangle\,d\lambda(t)\\ &=\sup_{\|x^*\|\leqq 1}\left\{\int\langle T^{t+s}y,\ x^*\rangle\,d\lambda(t)-\int\langle T^{t+s}y,\ x^*\rangle\,d(r_1{}^*\lambda)(t)\right\}\\ &\leqq\Big(\sup_{t,s\in N}\|T^{t+s}y\|\Big)\cdot\|\lambda-r_1{}^*\lambda\|<d\cdot\varepsilon^2/2d=\varepsilon^2/2.\end{aligned}$$

Fix α with $\alpha\geqq\alpha_0$ and let $\mu_\alpha=\sum_{j=1}^m b_j\delta_{s_j}$, where $b_1,\cdots,b_m\geqq 0$ with $\sum_{j=1}^m b_j=1$ and $s_1,\cdots,s_m\in N$. Put

$$A=\{j\ :\ 1\leqq j\leqq m\ \text{and}\ \|\mathcal{T}_\lambda T^{s_j}y-T\mathcal{T}_\lambda T^{s_j}y\|\geqq\varepsilon\}$$

and $B=\{1,2,\cdots,m\}\setminus A$. Then $\sum_{j\in A}b_j\leqq\varepsilon$. In fact, if $\sum_{j\in A}b_j>\varepsilon$, then we have

$$\begin{aligned}\varepsilon^2&>\int\|\mathcal{T}_\lambda T^s y-T\mathcal{T}_\lambda T^s y\|\,d\mu_\alpha(s)\\ &=\sum_{j=1}^m b_j\|\mathcal{T}_\lambda T^{s_j}y-T\mathcal{T}_\lambda T^{s_j}y\|\\ &\geqq\sum_{j\in A}b_j\|\mathcal{T}_\lambda T^{s_j}y-T\mathcal{T}_\lambda T^{s_j}y\|\\ &\geqq\Big(\sum_{j\in A}b_j\Big)\varepsilon>\varepsilon^2,\end{aligned}$$

which is a contradiction. Fix $f\in F(T)$ and define

$$x_1=\sum\{b_j f : j\in A\}+\sum\{b_j \mathcal{T}_\lambda T^{s_j} y : j\in B\}$$

and $x_2=\sum\{b_j(\mathcal{T}_\lambda T^{s_j}y-f) : j\in A\}$. Then since $\|\mathcal{T}_\lambda T^{s_j}y-T\mathcal{T}_\lambda T^{s_j}y\|<\varepsilon$ for all $j\in B$, we have $\mathcal{T}_\lambda T^{s_j}y\in W'$. Hence we have $x_1\in W'$, since W' is convex and $f\in W'$. We also obtain

$$\|x_2\|\leqq 2d\cdot\sum\{b_j : j\in A\}\leqq 2d\cdot\varepsilon<2\delta/3.$$

Thus, we have $\int\mathcal{T}_\lambda T^s y\,d\mu_\alpha(s)=x_1+x_2\in W'+B_{2\delta/3}$. While, we have

$$\begin{aligned}&\left\|\mathcal{T}_{\mu_\alpha}y-\int\mathcal{T}_\lambda T^s y\,d\mu_\alpha(s)\right\|\\ &\leqq\sum_{i=1}^n a_i\left\|\int(T^s y-T^{t_i+s}y)\,d\mu_\alpha(s)\right\|\\ &\leqq\sum_{i=1}^n a_i\cdot\sup_{s\in N}\|T^s y\|\cdot\|\mu_\alpha-r_{t_i}{}^*\mu_\alpha\|\\ &\leqq d\cdot(\varepsilon/\tau)\cdot\tau=d\cdot\varepsilon<\delta/3.\end{aligned}$$

Therefore, $\mathcal{T}_{\mu_\alpha}y=\mathcal{T}_{\mu_\alpha}T^k x\in W'+B_\delta\subset W$ for all $\alpha\geqq\alpha_0$.

4. Nonlinear ergodic theorems.

We prove in this section nonlinear ergodic theorems for nonexpansive mappings in a uniformly convex Banach space. First we obtain the existence of a nonexpansive "ergodic" retraction.

THEOREM 2. *Let C be a closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself with $F(T)\neq\emptyset$. Then there exists a nonexpansive retraction P of C onto $F(T)$ such that $PT=TP=P$ and $Px\in\overline{\mathrm{co}}\{T^n x : n=0, 1, 2, \cdots\}$ for each $x\in C$.*

PROOF. Consider $\mu_n=(1/n)\sum_{i=0}^{n-1}\delta_i\,(n=1, 2, \cdots)$. Then there is a subnet $\{\mu_{n_\alpha}\}$ of $\{\mu_n\}$ such that μ_{n_α} converges to a mean μ in the weak* topology. It is obvious that μ is invariant. Since μ_{n_α} converges to μ in the weak* topology, we have that $\mathcal{T}_{\mu_{n_\alpha}}x$ converges weakly to $\mathcal{T}_\mu x$ for each $x\in C$. So, put $Px=\mathcal{T}_\mu x$ for each $x\in C$. Then, for any x and y in C, we have

$$\begin{aligned}\|Px-Py\|^2&=\langle Px-Py, j\rangle\\ &=\lim_\alpha\langle\mathcal{T}_{\mu_{n_\alpha}}x-\mathcal{T}_{\mu_{n_\alpha}}y, j\rangle\\ &\leqq\sup_\alpha\|\mathcal{T}_{\mu_{n_\alpha}}x-\mathcal{T}_{\mu_{n_\alpha}}y\|\cdot\|j\|\\ &\leqq\|x-y\|\cdot\|Px-Py\|,\end{aligned}$$

where $j\in J(Px-Py)$. Hence P is nonexpansive on C. Since $\langle PTx, x^*\rangle =\langle \mathcal{T}_{\mu}Tx, x^*\rangle=\int\langle T^nTx, x^*\rangle d\mu(n)=\int\langle T^{n+1}x, x^*\rangle d\mu(n)=\int\langle T^nx, x^*\rangle d\mu(n)=\langle Px, x^*\rangle$ for every $x^*\in E^*$ and $x\in C$, we have $PT=P$. Since μ is a mean on N, we have

$$Px=\mathcal{T}_{\mu}x\in\overline{\mathrm{co}}\{T^nx : n=0, 1, 2, \cdots\}$$

for each $x\in C$. Finally we show $Px=\mathcal{T}_{\mu}x\in F(T)$ for each $x\in C$. Fix $x\in C$. Since $F(T)\neq\emptyset$, the set $\{x, Tx, T^2x, \cdots\}$ is bounded. So, we have a T-invariant bounded closed convex subset D of C with $D\supset\{x, Tx, T^2x, \cdots\}$. Let us apply Theorem 1 or [**3**, Cor. 1.1] to $\{\mu_n\}$, D and T_D, where T_D is the restriction of T to D. Then, we obtain $Px=\mathcal{T}_{\mu}x\in F(T_D)\subset F(T)$. This completes the proof.

The following theorem improves a result of Bruck [**3**].

THEOREM 3. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and let T be a nonexpansive mapping of C into itself with $F(T)\neq\emptyset$. Then there is a unique nonexpansive retraction P of C onto $F(T)$ such that $PT=TP=P$ and $Px\in\overline{\mathrm{co}}\{T^nx : n=0, 1, 2, \cdots\}$ for each $x\in C$. Further, if $\{\mu_\alpha\}$ is a net of finite means on N such that $\|\mu_\alpha-r_1^*\mu_\alpha\|\to 0$, then $\mathcal{T}_{\mu_\alpha}x$ converges weakly to Px for each $x\in C$.*

PROOF. By Theorem 2, there is a nonexpansive retraction P of C onto $F(T)$ such that $PT=TP=P$ and $Px\in\overline{\mathrm{co}}\{T^nx : n=0, 1, 2, \cdots\}$ for each $x\in C$. Fix $x\in C$ and $n\in N$. Then we have

$$Px=PT^nx\in\overline{\mathrm{co}}\{T^kT^nx : k=0, 1, 2, \cdots\}=\overline{\mathrm{co}}\{T^kx : k\geqq n\}.$$

Hence, $Px\in\bigcap_{n\in N}\overline{\mathrm{co}}\{T^kx : k\geqq n\}$. By [**3**], we also know that

$$F(T)\cap\bigcap_{n\in N}\overline{\mathrm{co}}\{T^kx : k\geqq n\}$$

consists of at most one point. Therefore we have

$$\{Px\}=F(T)\cap\bigcap_{n\in N}\overline{\mathrm{co}}\{T^kx : k\geqq n\}.$$

This implies that such a retraction is unique. Let $x\in C$ and let $\{\mathcal{T}_{\mu_{\alpha_\beta}}x\}$ be a subnet of $\{\mathcal{T}_{\mu_\alpha}x\}$ which converges weakly to $y\in C$. Then, by Theorem 1, we have $y\in F(T)$. Further $y\in\bigcap_{n\in N}\overline{\mathrm{co}}\{T^kx : k\geqq n\}$. In fact, fix $n\in N$. Then since $\|\mu_{\alpha_\beta}-r_n^*\mu_{\alpha_\beta}\|\to 0$, $\mathcal{T}_{\mu_{\alpha_\beta}}T^nx$ converges weakly to $y\in F(T)$. Hence $y\in\overline{\mathrm{co}}\{T^kx : k\geqq n\}$. So, we have the desired result.

REMARK. In Theorem 3, we know that $\mathcal{T}_{\mu_\alpha}T^kx$ converges weakly to Px

uniformly in $k=0, 1, 2, \cdots$.

References

[1] J.B. Baillon, Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert, C.R. Acad. Sci. Paris, **280** (1975), 1511-1514.
[2] F.E. Browder, "Nonlinear operators and nonlinear equations of evolutions in Banach spaces", Proc. Symposia in Pure Math., **18**-2, Amer. Math. Soc., 1976.
[3] R.E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, Israel J. Math., **32** (1979), 107-116.
[4] R.E. Bruck, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, Israel J. Math., **38** (1981), 304-314.
[5] K. Goebel and S. Reich, "Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings", Pure and Applied Mathematics, **83** (1984), Marcel Dekker.
[6] N. Hirano, A proof of the mean ergodic theorem for nonexpansive mappings in Banach space, Proc. Amer. Math. Soc., **78** (1980), 361-365.
[7] N. Hirano and W. Takahashi, Nonlinear ergodic theorems for an amenable semigroup of nonexpansive mappings in a Banach space, Pacific J. Math., **112** (1984), 333-346.
[8] K. Kido and W. Takahashi, Mean ergodic theorems for semigroups of linear continuous operators in Banach spaces, J. Math. Anal. Appl., **103** (1984), 387-394.
[9] K. Kido and W. Takahashi, Means on commutative semigroups and nonlinear ergodic theorems, to appear in J. Math. Anal. Appl.
[10] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl., **67** (1979), 274-276.
[11] G. Rodé, An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space, J. Math. Anal. Appl., **85** (1982), 172-178.
[12] W. Takahashi, Invariant functions for amenable semigroups of positive contractions on L^1, Kōdai Math. Sem. Rep., **23** (1971), 131-143.
[13] W. Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, Proc. Amer. Math. Soc., **81** (1981), 253-256.

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